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Assignment

1 If $y = e^{x^2+x}$, show that $y'' = y'(2x+1) + 2y$ and hence prove that $y^{(n+2)} = (2x+1)y^{n+1} + 2(n+1)y$

Solution

$y = e^{x^2+x}$ .... (i)

Using Chain rule; let $u = x^2 + x$; $y = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$; $\frac{du}{dx} = 2x+1$

$\frac{dy}{du} = e^u = e^{x^2+x}$

$\frac{dy}{dx} = (2x+1)e^{x^2+x} = y'$

Finding y''; using product rule

$y' = (2x+1)(e^{x^2+x})$

let $U = 2x+1$; $\frac{du}{dx} = 2$

let $V = e^{x^2+x}$; $\frac{dv}{dx} = (2x+1)\ e^{x^2+x}$

$\frac{d^2y}{dx^2} = y'' = U\frac{dv}{dx} + V\frac{du}{dx}$

$y'' = (2x+1)(2x+1)(e^{x^2+x}) + e^{x^2+x}(2)$

$y = e^{x^2+x}$; $y' = (2x+1)e^{x^2+x}$

Substitute the values of y and y' in y''

$y'' = (2x+1)(2x+1)(e^{x^2+x})$

$y'' = (2x+1)y' + 2y$ ..... (ii)

To prove $y^{(n+2)} = (2x+1)y^{(n+1)} + 2y^n$ .... (i

$(2x+1)y^{n+1} + 2(n+1)y^n - y^{(n+2)} = 0$

Relating equation (ii) and equations (i)

$y^{n+1} = y', \; y^{n+2} = y''$

Let $w_1 = y'(2x+1)$

$V = 2x+1$

$V' = 2$

$U = y'$

$U^n = y^{n+1}$

$U^{n-1} = y^{(n+1)-1} = y^n$

~~Let V = 2x+1~~

Applying Leibnitz theorem

$y^n = U^n V + n U^{(n-1)} V' + \frac{n(n-1)}{2!} U V^2 U^{(n-2)}$

$+ \frac{n(n-1)(n-2)}{3!} + \frac{n(n-1)(n-2)}{3} U^{(n-3)} V^3 + \ldots$

$y^n = y^{(n+1)}(2x+1) + n y^n (2) + 0$

$y^n = (2x+1) y^{(n+1)} + 2(n) y^n$

Let $w_2 = 2y$

$U = y \qquad V = 2$

$U^n = y^n \qquad V' = 0$

$y^n = y^n(2) + 0 = 2y^n$

Let $w_3 = -y''$

$-y'' = -(y^{(2)})$

$U = y^2 \qquad V = -1$

$U^n = y^{n+2}(-1) + 0 = -y^{n+2}$

$y^{n+1}(2x+1) + 2y^n(n+1) - y^{n+2} = 0$

$\therefore y^{n+1}(2x+1) + 2y^n(n+1) = y^{n+2}$

2 Using the Leibnitz theorem, given that

(i) $y = x^3 e^{4x}$, determine y^5

ii) $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$, show that

$x^2 y^{(n+2)} (2n+1) x y^{(n+1)} + (n^2+1) y^{(n)} = 0$

Solution

i $y = x^3 e^{4x}$

Using Leibnitz theorem

$U = e^{4x}$, $U' = 4e^{4x}$, $U'' = 16e^{4x}$, $U''' = 64e^{4x}$,

$U^{iv} = 256e^{4x}$, $V = x^3$, $V' = 3x^2$, $V'' = 6x$,

$V''' = 0$

$\therefore U^n = 4^n e^{4x}$

$y^n = U^n V + nU^{n-1}V' + \frac{n(n-1)}{2!}U^{n-2}V^2 + \frac{n(n-1)(n-2)}{3!}U^{n-3}V^3$

$+ \frac{n(n-1)(n-2)(n-3)}{4!}U^{n-4}V^4 + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}U^{n-5}V^5$

$= 4^n e^{4x}(x^3) e^{4x} + n4^{n-1}e^{4x}(3x^2) + \frac{n(n-1)4^{n-2}}{2!}e^{4x}6x +$

$\frac{n(n-1)(n-2)(n-3)}{4!} 4^{n-4}(e^{4x})(0)$

Since $n = 5$

$y^5 = 4^5 e^{4x} x^3 e^{4x} + 5(4^4 e^{4x}) 3x^2 + \frac{5(5-1)4^3 e^{4x}}{2!}6x +$

$\frac{5(5-1)(5-2)4^2 e^{4x}6}{3!} + 0$

$y^5 = 4^5 e^{4x} x^3 + 5(4^4 e^{4x})3x^2 + \frac{5(4)4^3 e^{4x}}{2!}6x +$

$\frac{5(4)(3)4^2 e^{4x}6}{3!} + 0$

$y^5 = 64e^{4x}(16x^3 + 60x^2 + 60x + 15)$

b) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

$x^2y'' + xy' + y = 0$, let $y_1 = x^2y''$.

$V = x^2$ $\quad U^{n} = y'' = y^{(n+2)}$

$V' = 2x$ $\quad U^n = y^{n+2}$

$V'' = 2$ $\quad U^{n-1} = y^{(n+2)-1}$

$V''' = 0$

Using Leibnitz Theorem

$y_1^n = U^nV + nU^{n-1}V' + \frac{n(n-1)}{2!}U^{(n-2)}V^2 + 0$

$y_1^n = y^{(n+2)}x^2 + ny^{n+1}(2x) + \frac{n(n-1)y^n}{2!}(2) + 0$

$y_1^n = x^2y^{n+2} + 2nxy^{n-1} + n^2 - ny^n$

Let $y_2 = xy'$.

$U = y'$, $U^n = y^{n+1}$

$V' = 1$ $\quad U^{n+1} = y^{n+1-n} = y^n$

$V^2 = 0$ $\quad U^{n-2} = y^{(n+1)-2} = y^{n-1}$

$y_2^n = U^nV + nU^{n-1}V' + 0$

$y_2^n = y^{(n+1)}x + n(y^n)(1)$

$y_2^n = xy^{n+1} + ny^n$

Let $y_3 = y$

$V = 1$, $V' = 0$ and $U = y$, $U^n = y^n$, $U^{n-1} = y^{n-1}$

$y_3^n = U^nV + 0$

$y^n(1)$

$y_3^n = y^n$

$y_1^n + y_2^n + y_3^n = 0$

$x^2y^{n+2} + (2nx)y^{n+1} + xy^{n+1} + (n^2-n)y^n + ny^n + y^n$

0

$\therefore x^2y^{n+2} + (2n+1)xy^{(n+1)} + (n^2+1)y^n = 0$