1 If $y = e^{x^2+x}$, show that $y'' = y'(2x+1) + 2y$ and hence prove that $y^{(n+2)} = (2x+1)\,y^{n+1} + 2y^{n}(n+1)$

$y = e^{x^2+x}$ ——— (1)

Using chain rule let $u = x^2 + x$ ∴ $y = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{du}{dx} = 2x+1$, $\frac{dy}{du} = e^u$

$\frac{dy}{dx} = (2x+1)\,e^{x^2+x} = y'$

To find y'' we use product rule

$y' = (2x+1) = u \times , (e^{x^2+x}) = v$

$y' = uv$

$y'' = u\frac{dv}{dx} + v\frac{du}{dx}$

$du = 2$

$= dv = (2x+1)\,e^{x^2+x}$

$y'' = (2x+1)(2x+1)e^{x^2+x} + (e^{x^2+x})\,2$

Recall $(2x+1)e^{x^2+x} = y'$, $e^{x^2+x} = y$

$y'' = (2x+1)y' + 2y$ ——— (1.1)

Applying Leibnitz

$y^n = u^n v^0 + n u^{n-1} v^1 + 0$

$y^n = y^{n+1}(2x+1) + n y^n (2) + 0$

$y^n = (2x+1) \, y^{(n+1)} + 2n y^n$

Let $u = 2x$ [illegible] $= 2$

$$\frac{n(n-1)(n-2)}{3!} 4e^{n-3} 4x \cdot 6 + 0$$

When $n = 5$

b) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

$x^2y'' + xy' + y = 0$

$\downarrow \quad \downarrow \quad \downarrow$

$w^1 \quad w^2 \quad w^3$

$w^1 = x^2y''$

$v = x^2 \quad v' = 2x \quad v'' = 2 \quad v''' = 0$

$u = y'' \quad u' = y''' \quad y'' = y^{iv} \quad y''' = y^v$

$u^n = y^{n+2}$

Using leibnitz

$w_1 = U_nV^0 + nu^{n-1}v' + n(n-1)u^{(n-2)}v^2 + 0$

~~$= y^{(n+2)} \quad x^2 + ny^{n+1} \quad 2'$~~

$w_1 \quad y^{n+2}x^2 + ny^{n+1}(2x) + \frac{n(n-1)y^n(2)}{2!} + 0$

$w_1 = x^2y^{n+2} + 2nxy^{n-1} + n^2 - ny^n$

let $w^2 = xy'$

$v = 1 \quad u' = y'' \quad u^n = y^{n+1}$

$v' = 0 \quad u'' = y''' \quad u^{n-1} = y^{n+1-1} = y^n$

$u^{n-2} = y^{n+1-2} = y^{n-1}$

$w_2 = u^nv + nu^{n-1}v' + 0$

$= y^{n+1}x + ny^n(1)$

$= xy^{n+1} + ny^n$

$w^3 = y$

$w^3 = u^nv + 0$

$= y^n$

$w^1 + w^2 + w^3$

$x^2y^{n+2} + 2nxy^{n+1} + xy^{n+1} + (n^2-n)y^n + ny^n + y^n = 0$

$\therefore x^2y^{n+2} + (2n+1)xy^{(n+1)} + (n^2+1)y^n = 0$