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### Assignment 2

1. If  $y = e^{x^2+x}$ , show that  $y'' = y'(2x+1) + 2y$  and hence prove that  $y^{(n+2)} = (2n+1)y^{n+1} + 2(n+1)y^n$

Solution:

$$y = e^{x^2+x} \quad \text{--- (i)}$$

Using chain rule; let  $u = x^2+x \therefore y = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}; \quad \frac{du}{dx} = 2x+1$$

$$\frac{dy}{du} = e^u = e^{x^2+x}$$

$$\frac{dy}{dx} = (2x+1)e^{x^2+x} = y'$$

Finding  $y''$ ; using product rule

$$y' = (2x+1)(e^{x^2+x})$$

$$\text{Let } u = 2x+1; \quad \frac{du}{dx} = 2$$

$$\text{Let } v = e^{x^2+x}; \quad \frac{dv}{dx} = (2x+1)e^{x^2+x}$$

$$\frac{d^2y}{dx^2} = y'' = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$y'' = (2x+1)(2x+1)(e^{x^2+x}) + e^{x^2+x}(2)$$

$$y = e^{x^2+x} \quad \& \quad y' = (2x+1)e^{x^2+x}$$

Substitute the values of  $y$  and  $y'$  in  $y''$

$$y'' = (2x+1)(2x+1)(e^{x^2+x})$$

$$y'' = (2x+1)y' + 2y \quad \text{--- (ii)}$$

To prove  $y^{(n+2)} = (2n+1)y^{n+1} + 2(n+1)y^n \quad \text{--- (iii)}$

$$(2n+1)y^{n+1} + 2(n+1)y^n - y^{(n+2)} = 0$$

Relating eqn (iii) and eqn (ii)

$$y^{n+1} = y', \quad y^{n+2} = y''$$

$$\text{Let } w_1 = y'(2x+1)$$

$$v = 2x+1$$

$$v' = 2$$

$$u = y'$$

$$u^n = y^{n+1}$$

$$u^{n-1} = y^{(n+1)-1} = y^n$$

Applying Leibnitz theorem;

$$y^n = u^n v + n u^{(n-1)} v^{(1)} + \frac{n(n-1)}{2!} v^{(2)} u^{(n-2)} + \frac{n(n-1)(n-2)}{3!} u^{(n-3)} v^3 + \dots$$

$$y^n = y^{(n+1)}(2x+1) + n y^n(2) + 0$$

$$y^n = (2x+1) y^{(n+1)} + 2(n) y^n$$

Let  $w_2 = 2y$

$$u = y \quad v = 2$$

$$u^n = y^n \quad v' = 0$$

$$y^n = y^n(2) + 0 = 2y^n$$

Let  $w_3 = -y''$

$$-y'' = -(y^{(2)})$$

$$u = y^2 \quad v = -1$$

$$u^n = y^{n+2} \quad v' = 0$$

$$y^n = y^{n+2}(-1) + 0 = -y^{n+2}$$

$$y^{n+2}(2x+1) + 2ny^n + 2yn - y^{n+2} = 0$$

$$y^{n+1}(2x+1) + 2y^n(n+1) - y^{n+2} = 0$$

$$y^{n+1}(2x+1) + 2y^n(n+1) = y^{n+2}$$

2 Using the Leibnitz theorem, given that (i)  $y = x^3 e^{4x}$ , determine  $y^5$ ,  
 (ii)  $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$ , show that  $x^2 y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$

Solution

i  $y = x^3 e^{4x}$

Using Leibnitz theorem

$$u = e^{4x}; u' = 4e^{4x}; u'' = 16e^{4x}; u''' = 64e^{4x}; u^{(4)} = 256e^{4x}$$

$$v = x^3; v' = 3x^2; v'' = 6x; v''' = 0$$

$$\therefore u^n = 4^n e^{4x}$$

$$y^n = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v'' + \frac{n(n-1)(n-2)}{3!} u^{n-3} v''' + \frac{n(n-1)(n-2)(n-3)}{4!} u^{n-4} v^{(4)} + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} u^{n-5} v^{(5)}$$

$$u^{n-5} v^5$$

$$= 4^n e^{4x} (x^3) e^{4x} + n 4^{n-1} e^{4x} (3x^2) + \frac{n(n-1) 4^{n-2} e^{4x} 6x}{2!} + \frac{(n)(n-1)(n-2) 4^{n-3} e^{4x} 6}{3!}$$

$$\frac{n(n-1)(n-2)(n-3) 4^{n-4} (e^{4x})(0)}{4!}$$

Since  $n=5$

$$y^5 = 4^5 e^{4x} x^3 e^{4x} + 5(4^4 e^{4x}) 3x^2 + \frac{5(5-1) 4^3 e^{4x} 6x}{2!} + \frac{5(5-1)(5-2) 4^2 e^{4x} 6}{3!} + 0$$

$$y^5 = 4^5 e^{4x} x^3 + 5(4^4 e^{4x}) 3x^2 + \frac{5(4) 4^3 e^{4x} 6x}{2!} + \frac{5(4)(3) 4^2 e^{4x} 6}{3!} + 0$$

$$y^5 = 1024 e^{4x} x^3 + 3840 e^{4x} x^2 + 3840 e^{4x} x + 960 e^{4x}$$

$$\underline{y^5 = 64 e^{4x} (16x^3 + 60x^2 + 60x + 15)}$$

b  $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$

$$x^2 y'' + x y' + y = 0$$

Let  $g_1 = x^2 y''$

$$v = x^2 \quad u^n = y'' = y^{(n+2)}$$

$$v' = 2x \quad u^n = y^{n+2}$$

$$v'' = 2 \quad u^{n-1} = y^{(n+2)-1} = y^{n+1}$$

$$v''' = 0 \quad u^{n-2} = y^{(n+2)-2} = y^n$$

Using Leibnitz Theorem

$$g_1^n = u^n v + n u^{n-1} v' + \frac{n(n-1) u^{n-2} v''}{2!} + 0$$

$$g_1^n = y^{(n+2)} x^2 + n y^{n+1} (2x) + \frac{n(n-1) y^n (2)}{2!} + 0$$

$$g_1^n = x^2 y^{n+2} + (2x)(n) y^{n+1} + n(n-1) y^n$$

$$g_1^n = x^2 y^{n+2} + 2nx y^{n+1} + n^2 - n y^n$$

Let  $g_2 = x y'$

$$u = y', \quad u^n = y^{(n+1)}$$

$$v' = 1$$
$$v^2 = 0$$

$$u^{n+1} = y^{n+1-1} = y^n$$
$$u^{n-2} = y^{(n+1)-2} = y^{n-1}$$

$$g_2^n = u^n v + n u^{n-1} v' + 0$$
$$g_2^n = y^{n+1} x + n(y^n)(1)$$
$$g_2^n = x y^{n+1} + n y^n$$

Let  $g_3 = y$

$$v = 1 \quad u = y, u^n = y^n$$
$$v' = 0 \quad u^{n-1} = y^{n-1}$$

$$g_3^n = u^n v + 0$$
$$g_3^n = y^n (1)$$
$$g_3^n = y^n$$

$$g_1^n + g_2^n + g_3^n = 0$$
$$x^2 y^{n+2} + 2n x y^{n+1} + x y^{n+1} + (n^2 - n) y^n + n y^n + y^n = 0$$
$$\therefore \underline{x^2 y^{n+2} + (2n+1) x y^{n+1} + (n^2+1) y^n = 0}$$