

Assignment

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Dept: Civil Engineering

ENG 351

1. If $y = e^{x^2+x}$ show that $y'' = y'(2x+1) + 2y$ and hence prove that $y^{(n+2)}$
 $= (2n+1)y^{(n+1)} + 2(n+1)y^{(n)}$

Soln

$$y = e^{x^2+x} \dots \textcircled{1}$$

using chain rule; let $u = x^2+x \therefore y = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}; \frac{du}{dx} = 2x+1$$

$$\frac{dy}{du} = e^u = e^{x^2+x}$$

$$\frac{dy}{dx} = (2x+1)e^{x^2+x} = y'$$

Finding y'' , using product rule

$$y' = (2x+1)(e^{x^2+x})$$

$$\text{let } u = 2x+1; \frac{du}{dx} = 2$$

$$\text{let } v = e^{x^2+x}; \frac{dv}{dx} = (2x+1)e^{x^2+x}$$

$$\frac{d^2y}{dx^2} = y'' = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$y'' = (2x+1)(2x+1)(e^{x^2+x}) + e^{x^2+x}(2)$$

$$y = e^{x^2+x} \quad y' = (2x+1)e^{x^2+x}$$

subst. the values of y into y''

$$y'' = (2x+1)(2x+1)(e^{x^2+x})$$

$$y'' = (2x+1)y' + 2y \dots \textcircled{2}$$

To prove that $y^{(n+2)} = (2n+1)y^{(n+1)} + 2(n+1)y^{(n)} \dots \textcircled{3}$

$$(2n+1)y^{(n+1)} + 2(n+1)y^{(n)} - y^{(n+2)} = 0$$

Relating eqn (2) and (3)

$$y^{(n+1)} = y', \quad y^{(n+2)} = y''$$

$$\text{Let } w = y'(2x+1)$$

$$v = 2x+1 \quad u = y'$$

$$v = 2 \quad u' = y^{(n+1)}$$

$$u^{n-1} = y^{(n+1)-1} = y^{(n)}$$

Applying Leibniz theorem

$$y^n = u^n v + n u^{(n-1)} v^{(1)} + \frac{n(n-1)}{2!} u^{(n-2)} v^{(2)} + \frac{n(n-1)(n-2)}{3!} u^{(n-3)} v^{(3)} + \dots$$

$$y^n = y^{(n+1)} (2x+1) + n y^n (2) + 0$$

$$y^n = (2x+1) y^{(n+1)} + 2(n) y^n$$

let $u_2 = 2y$

$$u = y \quad v = 2$$

$$u^n = y^n \quad v' = 0$$

$$y^n = y^n (2) + 0 = 2y^n$$

let $u_3 = -y''$

$$-y'' = -(y^{(2)})$$

$$u = y^2 \quad v = -1$$

$$u^n = y^{n+2} \quad v' = 0$$

$$y^n = y^{n+2} (1) + 0 = -y^{n+2}$$

$$y^{n+1} (2x+1) + 2ny^n + 2y^n - y^{n+2} = 0$$

$$y^{n+1} (2x+1) + 2y^n (n+1) - y^{n+2} = 0$$

$$\therefore y^{n+1} (2x+1) + 2y^n (n+1) = y^{n+2}$$

2) Using the Leibniz theorem given that (i) $y = x^3 e^{4x}$, determine $y^{(4)} x^2$

(ii) $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$, show that $x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2+1) y^{(n)} = 0$

Soln

i) $y = x^3 e^{4x}$

Using Leibniz theorem

$$u = e^{4x}, u' = 4e^{4x}, u'' = 16e^{4x}, u''' = 64e^{4x}, u^{(4)} = 256e^{4x}$$

$$v = x^3, v' = 3x^2, v'' = 6x, v''' = 6, v^{(4)} = 0$$

$$\therefore u^n = x^n e^{4x}$$

$$y^n = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v'' + \frac{n(n-1)(n-2)}{3!} u^{n-3} v''' +$$

$$\frac{n(n-1)(n-2)(n-3)}{4!} u^{n-4} v^{(4)} + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} u^{n-5} v^{(5)}$$

$$= x^n e^{4x} (x^3) e^{4x} + n x^{n-1} e^{4x} (3x^2) + \frac{n(n-1)}{2!} x^{n-2} e^{4x} (6x) + \frac{n(n-1)(n-2)}{3!} x^{n-3} e^{4x} (6) + \frac{n(n-1)(n-2)(n-3)}{4!} x^{n-4} e^{4x} (0) + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} x^{n-5} e^{4x} (0)$$

4!

Since $n=5$

$$y^5 = 4^5 e^{4x} x^5 + 5(4^4 e^{4x}) 3x^2 + \frac{5(5-1)}{2!} 4^3 e^{4x} 6x + \frac{5(5-1)(5-3)}{3!} 4^2 e^{4x} 10$$

$$y^5 = 10240 e^{4x} x^5 + 3940 e^{4x} x^2 + 3940 e^{4x} x + 960 e^{4x}$$

$$y^5 = 64 e^{4x} (16x^5 + 60x^2 + 60x + 15)$$

ii) $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$

$$x^2 y'' + x y' + y = 0$$

Let $g = x^2 y''$

$$v = x^2 \quad u^n = y'' = y^{(n+2)}$$

$$v' = 2x \quad u^n = y^{n+2}$$

$$v'' = 2 \quad u^{n-1} = y^{(n+2)-1} = y^{n+1}$$

$$v''' = 0 \quad u^{n-2} = y^{(n+2)-2} = y^n$$

Using Leibniz theorem

$$g_1^n = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v'' + 0$$

$$g_1^n = x^2 y^{n+2} + (2x)(n) y^{n+1} + n(n-1) y^n$$

$$g_2^n = x^2 y^{n+2} + 2nx y^{n+1} + n^2 y^n$$

Let $g_2 = x y'$

$$u = y'; \quad u^n = y^{(n+1)}$$

$$v' = 1 \quad u^{n-1} = y^{(n+1)-1} = y^n$$

$$v'' = 0 \quad u^{n-2} = y^{(n+1)-2} = y^{n-1}$$

$$g_2^n = u^n v + n u^{n-1} v' + 0$$

$$g_2^n = y^{(n+1)} x + n(y^n)(1)$$

$$g_2^n = x y^{n+1} + n y^n$$

Let $g_3 = y$

$$v = 1 \quad u = y; \quad u^n = y^n$$

$$v' = 0 \quad u^{n-1} = y^{n-1}$$

$$g_3^n = u^n v + 0$$

$$y^{n+1}$$

$$g_3^n = y^n$$

$$g_1^n + g_2^n + g_3^n = 0$$

$$x^2 y^{n+2} + 2xy^{n+1} + y^{n+1} + (n^2 - n) y^{n+1} y' + n y^n y'' = 0$$

$$\therefore x^2 y^{n+2} + (2n+1) x y^{n+1} + (n^2 + 1) y^n = 0$$