(1) $y = e^{x^2+x}$.

show that

$y'' = y'(2x+1) + 2y$

and hence prove that

$y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{(n)}$

solution.

$y = e^{x^2+x}$

Differentiate

$y = y'$ and $y = y''$

$y' = (2x+1)e^{x^2+x}$

$y'' = (2x+1)(2x+1)e^{x^2+x} + e^{x^2+x} \cdot 2$

$y'' = (2x+1)y' + y \cdot 2$

Therefore.

$y'' = y'(2x+1) + 2y$.

Finding the n^{th} derivative of y''

$y^{(n+2)} = y^{(n+1)}(2x+1) + ny^{n} \cdot 2 + 2y^{n}$

$y^{(n+2)} = y^{(n+1)}(2x+1) + 2y^{n}(n+1)$

Therefore

$y^{(n+2)} = (2x+1)y^{(n+1)} + 2y^{(n)}(n+1)$

$y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{(n)}$

(2) $y = x^3 e^{4x}$ determine $y^{(5)}$.

solution.

$y^{n} = U^{n}V + nU^{(n-1)}V' + \frac{n(n-1)}{2!}U^{(n-2)}V'' + \frac{n(n-1)(n-2)}{3!}U^{(n-3)}V''' + \frac{n(n-1)(n-2)(n-3)}{3!}$

$U^{(n-4)}V^{iv} + \dots$

taking

$U = e^{4x}$ $\quad V = x^3$

$U^{n} = 4^{n}e^{4x}$ $\quad V' = 3x^2$

$U^{(n-1)} = 4^{(n-1)}e^{4x}$ $\quad V'' = 6x$

$U^{(n-2)} = 4^{(n-2)}e^{4x}$ $\quad V''' = 6$

$U^{(n-3)} = 4^{(n-3)} e^{4x}$ $\quad V^4 = 0$

$U^{(n-4)} = 4^{(n-4)} e^{4x}$ $\quad V^5 = 0$

$U^{(n-5)} = 4^{(n-5)} e^{4x}$

Substitute values in general solution.

$$y^n = 4^n e^{4x} \cdot x^3 + n\left[4^{(n-1)} e^{4x} \cdot 3x^2\right] + \frac{n(n-1)}{2!} \cdot 4^{(n-2)} e^{4x} \cdot 6x + \frac{n(n-1)(n-2)}{3!} 4^{(n-3)} e^{4x}$$

$$\cdot 6 + \frac{n(n-1)(n-2)(n-3)}{4!} 4^{(n-4)} e^{4x} \cdot 0$$

$$y^n = x^3 4^n e^{4x} + 3nx^2 4^{(n-1)} e^{4x} + \frac{n(n-1)}{2\times1} 6x \cdot 4^{(n-2)} e^{4x} + \frac{n(n-1)(n-2)}{3\times2\times1} \cdot 6 \cdot 4^{(n-3)} e^{4x} + 0$$

$$y^n = x^3 4^n e^{4x} + 3nx^2 + 4^{(n-1)} e^{4x} + \frac{n(n-1)}{2} 6x \cdot 4^{(n-2)} e^{4x} + \frac{6n(n-1)(n-2)}{6} \cdot 4^{(n-3)} e^{4x} + 0$$

$$y^n = x^3 4^n e^{4x} + 3nx^2 4^{(n-1)} e^{4x} + 3nx(n-1) 4^{(n-2)} e^{4x} + n(n-1)(n-2) 4^{(n-3)} e^{4x} + 0.$$

$y^n = y^5$, i.e $n = 5$.

$$y^5 = x^3 4^{(5)} e^{4x} + 3(5)x^2 4^{(5-1)} e^{4x} + 3(5)(5-1)x \cdot 4^{(5-2)} e^{4x} + 5(5-1)(5-2) 4^{(5-3)} e^{4x}$$

$$y^5 = 1024 x^3 e^{4x} + 15(256) x^2 e^{4x} + 15(4)(64) x e^{4x} + 5(4)(3)(16) e^{4x} + 0$$

$$y^5 = 1024 x^3 e^{4x} + 3840 x^2 e^{4x} + 3840 x e^{4x} + 960 e^{4x}.$$

(e(i)) $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$.

Show that

$$x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2+1) y^{(n)} = 0$$

solution

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$$

$$\underbrace{x^2 y''}_{a_1} + \underbrace{x y'}_{a_2} + \underbrace{y}_{a_3} = 0$$

Taking a_1

$u = y''$ $\quad V = x^2$

$U^n = y^{(n+2)}$ $\quad V' = 2x$

$U^{(n-1)} = y^{(n+1)}$ $\quad V'' = 2$

$U^n = y^n$ $\quad V''' = 0.$

Taking a_2.

$U = y'$ $\quad V = x$

$U^n = y^{(n+1)}$ $\quad V' = 1$

$U^{(n-1)} = y^n$ $\quad V'' = 0$

Taking a_3

$U = y$ $\quad V = 1$

$U^n =$ $\quad V' = 0$

$$y^{(n)} = U^n V + nU^{(n-1)} V' + \frac{n(n-1)}{2!} U^{(n-2)} V'' + \frac{n(n-1)(n-2)}{3!} U^{(n-3)} V''' + \frac{n(n-1)(n-2)(n-3)}{4!}$$

$$U^{(n-4)} V^{(4)} + \ldots$$

Therefore.

$$a_1^{(n)} = y^{(n+2)} \cdot x^2 + n\left[y^{(n+1)} \cdot 2x\right] + \frac{n(n-1)}{2!} \cdot y^n \cdot 2 + 0$$

$$a_2^{(n)} = y^{(n+1)} \cdot x + n\left[y^n \cdot 1\right] + 0.$$

$$a_3^{(n)} = y^n \cdot 1 + 0$$

Substitute values of $a_1^{(n)}$; $a_2^{(n)}$, $a_3^{(n)}$ into general equation

$$x^2 y'' + xy' + y = 0.$$

$$y^{(n+2)} \cdot x^2 + 2nx\, y^{(n+1)} + \frac{n(n-1)}{2 \times 1} \cdot 2y^n + y^{(n+1)} \cdot x + ny^n + y^n = 0$$

$$x^2 y^{(n+2)} + 2nx\, y^{(n+1)} + \frac{n(n-1)}{\cancel{2}} \cdot \cancel{2} y^n + xy^{(n+1)} + ny^n + y^n = 0$$

$$x^2 y^{(n+2)} + 2nx\, y^{(n+1)} + n(n-1)\, y^n + xy^{(n+1)} + ny^n + y^n = 0$$

$$x^2 y^{(n+2)} + 2nx\, y^{(n+1)} + xy^{(n+1)} + n(n-1)y^n + ny^n + y^n = 0$$

$$x^2 y^{(n+2)} + (2n+1)x\, y^{(n+1)} + (n^2 - \cancel{n} + \cancel{n} + 1)\, y^n = 0$$

$$x^2 y^{(n+2)} + (2n+1)\, x\, y^{(n+1)} + (n^2+1) y^n = 0$$