

ENG 381

~~MKS~~

Assignment 2

1. If $y = e^{x^2+x}$ show that $y'' = y'(2x+1) + 2y$ and hence prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y$

Solution

$$y = e^{x^2+x} \quad \text{--- (1)}$$

using chain rule; let $u = x^2+x \therefore y = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}; \quad \frac{du}{dx} = 2x+1$$

$$\frac{dy}{dx} = e^u = e^{x^2+x}$$

$$\frac{dy}{dx} = (2x+1)e^{x^2+x} = y'$$

finding y'' using Product rule

$$y' = (2x+1)(e^{x^2+x})$$

$$\text{let } u = 2x+1; \quad \frac{du}{dx} = 2$$

$$\text{let } v = e^{x^2+x}; \quad \frac{dv}{dx} = (2x+1)e^{x^2+x}$$

$$\frac{dy'}{dx} = y'' = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$y'' = (2x+1)(2x+1)(e^{x^2+x}) + e^{x^2+x}(2)$$

$$y = e^{x^2+x} \quad \& \quad y' = (2x+1)e^{x^2+x}$$

Substitute the values of y and y' in y''

$$y'' = (2x+1)(2x+1)(e^{x^2+x}) + (1+2x)(1+2x)y$$

$$y'' = (2x+1)y' + 2y \quad \text{--- (2)}$$

To prove $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y$ --- (3)

$$(2x+1)y^{(n+1)} + 2(n+1)y = y^{(n+2)} = 0$$

Relating eqn (2) and (1)

$$y^{(n+1)} = y' y^{(n+2)} = y''$$

$$\text{let } w_1 = y'(2x+1)$$

$$\text{let } w_2 = y(2x+1)$$

$$\text{Let } v = 2x+1 \quad u = y$$

$$v' = 2 \quad y^n = y^{n+1}$$

$$u^{n-1} = y^{(n+1)-1} = y^n$$

Applying Leibnitz theorem:

$$y^n = u^n v + n u^{(n-1)} v^{(1)} + \frac{n(n-1)}{2!} v^{(2)} u^{(n-2)} + \frac{n(n-1)(n-2)}{3!} +$$

$$\frac{n(n-1)(n-2)}{3!} u^{(n-3)} v^{(3)} + \dots$$

$$y^n = y^{(n+1)} (2x+1) + n y^n (2) + 0$$

$$y^n = (2x+1) y^{(n+1)} + 2(n) y^n$$

$$\text{let } w_2 = 2y$$

$$u = y \quad v = 2$$

$$u' = y^n \quad v' = 0$$

$$y^n = y^n (2) + 0 = 2y^n$$

$$\text{let } w_3 = -y''$$

$$-y'' = -(y^{(2)})$$

$$u = y^2 \quad v = -1$$

$$u' = y^{n+2} \quad v' = 0$$

$$y^n = y^{n+2} (-1) + 0 = -y^{n+2}$$

$$y^{n+2} (2x+1) + 2y^n (n+1) - y^{n+2} = 0$$

$$\therefore y^{n+1} (2x+1) + 2y^n (n+1) = y^{n+2}$$

2. Using the Leibnitz theorem, given that (i) $y = x^3 e^{4x}$, determine y^5 , (ii) $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$, show that $x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2+1) y^{(n)} = 0$

Solution

$$(i) \quad y = x^3 e^{4x}$$

Using Leibnitz theorem

$$u = e^{4x}, \quad u' = 4e^{4x}, \quad u'' = 16e^{4x}, \quad u''' = 64e^{4x}, \quad u^{(4)} = 256e^{4x}$$

$$v = x^3; v' = 3x^2, v'' = 6x, v''' = 0$$

$$\therefore u^n = 4^n e^{4x}$$

$$y^n = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v^2 + \frac{n(n-1)(n-2)}{3!} u^{n-3} v^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} u^{n-5} v^5$$

$$= 4^n e^{4x} (x^3) e^{4x} + n 4^{n-1} e^{4x} (3x^2) + \frac{n(n-1)}{2!} 4^{n-2} e^{4x} (6x) + \frac{n(n-1)(n-2)}{3!} 4^{n-3} e^{4x} (6) + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} 4^{n-5} e^{4x} (0)$$

$$n(n-1)(n-2)(n-3) 4^{n-1} (e^{4x}) (0)$$

Since $n=5$

$$y^5 = 4^5 e^{4x} x^3 e^{4x} + 5(4^4 e^{4x}) 3x^2 + 5(5-1) 4^3 e^{4x} 6x + 5(5-1)(5-2) 4^2 e^{4x} 6 + 0$$

$$y^5 = 4^5 e^{4x} x^3 + 5(4^4 e^{4x}) 3x^2 + 5(4) 4^3 e^{4x} 6x + 5(4)(3) 4^2 e^{4x} 6 + 0$$

$$y^5 = 1024 e^{4x} x^3 + 3840 e^{4x} x^2 + 3840 e^{4x} x + 960 e^{4x}$$

$$y^5 = 64 e^{4x} (16x^3 + 60x^2 + 60x + 15)$$

(ii) $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$

$$x^2 y'' + x y' + y = 0$$

$$\text{Let } y_1 = x^2 y''$$

$$v = x^2 \quad u' = y'' = y^{(n+2)}$$

$$v' = 2x \quad u^n = y^{n+2}$$

$$v'' = 2 \quad u^{n-1} = y^{(n+2)-1} = y^{n+1}$$

$$v''' = 0 \quad u^{n-2} = y^{(n+2)-2} = y^n$$

Using Leibnitz Theorem

$$y_1^n = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v^2 + 0$$

$$y_1^n = y^{(n+2)} x^2 + n y^{n+1} (2x) + \frac{n(n-1)}{2!} y^n (2) + 0$$

$$g_1' = x^2 y^{n+2} + (2x)(n) y^{n+1} + n(n-1) y^n$$

~~$$g_1^n = x^2 y^{n+2} x^2 + n y^{n+1} (2x) + n(n-1) y^n (2) + 0$$~~

$$g_1^n = x^2 y^{n+2} + 2nx y^{n+1} + n^2 - n y^n$$

$$4 + g_2 = x y'$$

$$u = y', u^n = y^{(n+1)}$$

$$v' = 1$$

$$v^2 = 0$$

$$u^{n+1} = y^{n+1-1} = y^n$$

$$u^{n-2} = y^{(n+1)-2} = y^{n-1}$$

$$g_1^n = u^n v + n u^{n-1} v' + 0$$

$$g_1^n = y^{(n+1)} x + n (y^n) (1)$$

$$g_2^n = x y^{n+1} + n y^n$$

let v

$$v = 1$$

$$v' = 0$$

$$u = y, u^n = y^n$$

$$u^{n-1} = y^{n-1}$$

$$g_3^n = u^n v + 0$$

$$y^n (1)$$

$$g_3^n = y^n$$

$$g_1^n + g_2^n + g_3^n = 0$$

$$x^2 y^{n+2} + 2nx y^{n+1} + x y^{n+1} + (n^2 - n) y^n + n y^n + y^n = 0$$

$$\therefore x^2 y^{n+2} + (2n+1) x y^{n+1} + (n^2+1) y^n = 0$$