$G_2 = U^n V + n U^{n-1} V' + \frac{n(n-1)}{2!} U^{n-2} V''$

$= y^{n+1}(2x+1) + n y^n (2) + \frac{n(n-1)}{2!} y^{n-1} (0)$

$= y^{n+1}(2x+1) + 2n y^n$ —— (5)

For G_3

$u = y,\ u' = y',\ \therefore u^n = y^n$

$v = 2,\ v' = 0$

$G_3 = U^n V + n U^{n-1} V'$

$= y^n (2) + n y^{n-1} (0)$

$= 2y^n$ —— (6)

Putting eq (4), (5), (6) in (3)

$G_1 - G_2 - G_3 = 0$

$y^{n+2} - (y^{n+1}[2x+1) + 2n y^n) - 2y^n = 0$

$y^{n+2} - y^{n+1}(2x+1) - 2n y^n - 2y^n = 0$

$\Leftrightarrow y^{n+2} = y^{n+1}(2x+1) + 2n y^n + 2y^n$

$= y^{n+1}(2x+1) + 2y^n (n+1)$

$\therefore y^{n+2} = y^{n+1}(2x+1) + 2y^n(n+1)$

(2)

(1) $y = x^3 e^{4x}$

Using Leibnitz Theorem

$u = e^{4x},\ u' = 4e^{4x},\ u'' = 16e^{4x},\ u''' = 64e^{4x},\ u^{iv} = 256e^{4x}$

$v = x^3,\ v' = 3x^2,\ v'' = 6x,\ v''' = 6,\ v^{iv} = 0$

$\therefore u^n = 4^n e^{4x}$

$y^n = U^n V + n U^{n-1} V' + \frac{n(n-1)}{2!} U^{n-2} V'' + \frac{n(n-1)(n-2)}{3!} U^{n-3} V''' +$

$\frac{n(n-1)(n-2)(n-3)}{4!} U^{n-4} V^{iv} + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} U^{n-5} V^{v} \ldots$

$= 4^n e^{4x}(x^3) + n 4^{n-1} e^{4x}(3x^2) + \frac{n(n-1)}{2!} 4^{n-2}(6x) + \frac{n(n-1)(n-2)}{3!}$

$4^{n-3}(6) + \frac{n(n-1)(n-2)(n-3)}{4!} 4^{n-5}(0) \ldots$

$\therefore$

$y^5 = 4^5 e^{4x}(x^3) + 5(4^{5-1})(e^{4x})(3x^2) + \frac{5(5-1)}{2!}(4^{5-2})(e^{4x})(6x) + \frac{5(5-1)(5-2)}{3!}(4^{5-3})(e^{4x})(6)$

$y = e^{x^2+x}$ ——— ①

$\frac{dy}{dx} = y' = (2x+1)e^{x^2+x}$ ——— ②

$\frac{d^2y}{dx^2} = y'' = (2x+1)(2x+1)e^{x^2+x} + 2e^{x^2+x}$ ... Product Rule

Put equ ① and ② in y''

$y'' = y'(2x+1) + 2y$

∴ Proven

From above

$y'' - y'(2x+1) - 2y = 0$

Using Leibnitz theorem

$G_1 = y''$

$G_2 = y'(2x+1)$

$G_3 = 2y$

∴ $G_1 - G_2 - G_3 = 0$ ——— ③

For G_1

$U = y''$

$U' = y'''$

hence $U^n = y^{n+2}$

$V = 1$

$V' = 0$

$G_1 = U^nV + nU^{n-1}V'$

$= y^{n+2}(1) + ny^{n+1}(0)$

$= y^{n+2}$ ——— ④

For G_2

$G_2 = y'(2x+1)$

$U = y'$, $U' = y''$, $U'' = y'''$, hence $U^n = y^{n+1}$

$V = 2x+1$, $V' = 2$, $V'' = 0$

$G_2 = U^nV + nU^{n-1}V' + \underline{n(n-1)U^{n-2}V''}$

$$G_2 = U^n V + nU^{n-1}V' + \frac{n(n-1)}{2!}U^{n-2}V''$$

$$= y^{n+1}(2x+1) + ny^n(2) + \frac{n(n-1)}{2!}y^{n-1}(0)$$

$$= y^{n+1}(2x+1) + 2ny^n \quad \text{——— (5)}$$

For G_3

$U = y$, $U' = y'$, $\therefore U^n = y^n$

$V = 2$, $V' = 0$

$$G_3 = U^n V + nU^{n-1}V'$$

$$= y^n(2) + ny^{n-1}(0)$$

$$= 2y^n \quad \text{——— (6)}$$

Putting equ (4), (5), (6) in (3)

$$G_1 - G_2 - G_3 = 0$$

$$y^{n+2} - (y^{n+1}[2x+1) + 2ny^n) - 2y^n = 0$$

$$y^{n+2} - y^{n+1}(2x+1) - 2ny^n - 2y^n = 0$$

$$\Leftrightarrow y^{n+2} = y^{n+1}(2x+1) + 2ny^n + 2yn$$

$$= y^{n+1}(2x+1) + 2y^n(n+1)$$

$$\therefore y^{n+2} = y^{n+1}(2x+1) + 2y^n(n+1)$$

(2)

(1) $y = x^3 e^{4x}$

Using Leibnitz theorem

$U = e^{4x}$, $U' = 4e^{4x}$, $U'' = 16e^{4x}$, $U''' = 64e^{4x}$, $U^{iv} = 256e^{4x}$

$V = x^3$, $V' = 3x^2$, $V'' = 6x$, $V''' = 6$, $V^{iv} = 0$

$\therefore U^n = 4^n e^{4x}$

$$y^n = U^n V + nU^{n-1}V' + \frac{n(n-1)}{2!}U^{n-2}V'' + \frac{n(n-1)(n-2)}{3!}U^{n-3}V''' + \frac{n(n-1)(n-2)(n-3)}{4!}U^{n-4}V^{iv} + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}U^{n-5}V^{v} \ldots$$

$$= 4^n e^{4x}(x^3) + n4^{n-1}e^{4x}(3x^2) + \frac{n(n-1)}{2!}4^{n-2}(6x) + \frac{n(n-1)(n-2)}{3!}4^{n-3}(6) + \frac{n(n-1)(n-2)(n-3)}{4!}U^{n-5}(0) \ldots$$

$\therefore$

$$y^5 = 4^5 e^{4x}(x^3) + 5(4^{5-1})(e^{4x})(3x^2) + 5(5-1)(4^{5-2})(e^{4x})(6x) + 5(5-1)(5-2)(4^{5-3})(e^{4x})(6)$$

$\therefore$
$$y^5 = 4^5 e^{4x}(x^3) + 5(4^{5-1})(e^{4x})(3x^2) + \frac{5(5-1)}{2!}(4^{5-2})(e^{4x})(6x) + \frac{5(5-1)(5-2)}{3!}(4^{5-3})(e^{4x})(6)$$
$$+ \frac{5(5-1)(5-2)(5-3)}{4!}(4^{5-4})(e^{4x})(0)$$
$$= 1024x^3e^{4x} + 3840x^2e^{4x} + \cancel{\ldots}\,10(4^3e^{4x})(6x) + \cancel{760}(4^2e^{4x})$$
$$\text{(or) } 10(4^2e^{4x})(6)$$
$$= 1024x^3e^{4x} + 3840x^2e^{4x} + 3840xe^{4x}\cancel{6x} + 960e^{4x}$$
$$= 16e^{4x}(64x^3 + 240x^2 + 240x + 60)$$

$$\therefore\ y^5 = 16e^{4x}(64x^3 + 240x^2 + 240x + 60)$$

b) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

$x^2y'' + xy' + y = 0$

$G_1 + G_2 + G_3 = 0$

For G_1

$u = y''$, $u' = y'''$, $y^{n+2} = u^n$

$v = x^2$, $v' = 2x$, $v'' = 2$, $v''' = 0$

$\therefore G_1 = u^nv + nu^{n-1}v' + \frac{n(n-1)}{2!}u^{n-2}v''$

$= y^{n+2}(x^2) + 2xy^{n+1} + y^n n(n-1)$

For G_2

$u = y'$, $u' = y''$, $u'' = y'''$, $u^n = y^{n+1}$

$v = x$, $v' = 1$, $v'' = 0$

$G_2 = u^nv + nu^{n-1}v'$

$= xy^{n+1} + ny^n$

For G_3

$u = y$, $u' = y'$, $u^n = y^n$

$v = 1$, $v' = 0$

$G_3 = u^nv$

$= y^n(1)$

$= y^n$

$G_1 + G_2 + G_3 = 0$

$(x^2y^{n+2} + 2xy^{n+1} + n(n-1)y^n) + (xy^{n+1} + ny^n) + y^n = 0$

$x^2y^{n+2} + xy^{n+1}(2n+1) + y^n(n(n-1) + n + 1) = 0$

$x^2y^{n+2} + (2n+1)xy^{n+1} + (n^2+1)y^n = 0$