17ENG04/023
ECSANG ROSEMARY

ENG 381

ASSIGNMENT II

(1) If $y = e^{x^2+x}$

Show that

$$y'' = y'(2x+1) + 2y$$

Hence prove that

$$y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^n$$

(2) Using Leibnitz theorem, given that

(i) ~~$y = x^3 4x$~~ $y = x^3 e^{4x}$, determine $y^{(5)}$

(ii) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Show that

$$x^2 y^{(n+2)} + (2n+1)x y^{(n+1)} + (n^2+1)y^{(n)} = 0$$

Solution

(1) $y = e^{x^2+x}$

Let $u = x^2 + x$

$\therefore y = e^u$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\frac{du}{dx} = 2x+1$$

$$\frac{dy}{du} = e^{x^2+x} \times (2x+1)$$

$$= 2xe^{x^2+x} + e^{(x^2+x)}$$

$$\frac{d^2y}{dx^2} = e^{x^2+x}(2x+1)$$

Let $R = e^{x^2+x}$

and

$S = (2x+1)$

$$\frac{d^2y}{dx^2} = R\frac{ds}{dx} + S\frac{dR}{dx}$$

For

$$\frac{ds}{dx} = 2$$

$$\frac{dR}{dx} = \frac{d(e^{(x^2+x)})}{dx} = e^{x^2+x}(2x+1)$$

$$\frac{d^2y}{dx^2} = e^{x^2+x}(2) + (2x+1)^2 \cdot e^{x^2+x}$$

$$= 2e^{x^2+2} + e^{x^2+x}(2x+1)^2$$

Recall,

$$y = e^{x^2+x}$$

$$\frac{dy}{dx} = 2y^{u} + (2x+1)\frac{dy}{dx}$$

$$\frac{d^2y^{(2)}}{dx^2} = (2x+1)\frac{dy^{(1)}}{dx} + 2y$$

$$y^{(2)} = (2x+1)y^{(1)} + 2y$$

$$y^{u} = (2x+1)y^{(1)} + 2y$$

Finding the Nth term using Leibnitz's theorem

$$y^{(n)} = u^{n}v^{(0)} + n u^{(n-1)} v^{(1)} + \frac{n(n-1)}{2!} u^{(n-2)} v^{(2)} + \frac{n(n-1)(n-2)}{3!} u^{(n-3)} v^{(3)}$$

$$+ \frac{n(n-1)(n-2)(n-3)}{4!} u^{(n-4)} v^{(4)} + \dots$$

$$(2x+1)y' + 2y - y'' = 0$$

$w_1 = (2x+1)y'$

$u = y^{(1)}$ $\quad\quad v = 2x+1$

$u^{n} = y^{(n+1)}$ $\quad\quad v^{(1)} = 2$

$u^{(n-1)} = y^{(n)}$ $\quad\quad v^{(2)} = 0$

$w_2 = 2y$

$u = 2y$ $\quad\quad v = 1$

$u^{n} = 2y^{(n)}$ $\quad\quad v^{(1)} = 0$

$w_3 = -y''$

$u^{m} = -y^{(2)}$ $\quad\quad v = 1$

$u^{n} = -y^{(n+2)}$ $\quad\quad v^{(1)} = 0$

$$y^{(n)} = y^{(n+1)} \cdot (2x+1) + n y^{(n)} \cdot 2 + 2y^{(n)} - y^{(n+2)}$$

$$0 = (2x+1)y^{(n+1)} + 2ny^{(n)} + 2y^{(n)} - y^{(n+2)}$$

$$y^{(n+2)} = (2x+1)y^{(n+1)} + 2y^{(n)}(n+1)$$

② i) $y^{(n)} = u^n v + n u^{(n-1)} v^{(n)} + \frac{n(n-1)}{2!} u^{(n-2)} \cdot v^{(2)} + \frac{n(n-1)(n-2)}{3!} u^{(n-3)}$

$\cdot v^{(3)} + \frac{n(n-1)(n-2)(n-3)}{4!} u^{(n-4)} \cdot v^{(4)} + \cdots$

$n=5$

$y^{(5)} = u^5 v + 5 u^{(5-1)} v^{(5)} + \frac{5(5-1)}{2!} u^{(5-2)} \cdot v^{(2)} + \frac{5(5-1)(5-2)}{3!}$

$\cdot y^{(5-3)} \cdot v^{(3)} + \frac{5(5-1)(5-2)(5-3)}{4!} \cdot u^{(5-4)} \cdot v^{(4)} +$

$\frac{5(5-1)(5-2)(5-3)(5-4)}{5!} u^{(5-5)} \cdot v^{(5)}$ ~~$+ 5(5-1)(5-2)(5-3)(5-4)$~~

$\frac{5(5-1)(5-2)(5-3)(5-4)(5-5)}{6!} u^{(5-6)} \cdot v^{(6)}$

$y^{(5)} = u^{(5)} v^{(0)} + 5 u^{(4)} + v^{(1)} + 10 u^{(3)} v^{(2)} + 10 u^{(2)} \cdot v^{(3)}$
$+ 5 u^{(1)} \cdot v^{(4)} + u^{(0)} \cdot v^{(5)}$

$u^{(0)} = e^{4x}$	$v^{(0)} = x^3$
$u^{(1)} = 4e^{4x}$	$v^{(1)} = 3x^2$
$u^{(2)} = 16e^{4x}$	$v^{(2)} = 6x$
$u^{(3)} = 64e^{4x}$	$v^{(3)} = 6$
$y^{(4)} = 256e^{4x}$	$v^{(4)} = 0$
$u^{(5)} = 1024e^{4x}$	$v^{(5)} = 0$

Substituting into Leibnitz's theorem formula

$y^{(5)} = 1024e^{4x} \cdot x^3 + 5(256e^{4x}) \cdot 3x^2 + 10(64e^{4x}) \cdot 6x + 10(16e^{4x}) \cdot 6$
$+ 5(4e^{4x}) \cdot 0 + e^{4x} \cdot 0$

$= x^3 1024e^{4x} + x^2 3840e^{4x} + x 3840e^{4x} + 960e^{4x} + 0 + 0$

$y^{(5)} = x^3 1024e^{4x} + (x^2 + x) 3840e^{4x} + 960e^{4x}$

2.5 $\dfrac{x^2 d^2y}{dx^2} + \dfrac{x\,dy}{dx} + y = 0$

$x^2 y'' + xy' + y = 0$

$x^2 y^{(2)} + xy^{(1)} + y^{(0)} = 0$

Finding the nth term

$y^{(n)} = U^n V^{(0)} + n U^{(n-1)} V^{(1)} + \dfrac{n(n-1)}{2!} U^{(n-2)} V^{(2)} + \dfrac{n(n-1)(n-2)}{3!} U^{(n-3)} V^{(3)}$

$+ \dfrac{n(n-1)(n-2)(n-3)}{4!} U^{(n-4)} V^{(4)} + \dots$

$W_1 = x^2 y^{(2)}$

$U = y^{(2)}$ $\quad V = x^2$

$U^n = y^{(n+2)}$ $\quad V^{(1)} = 2x$

$U^{(n-1)} = y^{(n+1)}$ $\quad V^{(2)} = 2$

$U^{(n-2)} = y^{(n)}$ $\quad V^{(3)} = 0$

$W_2 = xy'$

$U = y^{(1)}$ $\quad V = x$

$U^n = y^{(n+1)}$ $\quad V^{(1)} = 1$

$U^{(n-1)} = y^{(n)}$ $\quad V^{(2)} = 0$

$W_3 = y^{(0)}$

$U = y^{(0)}$ $\quad V = 1$

$U^n = y^{(n)}$ $\quad V^{(1)} = 0$

$y^{(n)} = y^{(n+2)} x^2 + n y^{(n+1)} \cdot 2x + \dfrac{n(n-1)}{\cancel{2!}} y^{(n)} \cdot \cancel{2}$

$0 = x^2 y^{(n+2)} + 2xn y^{(n+1)} + n^2 y^{(n)} - \cancel{n y^{(n)}} + x y^{(n+1)} +$

$y^{(n)} + \cancel{n y^{(n)}}$

$0 = x^2 y^{(n+2)} + x y^{(n+1)} \cdot (2n+1) + y^{n} (n^2+1)$