1. $y = e^{x^2+x}$

$y' = (2x+1)e^{x^2+x}$

$y'' = 2e^{x^2+x} + (2x+1)(2x+1)e^{x^2+x}$

$y'' = 2e^{x^2+x} + (2x+1)^2 e^{x^2+x}$

$y'(2x+1) + 2y$

$= (2x+1)e^{x^2+x} \cdot (2x+1) + 2(x^2+x)$

$= (2x+1)^2 e^{x^2+x} + 2e^{x^2+x}$

but $y'' = 2e^{x^2+x} + (2x+1)^2 e^{x^2+x}$

$\therefore y'' = y'(2x+1) + 2y$

from the equation above,

Part A,

$A = y''$, $A' = y'''$, $A^n = y^{2+n}$

Part B

$B = y'(2x+1)$

$u = y'$, $u^n = y^{n+1}$

$v = 2x+1$, $v' = 0$

$\therefore B^n = (y^{n+1})(2x+1) + n(y^n)(2) + 0$

$B^n = (2x+1)y^{n+1} + 2ny^n$

Part C,

$C = 2y$

$C^n = 2y^n$

$\therefore A^n = B^n + C^n$

$y^{n+2} = (2x+1)y^{n+1} + 2ny^n + 2y^n$

$y^{n+2} = (2x+1)y^{n+1} + 2y^n(n+1)$

$y^{n+2} = (2x+1)y^{n+1} + 2(n+1)y^n$

2. $y = x^3 e^{4x}$ $\quad y^{(5)}$

Let $u = e^{4x}$, $u' = 4e^{4x}$, $u'' = 16e^{4x}$, $u^n = 4^n e^{4x}$

Let $v = x^3$, $v' = 3x^2$, $v'' = 6x$, $v''' = 0$

By Leibniz theorem,

$y^n = 4^n e^{4x} \cdot x^3 + n \cdot 4^{n-1} e^{4x} \cdot 3x^2 + \frac{n(n-1)}{2!} \cdot 4^{n-2} e^{4x} \cdot 6x + \frac{n(n-1)(n-2)}{3!} \cdot$
$4^n e^{4x} \cdot 6 + c$

$y^n = 4^n e^{4x} \cdot x^3 + 3x^2 n \cdot 4^{n-1} e^{4x} + 3n(n-1) \cdot 4^{n-2} e^{4x} x$
$n(n-1)(n-2) \cdot 4^{n-3} e^{4x}$

$\therefore y^5 = 4^5 e^{4x} \cdot x^3 + 3x^2(5) \cdot 4^4 e^{4x} + 3(5)(4) \cdot 4^3 e^{4x} \cdot x + (5)(4)$
$(3) \cdot 4^2 e^{4x}$

$y^5 = 1024 e^{4x} \cdot x^3 + 3840 e^{4x} \cdot x^2 + 3840 e^{4x} \cdot x + 960 e^{4x}$

$y^5 = 64e^{4x}(16x^3 + 60x^2 + 60x + 15)$.

(ii) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

For Part A, $A = x^2 y''$

$u = y''$, $u^n = y^{n+2}$

$v = x^2$, $v' = 2x$, $v'' = 2$, $v''' = 0$

~~$u = y''$, $u^n = y^{n+2}$~~ $A^n = (y^{n+2})x^2 + n(y^{n+1}) \cdot 2x + \frac{n(n-1)}{2!} \cdot$
~~v~~ $(y^n) \cdot 2 + 0$

$A^n = x^2 y^{(n+2)} + 2xny^{(n+1)} + n(n-1)y^n$

For Part B,

$B = xy'$

$u = y'$, $u^n = y^{n+1}$

$v = x$, $v' = 1$, $v'' = 0$

$B^n = (y^{n+1}) \cdot x + n(y^n) \cdot 1 + 0$

$= xy^{(n+1)} + ny^n$

For Part c,

$c = y$

$C^n = y^n$

$\therefore A^n + B^n + C^n = 0$

$= x^2 y^{(n+2)} + 2xny^{(n+1)} + (n^2 - n)y^n + 2xy^{(n+1)} + ny^n + y^n = 0$

$= x^2 y^{(n+2)} + xy^{(n+1)}(2n+1) + y^n(n^2 - n + n + 1) = 0$

$= x^2 y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^n = 0$