(1) If $y = e^{x^2+x}$, show that

$y'' = y'(2x+1) + 2y$ and hence,

prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^n$

Soln

$y = e^{x^2+x}$

$y' = (2x+1)e^{x^2+x}$

$u = 2x+1 \qquad u' = 2$

$v = e^{x^2+x} \qquad v' = (2x+1)e^{x^2+x}$

$y'' = uv' + vu'$

$y'' = (2x+1)\left[(2x+1)e^{x^2+x}\right] + e^{x^2+x} \times 2$

but $y' = (2x+1)e^{x^2+x}$ & $y = e^{x^2+x}$

$y'' = y'(2x+1) - 2y$

$y'' = y'(2x+1) - 2u = 0$

Let $A = y''$

$u = y''$, $u^n = y^{n+2}$

$v = 1$, $v' = 0$

$y^n = u^n v + n v^{n-1} y'$

$y^n = y^{n+2} + n y^{n+1} \times 0$

$y^n = y^{n+2} \Rightarrow A$

$Y^n = Y^{n+1}(2x+1) + nY^n \times 2$

$Y^n = Y^{n+1}(2x+1) + 2nY^n \Rightarrow B'$

$u = Y$ Let $C = 2u$

$u = Y \qquad u^n = Y^n$

$V = 2 \qquad V' = 0$

$A - B - C = 0$

$A' - B' - C' = 0$

$Y^{n+2} = \left[Y^{n+1}(2x+1) + 2nY^n\right] - 2Y^n = 0$

$Y^{n+1} = Y^{n+1}(2x+1) + Y^n(2n+1)$

$Y^{n+1} = (2x+1)Y^{n+1} + 2(n+1)Y^n$

2 Using Leibnitz theorem

(a) $Y = \cancel{x^3 e^{4x}}\; x^3 e^{4x}$ determine Y^5

(b) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$ show that $x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2+1)y^n$

Soln

$y^n = 4^n e^{4x} \cdot x^3 + 3x^2 \cdot n \cdot 4^{n-1} e^{4x} + 3x(n^2-n)$
$+ n(n^2-3n+2)4^{n-3}e^{4x}$

$y^5 = 1024\, e^{4x}\, x^3 + 3 \cdot 256 \times 5 x^2 e^{4x}$

$y^5 = 1024x^3 e^{4x} + 3840x^2 e^{4x} + 3840x + 240 e^{4x}$

$y^5 = 64 e^{4x} x^3 \left[16 + \frac{60}{x} + \frac{60}{x^2} + \frac{15}{x^3} \right]$

$x^2 y'' + xy' + y = 0$

$y^n = u^n v + n u^{n-1} v' + \frac{n(n-1) u^{n-1} v^2}{2!}$
$+ \frac{n(n-1)(n-2)}{3!} \cdot u^{n-3} v^3$

$y = uv$

Let $A = x^2 y''$

$u = y''$ $\quad u^n = y^{n+2}$

$v = x^2$ $\quad v' = 2x$ $\quad v'' = 2$ $\quad v''' = 0$

$y^n = y^{n+2}$

$A' \Rightarrow y^n = y^{n+2} x^2 + n y^{n+1} \cdot 2x + n(n-1)$

Let $B = xy'$

$u = y'$ $\quad u^n = y^{n+1}$

$v = x$ $\quad v' = 1$ $\quad , v'' = 0$

$y^n = y^{n+1} \cdot x + n y^n$

Let $C = y$

M7 Power $\Rightarrow y^n = y^n$

$$y^{n+2} \cdot x^2 + y^{n+1} \cdot x(2n+1) + y^n(n^2 \cancel{-n+n} + 1) = 0$$

$$y^{n+2} \cdot x^2 + y^{n+1} \cdot x(2n+1) + y^n(n^2+1) = 0$$