Answers

1 $y = e^{x^2+x}$

$\ln y = x^2 + x$

Differentiating both sides

$\frac{1}{y}\frac{dy}{dx} = 2x+1$

Multiply both sides by y

$\frac{dy}{dx} = [2x+1]y$

$\frac{d^2y}{dx^2} = u\frac{dv}{dx} + v\frac{du}{dx}$

$v = 2x+1$ $\quad dv/dx = 2$

$u = y$ $\quad du/dx = \frac{du}{dy}\cdot\frac{dy}{dx} = 1\cdot\frac{dy}{dx}$

So that

$\frac{d^2y}{dx^2} = (2x+1)\cdot 1\frac{dy}{dx} + 2y$

$\frac{d^2y}{dx^2} = (2x+1)\frac{dy}{dx} + 2y$

$y'' = y'(2x+1) + 2y$

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$y^{(2)} = y^{(1)}(2x+1) + 2y$

Differentiating $y^{(1)}(2x+1)$

Let $v = 2x+1$ $\quad u = y'$

$v' = 2$ $\quad u^n = y^{(n+1)}$

$v^2 = 0$ $\quad u^{(n-1)} = y^n$

Recall from Leibnitz theorem

$u^n + nu^{(n-1)}v' + \cancel{\frac{n(n-1)u^{(n-2)}v^2}{2!}}$ $\quad$ But $v^2 = 0$

$= y^{(n+1)} \cdot (2x+1) + n\,(y^n) \cdot 2$

Differentiating $y^{(n)}$ we have $y^{(n+2)}$

$y^{(n+2)} = y^{(n+1)} \cdot (2x+1) + 2n\,y^n + 2y^n$

$y^{(n+2)} = (2x+1)\,y^{(n+1)} + 2y^n\,(n+1)$

2 Using Leibnitz theorem, given that

(i) $y = x^2 e^{4x}$, determining $y^{(5)}$

$u = e^{4x}$	$v = x^3$
$u^n = 4^n e^{4x}$	$v^1 = 3x^2$
$u^{(n-1)} = 4^{(n-1)} e^{4x}$	$v^2 = 6x$
$u^{(n-2)} = 4^{(n-2)} e^{4x}$	$v^3 = 6$
$u^{(n-3)} = 4^{(n-3)} e^{4x}$	$v^4 = 0$

where $n = 5$

$4^5 e^{4x} = 1024 e^{4x}$

$u^{(n-1)} = 4^{(5-1)} e^{4x} = 256 e^{4x}$

$u^{(n-2)} = 4^{(5-2)} e^{4x} = 64 e^{4x}$

$u^{(n-3)} = 4^{(5-3)} e^{4x} = 16 e^{4x}$

$y^{(5)} = [1024 e^{4x} \cdot x^3] + n(256 e^{4x} 3x^2) + \frac{n(n-1) 64 e^{4x} \cdot 6x}{2!} + \frac{n(n-1)(n-2)(16 e^{4x}) 6}{3!}$

we have

$= [1024x^3 + 384 0 x^2 + 384 0 x + 960)\, e^{4x}$

$= y^{(5)} = e^{4x}[1024x^3 + 3840x^2 + 3840x + 960]$

2 ii $\quad x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Show that $x^2 y^{(n+2)} + (2n+1)\,x\,y^{(n+1)} + (n^2+1)\,y^n = 0$

$x^2 y'' + xy' + y = 0$

$u = y^{(2)}$ $\quad v = x^2$

$u^n = y^{(n+2)}$ $\quad v^1 = 2x$ $\Rightarrow$ $y^{(n+1)}(x^2) + n(y^{n+1}) 2x + \frac{n(n-1)\,y^n (2)}{2!}\,v^3 = 0$

$y^{(n-?)} = y^{(n+1)}$

$y^{(n-2)} = y^{n}$

For (xy')

$V = x$ $\quad$ $u = y'$

$V' = 1$ $\quad$ $u^{n} = y^{(n+1)}$

$V^2 = 0$ $\quad$ $u^{(n-2)} = y$

Applying

$y^{n} = u^{n} + n u^{(n-1)} v^{(1)} + \cancel{\frac{n(n-1)}{2!} u^{(n-2)} v^{2}} + \ldots$

Since $v^2 = 0$

For (xy')

$y^{n} = y^{(n+1)} \cdot x + n y^{(n)}$

$y^{n} = x^2 y^{(n+2)} + 2x n y^{(n+1)} + n(n-1) y^{n} + x y^{(n+1)} + n y + y$

then

$y^{n} = x^2 y^{(n+2)} + x y^{(n+1)} \cdot (2n+1) + y^{n}(n^2 - n + n + 1) = 0$

$= x^2 y^{(n+2)} + x y^{(n+1)} (2n+1) + y^{n}(n^2+1) = 0$

$= y^{n} = x^2 y^{(n+2)} + x y^{(n+1)} (2n+1) + y^{n}(n^2+1) = 0$

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