

ASSIGNMENT VII

1. The model for the temperature distribution in a rod of length $L = 6\text{cm}$ is as given in eqn 1.

$$\frac{dT(x,t)}{dt} = \frac{C d^2 T(x,t)}{dx^2}$$

where $C = 2.2 \text{ cm}^2/\text{hr}$

With the conditions and the temperature ($^{\circ}\text{C}$)

$$T(x,0) = 3x^2$$

$$T(0,t) = 0$$

$$T(L,t) = -108$$

Using $\Delta t = 0.02\text{hr}$ and $\Delta x = 0.3\text{cm}$, obtain the temperature profile of the system for $0 \leq t \leq 0.3\text{hr}$

- a. Manually, in tabular form solving up to $t = 0.02\text{hr}$ and $x = 6\text{cm}$

Soln

Using Explicit Forward Euler Method

$$U_i^{k+1} = \gamma [U_i^k + U_{i-1}^k] + [1-2\gamma] U_i^k$$

when $i = 1$

$$U_{1,j+1} = \gamma U_{2,j} + \gamma U_{0,j} + (1-2\gamma) U_{1,j}$$

$$\text{but } \gamma = \frac{C \cdot dt}{(\Delta x)^2} = \frac{2.2 \times 0.02}{0.3^2} = 0.49$$

$$[1 - (2 \times 0.49)] = 0.02$$

Rewriting the explicit forward euler method

for when $i = 1$ to 19

$$U_{1,j+1} = 0.49 U_{0,j} + 0.49 U_{2,j} + 0.02 U_{1,j}$$

$$U_{2,j+1} = 0.49 U_{1,j} + 0.49 U_{3,j} + 0.02 U_{2,j}$$

$$\begin{aligned}
U_{3,j+1} &= 0.49U_{2,j} + 0.49U_{4,j} + 0.02U_{3,j} \\
U_{4,j+1} &= 0.49U_{3,j} + 0.49U_{5,j} + 0.02U_{4,j} \\
U_{5,j+1} &= 0.49U_{4,j} + 0.49U_{6,j} + 0.02U_{5,j} \\
U_{6,j+1} &= 0.49U_{5,j} + 0.49U_{7,j} + 0.02U_{6,j} \\
U_{7,j+1} &= 0.49U_{6,j} + 0.49U_{8,j} + 0.02U_{7,j} \\
U_{8,j+1} &= 0.49U_{7,j} + 0.49U_{9,j} + 0.02U_{8,j} \\
U_{9,j+1} &= 0.49U_{8,j} + 0.49U_{10,j} + 0.02U_{9,j} \\
U_{10,j+1} &= 0.49U_{9,j} + 0.49U_{11,j} + 0.02U_{10,j} \\
U_{11,j+1} &= 0.49U_{10,j} + 0.49U_{12,j} + 0.02U_{11,j} \\
U_{12,j+1} &= 0.49U_{11,j} + 0.49U_{13,j} + 0.02U_{12,j} \\
U_{13,j+1} &= 0.49U_{12,j} + 0.49U_{14,j} + 0.02U_{13,j} \\
U_{14,j+1} &= 0.49U_{13,j} + 0.49U_{15,j} + 0.02U_{14,j} \\
U_{15,j+1} &= 0.49U_{14,j} + 0.49U_{16,j} + 0.02U_{15,j} \\
U_{16,j+1} &= 0.49U_{15,j} + 0.49U_{17,j} + 0.02U_{16,j} \\
U_{17,j+1} &= 0.49U_{16,j} + 0.49U_{18,j} + 0.02U_{17,j} \\
U_{18,j+1} &= 0.49U_{17,j} + 0.49U_{19,j} + 0.02U_{18,j} \\
U_{19,j+1} &= 0.49U_{18,j} + 0.49U_{20,j} + 0.02U_{19,j}
\end{aligned}$$

For the boundary Condition

$T(x, 0) = 3x^2$ with x ranging from 0 to 6cm with step size of 0.3

$$T(x_1, 0) = 3x^2 = 3(0.3)^2 = 0.27$$

$$T(x_2, 0) = 3x^2 = 3(0.6)^2 = 1.08$$

$$T(x_3, 0) = 3x^2 = 3(0.9)^2 = 2.43$$

$$T(x_4, 0) = 3x^2 = 3(1.2)^2 = 4.32$$

$$T(x_5, 0) = 3x^2 = 3(1.5)^2 = 6.75$$

$$T(x_6, 0) = 3x^2 = 3(1.8)^2 = 9.72$$

$$T(x_7, 0) = 3x^2 = 3(2.1)^2 = 13.23$$

$$T(x_8, 0) = 3x^2 = 3(2.4)^2 = 17.28$$

$$T(x_9, 0) = 3x^2 = 3(2.7)^2 = 21.87$$

$$T(x_{10}, 0) = 3x^2 = 3(3)^2 = 27$$

$$T(x_{11}, 0) = 3x^2 = 3(3.3)^2 = 32.67$$

$$T(x_{12}, 0) = 3x^2 = 3(3.6)^2 = 38.88$$

$$T(x_{13}, 0) = 3x^2 = 3(3.9)^2 = 45.63$$

$$T(x_{14}, 0) = 3x^2 = 3(4.2)^2 = 52.92$$

$$T(x_{15}, 0) = 3x^2 = 3(4.5)^2 = 60.75$$

$$T(x_{16}, 0) = 3x^2 = 3(4.8)^2 = 69.12$$

$$T(x_{17}, 0) = 3x^2 = 3(5.1)^2 = 78.03$$

$$T(x_{18}, 0) = 3x^2 = 3(5.4)^2 = 87.48$$

$$T(x_{19}, 0) = 3x^2 = 3(5.7)^2 = 97.47$$

$$T(x_{20}, 0) = 3x^2 = 3(6)^2 = 108$$

$$T(0, t) = 0, \quad T(L, t) = 108$$

temperature has a range of 0 to 0.3hr with step size of 0.02hr. To get to 0.02hr, $j=0$

When $j=0$ (replacing U with T)

$$\begin{aligned} T_{1,1} &= 0.49U_{0,0} + 0.49U_{2,0} + 0.02U_{1,0} \\ &= 0.49(0) + 0.49(1.08) + 0.02(0.27) \\ &= 0.5346 \end{aligned}$$

$$\begin{aligned} T_{2,1} &= 0.49U_{1,0} + 0.49U_{3,0} + 0.02U_{2,0} \\ &= 0.49(0.27) + 0.49(2.43) + 0.02(1.08) \\ &= 1.3446 \end{aligned}$$

$$\begin{aligned} T_{3,1} &= 0.49U_{2,0} + 0.49U_{4,0} + 0.02U_{3,0} \\ &= 0.49(1.08) + 0.49(4.32) + 0.02(2.43) \\ &= 2.6946 \end{aligned}$$

$$\begin{aligned} T_{4,1} &= 0.49U_{3,0} + 0.49U_{5,0} + 0.02U_{4,0} \\ &= 0.49[2.43] + 0.49[6.75] + 0.02[4.32] \\ &= 4.5846 \end{aligned}$$

$$\begin{aligned} T_{5,1} &= 0.49U_{4,0} + 0.49U_{6,0} + 0.02U_{5,0} \\ &= 0.49[4.32] + 0.49[9.72] + 0.02[6.75] \\ &= 7.0146 \end{aligned}$$

$$\begin{aligned} T_{6,1} &= 0.49T_{5,0} + 0.49U_{7,0} + 0.02T_{6,0} \\ &= 0.49[6.75] + 0.49[13.23] + 0.02[9.72] \\ &= 9.9846 \end{aligned}$$

$$\begin{aligned} T_{7,1} &= 0.49T_{6,0} + 0.49T_{8,0} + 0.02T_{7,0} \\ &= 0.49[9.72] + 0.49[17.25] + 0.02[13.23] \\ &= 13.4946 \end{aligned}$$

$$\begin{aligned} T_{8,1} &= 0.49T_{7,0} + 0.49T_{9,0} + 0.02T_{8,0} \\ &= 0.49[13.23] + 0.49[21.87] + 0.02[17.25] \\ &= 17.5446 \end{aligned}$$

$$\begin{aligned} T_{9,1} &= 0.49T_{8,0} + 0.49T_{10,0} + 0.02T_{9,0} \\ &= 0.49[17.25] + 0.49[22] + 0.02[21.87] \\ &= 22.1346 \end{aligned}$$

$$\begin{aligned}
 T_{10,1} &= 0.49T_{9,0} + 0.49T_{11,0} + 0.02T_{10,0} \\
 &= 0.49(21.87) + 0.49(32.67) + 0.02(27) \\
 &= 27.2646
 \end{aligned}$$

$$\begin{aligned}
 T_{11,1} &= 0.49T_{10,0} + 0.49T_{12,0} + 0.02T_{11,0} \\
 &= 0.49(27) + 0.49(38.88) + 0.02(32.67) \\
 &= 32.9346
 \end{aligned}$$

$$\begin{aligned}
 T_{12,1} &= 0.49T_{11,0} + 0.49T_{13,0} + 0.02T_{12,0} \\
 &= 0.49(32.67) + 0.49(45.63) + 0.02(38.88) \\
 &= 39.1446
 \end{aligned}$$

$$\begin{aligned}
 T_{13,1} &= 0.49T_{12,0} + 0.49T_{14,0} + 0.02T_{13,0} \\
 &= 0.49(38.88) + 0.49(59.25) + 0.02(45.63) = 39.45.8946
 \end{aligned}$$

$$\begin{aligned}
 T_{14,1} &= 0.49T_{13,0} + 0.49T_{15,0} + 0.02T_{14,0} \\
 &= 0.49(45.63) + 0.49(60.75) + 0.02(52.92) \\
 &= 53.1846
 \end{aligned}$$

$$\begin{aligned}
 T_{15,1} &= 0.49T_{14,0} + 0.49T_{16,0} + 0.02T_{15,0} \\
 &= 0.49(52.92) + 0.49(69.12) + 0.02(60.75) = 61.0146
 \end{aligned}$$

$$\begin{aligned}
 T_{16,1} &= 0.49T_{15,0} + 0.49T_{17,0} + 0.02T_{16,0} \\
 &= 0.49(60.75) + 0.49(78.03) + 0.02(69.12) \\
 &= 69.3846
 \end{aligned}$$

$$\begin{aligned}
 T_{17,1} &= 0.49T_{16,0} + 0.49T_{18,0} + 0.02T_{17,0} \\
 &= 0.49(69.12) + 0.49(87.48) + 0.02(78.03) \\
 &= 78.2946
 \end{aligned}$$

$$\begin{aligned}
 T_{18,1} &= 0.49T_{17,0} + 0.49T_{19,0} + 0.02T_{18,0} \\
 &= 0.49(78.03) + 0.49(97.47) + 0.02(87.48) \\
 &= 87.7446
 \end{aligned}$$

$$\begin{aligned}
 T_{19,1} &= 0.49T_{18,0} + 0.49T_{20,0} + 0.02T_{19,0} \\
 &= 0.49(87.48) + 0.49(108) + 0.02(97.47) \\
 &= 97.7346
 \end{aligned}$$

Table For Solving up to $t = 0.02 \text{ hr}$ and $x = 6 \text{ cm}$

x	0	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7	3.0	3.3	3.6
$T(0)$	0	0.27	1.08	2.43	4.32	6.75	9.72	13.23	17.28	21.87	27.00	32.67	38.88
0.02	0	0.5846	1.3446	2.6946	4.5846	7.0146	9.9846	13.4946	17.5446	22.1346	27.2646	32.9346	39.1446

x	3.9	4.2	4.5	4.8	5.1	5.4	5.7	6.0
$T(0)$	45.63	52.92	60.75	69.12	78.03	87.48	97.47	108
0.02	45.8946	53.1846	61.0146	69.3846	78.2946	87.7446	97.7346	108