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DEPARTMENT: COMPUTER ENGINEERING

1) If $y = e^{x^2+x}$, show that $y'' = y'(2x+1) + 2y$ and hence, prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^n$

$y = e^{x^2+x}$ ...(i)

Applying the chain rule, Let $u = x^2+x$; $y = e^u$

$du/dx = 2x+1$; $dy/du = e^u = e^{x^2+x}$

$\therefore dy/dx = dy/du \times du/dx = e^{x^2+x}(2x+1)$

$\therefore y' = e^{x^2+x}(2x+1)$

To find y'', applying the product rule

Let $u = e^{x^2+x}$, $du/dx = e^{x^2+x}(2x+1)$

$v = 2x+1$, $dv/dx = 2$

$dy/dx = u\,dv/dx + v\,du/dx$

$= (e^{x^2+x})(2) + (2x+1)(2x+1)e^{x^2+x}$

Since $y = e^{x^2+x}$ and $y' = (2x+1)e^{x^2+x}$

Substituting e^{x^2+x} and $(2x+1)e^{x^2+x}$ with y and y' respectively

$\therefore y'' = 2y + y'(2x+1)$

$y'' = y'(2x+1) + 2y$ ...(ii)

To prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^n$

Taking $y^{(n+2)}$ to the R.H.S,

$(2x+1)y^{(n+1)} + 2(n+1)y^n - y^{(n+2)} = 0$ ...(iii)

Relating eqn (iii) to eqn (ii)

$y^{n+1} = y'$, $y^{n+2} = y''$

Let $W_1 = y'(2x+1)$

$v = 2x+1$ $\quad u = y'$

$v^1 = 2$ $\quad u^n = y^{n+1}$

$u^{n-1} = y^{(n+1)-1} = y^n$

Applying Leibnitz theorem:

$y^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \dots$

$y^{(n)} = y^{(n+1)}(2x+1) + ny^{n}(2) + 0$

$y^{(n)} = (2x+1)y^{n+1} + 2(n)y^{n}$

Let $w_2 = 2y$

$u = y \qquad v = 2$

$u^{(n)} = y^{n} \qquad v^{(1)} = 0$

$y^{(n)} = (y^{n})(2) + 0 = 2y^{n}$

Let $w_3 = -y''$

$-y'' = -(y^{(2)})$

$u = y^{(2)} \qquad v = -1$

$u^{n} = y^{n+2} \qquad v^{(1)} = 0$

$y^{n} = y^{n+2}(-1) + 0 = -y^{n+2}$

$y^{n+1}(2x+1) + 2ny^{n} + 2y^{n} - y^{n+2} = 0$

$y^{n+1}(2x+1) + 2y^{n}(n+1) - y^{n+2} = 0$

$\therefore y^{n+1}(2x+1) + 2y^{n}(n+1) = y^{n+2}$

(2) Using the Leibnitz theorem, given that

(i) $y = x^3e^{4x}$, determine $y^{(5)}$

(ii) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$; show that $xy^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$

(i) $y = x^3e^{4x}$

$v = x^3 \qquad u^{n} = a^{n}e^{ax}$ where $a = 4$

$v^{(1)} = 3x^2 \qquad \therefore u^{n} = 4^{n}e^{4x}$

$v^{(2)} = 6x \qquad u^{n-1} = 4^{n-1}e^{4x}$

$v^{(3)} = 6 \qquad u^{n-2} = 4^{n-2}e^{4x}$

$u^{n-3} = 4^{n-3}e^{4x}$

$$y^{(n)} = u^n v + n u^{(n-1)} v^{(1)} + \frac{n(n-1) u^{(n-2)} v^{(2)}}{2!} + \frac{(n)(n-1)(n-2) u^{(n-3)} v^{(3)}}{3!} + \ldots$$

$$y^{(n)} = 4^n e^{4x}(x^3) + n(4^{(n-1)} e^{4x})(3x^2) + \frac{n(n-1) 4^{(n-2)} e^{4x}(6x)}{2!} + \frac{(n)(n-1)(n-2) 4^{(n-3)} e^{4x}(6)}{3!}$$

Since $n = 5$

$$y^5 = 4^5 e^{4x}(x^3) + (5)(4^{(5-1)} e^{4x})(3x^2) + \frac{(5)(5-1) 4^{(5-2)} e^{4x}(6x)}{2!} + \frac{(5)(5-1)(5-2) 4^{(5-3)} e^{4x}(6)}{3!}$$

$$y^5 = 4^5 e^{4x}(x^3) + (5)(4^4 e^{4x})(3x^2) + \frac{(5)(4) 4^{(3)} e^{4x}(6x)}{2!} + \frac{(5)(4)(3) 4^{(2)} e^{4x}(6)}{3!}$$

$$y^5 = 1024 e^{4x} x^3 + 3840 e^{4x} x^2 + 3840 e^{4x} x + 960 e^{4x}$$

$$y^5 = 64 e^{4x}[16x^3 + 60x^2 + 60x + 15]$$

(ii) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Rewriting as $x^2 y'' + xy' + y = 0$

Let $w_1 = x^2 y''$

$V^{(0)} = x^2$ $\quad$ $u^n = y'' = y^{(n+2)}$

$V^{(1)} = 2x$ $\quad$ $u^n = y^{(n+2)}$

$V^{(2)} = 2$ $\quad$ $u^{n-1} = y^{(n+2)-1} = y^{n+1}$

$V^{(3)} = 0$ $\quad$ $u^{n-2} = y^{(n+2)-2} = y^n$

Applying Leibnitz theorem,

$$w_1^n = u^n v + n u^{(n-1)} v^{(1)} + \frac{n(n-1) u^{(n-2)} v^{(2)}}{2} + 0$$

$$w_1^n = y^{(n+2)}(x^2) + (n) y^{(n+1)}(2x) + \frac{n(n-1)(y^n)(2)}{2!} + 0$$

$$w_1^n = x^2 y^{(n+2)} + (2x)(n) y^{(n+1)} + n(n-1) y^n$$

$$w_1^n = x^2 y^{(n+2)} + (2nx) y^{(n+1)} + n(n-1) y^n$$

$$w_1^n = x^2 y^{(n+2)} + (2nx) y^{(n+1)} + (n^2 - n) y^n$$

Let $W_2 = xy'$

$V = x$, $\quad u = y'$, $u^n = y^{(n+1)}$

$V^{(1)} = 1$, $\quad u^{n-1} = y^{(n+1)-1} = y^n$

$V^{(2)} = 0$, $\quad u^{n-2} = y^{(n+1)-2} = y^{n-1}$

$W_2^n = u^n V + n\, u^{(n-1)} V^{(1)} + 0$

$W_2^n = y^{(n+1)}(x) + n(y^n)(1)$

$W_2^n = xy^{(n+1)} + ny^n$

Let $W_3 = y$

$V = 1$, $\quad u = y$, $u^n = y^n$

$V^{(1)} = 0$, $\quad u^{(n-1)} = y^{(n-1)}$

$W_3^n = u^n V + 0$

$= (y^n)(1)$

$W_3^n = y^n$

$W_1^n + W_2^n + W_3^n = 0$

$x^2y^{(n+2)} + (2nx)y^{(n+1)} + xy^{(n+1)} + (n^2-n)y^n + ny^n + y^n = 0$

$x^2y^{(n+2)} + (2nx + x)y^{(n+1)} + (n^2 - n + n + 1)y^n = 0$

$\therefore x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^n = 0$