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17/Engo3/048
Civil Engineering

1. $y = e^{x^2+x}$ show that

$y'' = y'[2x+1] + 2y$

and hence prove that

$y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{(n)}$

Soln

$y = e^{x^2+x}$

Diff $y = y'$ & y''

$y' = (2x+1)e^{x^2+x}$

$y'' = (2x+1)(2x+1)e^{x^2+x} + e^{x^2+x}\cdot 2$

$y'' = (2x+1)y' + y\cdot 2$

$\therefore$ Therefore,

$y'' = y'(2x+1) \cancel{y' + y\cdot 2} + 2y$

finding the nth derivative of y''

i.e $y^{(n+2)} = y^{n+1}(2x+1) + n y^{n}\cdot 2 + 2y^{n}$

$y^{(n+2)} = y^{n+1}(2x+1) + 2y^{n}(n+1)$

$\therefore$ $y^{n+2} = (2x+1)y^{n+1} + 2y^{n}(n+1)$

$y^{n+2} = (2x+1)y^{n+1} + 2(n+1)y^{n}$.

2. $y = x^3 e^{4x}$, determine $y^{(5)}$.

Soln

$\therefore$ $y^{n} = U^{n}V + nU^{(n-1)}V' + \frac{n(n-1)}{2!}U^{(n-2)}V'' + \frac{n(n-1)(n-2)}{3!}U^{n-3}V'''$

$+ \frac{n(n-1)(n-2)(n-3)}{4!}U^{(n-4)}V^{4} + \ldots$

Therefore $U = e^{4x}$; $V = x^3$

$U^{n} = 4^{n}e^{4x}$, $V' = 3x^2$

$U^{n-1} = 4^{(n-1)}e^{4x}$ $V'' = 6x$

$U^{n-2} = 4^{(n-2)}e^{4x}$ $V''' = 6$

$U^{n-3} = 4^{(n-3}e^{4x}$ $V^{4} = 0$

$u^{(n-4)} = 4^{(n-4)} e^{4x}$, $v^{5} = 0$.

$u^{n-5} = 4^{(n-5)} e^{4x}$

substitute values in general solution.

$$y^n = 4^n e^{4x} \cdot x^3 + n\left(4^{(n-1)} e^{4x} \cdot 3x^2\right) + \frac{n(n-1)}{2!} \cdot 4^{(n-2)} e^{4x} \cdot 6x + \ldots$$

$$+ \frac{n(n-1)(n-2)}{3!} 4^{(n-3)} e^{4x} \cdot 6 + \frac{n(n-1)(n-2)(n-3)}{4!} 4^{(n-4)} e^{4x} \cdot 0.$$

$$= y^n = x^3 4^n e^{4x} + 3nx^2 4^{(n-1)} e^{4x} + \frac{n(n-1)}{2\times1} 6x \cdot 4^{(n-2)} e^{4x} + \frac{n(n-1)(n-2) \cdot 6}{3\times2\times1} 4^{n-3} e^{4x} + 0.$$

$$= y^n = x^3 4^n e^{4x} + 3nx^2 4^{(n-1)} e^{4x} + \left(\frac{n(n-1) \cancel{6x}}{\cancel{2}}\right) \cdot 4^{(n-2)} e^{4x} + \frac{6n(n-1)(n-2)}{6} 4^{(n-3)} e^{4x} + 0.$$

$$= y^n = x^3 4^n e^{4x} + 3nx^2 4^{(n-1)} e^{4x} + 3nx(n-1) 4^{n-2} e^{4x} + n(n-1)(n-2) 4^{(n-3)} e^{4x}$$

$= y^n = y^5$, i.e

$n = 5$; subs value $n = 5$ into equ above

$$y^5 = x^3 4^5 e^{4x} + 3(5)x^2 4^{(5-1)} e^{4x} + 3(5)(5-1)x 4^{(5-2)} e^{4x} + 5(5-1)(5-2) 4^{(5-3)} e^{4x} + 0.$$

$$y^5 = 1024x^3 e^{4x} + 15(256)x^2 e^{4x} + 15(4)(64)x e^{4x} + 5(4)(3)(16) e^{4x} + 0.$$

$$y^5 = 1024x^3 e^{4x} + 3840x^2 e^{4x} + 3840x e^{4x} + 960 e^{4x} + 0$$

(2) ii

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0,$$

show that

$$x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2+1) y^{(n)} = 0.$$

solution

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

$$x^2 y'' + xy' + y = 0.$$

Taking $a_1 = x^2 y''$

$a_2 = xy'$

$a_3 = y$

Considering a_1 , a_2 , a_3

$u = y''$ $\quad v = x^2$ $\quad u = y'$ $\quad v = x$ $\quad u = y$ $\quad v = 1$

$u^n = y^{n+2}$ $\quad v' = 2x$ $\quad u^n = y^{n+1}$ $\quad v' = 1$ $\quad u^n = y^n$ $\quad v' = 0$

$u^{n-1} = y^{n+1}$ $\quad v'' = 2$ $\quad u^{n-1} = y^n$ $\quad v'' = 0$

$u^{n-2} = y^n$ $\quad v''' = 0$

$$y^{(n)} = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{(n-2)} v'' + \frac{n(n-1)(n-2)}{3!} u^{(n-3)} v'''$$

$$+ \frac{n(n-1)(n-2)(n-3)}{4!} u^{(n-4)} v^{(4)} + \dots$$

Therefore,

$$a_1^{(n)} = y^{n+2} \cdot x^2 + n(y^{n+1} \cdot 2x) + \frac{n(n-1)}{2!} \cdot y^n \cdot 2 + 0$$

$$a_2^{(n)} = y^{n+1} \cdot x + n(y^n \cdot 1) + 0.$$

$$a_3^{(n)} = y^n \cdot 1 + 0.$$

Substitute values of a_1^n, a_2^n, a_3^n into general eq

$$x^2 y'' + xy' + y = 0$$

$$= y^{(n+2)} \cdot x^2 + 2nx\, y^{(n+1)} + \frac{n(n-1)}{2\times 1} \cdot 2y^n + y^{(n+1)} \cdot x + n y^n + y^n = 0$$

$$x^2 y^{(n+2)} + 2nx\, y^{(n+1)} + \frac{n(n-1)}{\cancel{2}} \cdot \cancel{2} y^n + x y^{(n+1)} + n y^n + y^n = 0$$

$$x^2 y^{(n+2)} + 2nx\, y^{(n+1)} + n(n-1) y^n + x y^{(n+1)} + n y^n + y^n = 0$$

$$x^2 y^{(n+2)} + 2nx\, y^{(n+1)} + x y^{(n+1)} + n(n-1) y^n + n y^n + y^n = 0$$

$$x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + [n^2 \cancel{-n+n} + 1] y^n = 0$$

$$x^2 y^{(n+2)} + (2n+1) x\, y^{(n+1)} + [n^2+1] y^n = 0.$$