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17/ENG04/051 ENG MATHS III

Question 1

If $y = e^{x^2+x}$, show that $y'' = y'(2x+1) + 2y$, and hence, prove that $y^{(n+2)} = (2x+1)\,y^{(n+1)} + 2(n+1)\,y^{n}$.

Soln

Since $y = e^{(x^2+x)}$ $\therefore \frac{dy}{dx} = (2x+1)\,e^{(x^2+x)}$, $\frac{d^2y}{dx^2}$ = let $u = (2x+1)$, $v = e^{x^2+x}$

$\therefore \frac{du}{dx} = 2$ and $\frac{dv}{dx} = (2x+1)\,e^{(x^2+x)}$

$$\therefore \frac{d^2y}{dx^2} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (2x+1)\cdot(2x+1)\,e^{(x^2+x)} + e^{(x^2+x)}\cdot 2$$

$$\therefore \frac{d^2y}{dx^2} = (2x+1)^2\,e^{(x^2+x)} + 2\left(e^{(x^2+x)}\right)$$

Recall $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$ substituting the following values

$$\therefore y'' = (2x+1).(2x+1)\,e^{(x^2+x)} + 2\left(e^{(x^2+x)}\right)$$

Recalling $y' = (2x+1)\,e^{(x^2+x)}$ and $y = e^{(x^2+x)}$

$$\therefore y'' = y'(2x+1) + 2y.$$

Let $y'' = A$, $B = y'(2x+1)$, $C = 2y$

$A = y''$ $B = y'(2x+1)$ $C = 2y$

$A^n = y^{n+2}$ $u = y'$ $v = 2x+1$ $u = 2y$ $v = 2$

$u^n = y^{n+1}$ $v' = 2$ $u^n = y^n$ $v' = 0$

$u^{n-1} = y^n$ $v'' = 0$

$$\Longrightarrow \therefore B^n = u^n v + n u^{(n-1)} v' + \frac{n(n-1)}{2!} u^{(n-2)} v'' + \dots$$

$$= (y^{n+1})(2x+1) + n(y^n)\cdot 2 + 0$$

$$= (2x+1)\,y^{(n+1)} + 2n(y^{n})$$

$$\Longrightarrow \therefore C^n = u^n v + n u^{(n-1)} v' + \dots$$

$$= y^{(n)}\cdot 2 + 0$$

Recalling $A^n = B^n + C^n$

$$\therefore y^{(n+2)} = (2x+1)\,y^{(n+1)} + 2n(y^{(n)}) + 2\cdot y^{(n)}$$

$$\therefore y^{(n+2)} = (2x+1)\,y^{(n+1)} + 2(n+1)\,y^{(n)}$$

QUESTION 2

Using the Leibnitz theorem given that

(i) $y = x^3 e^{4x}$; determine $y^{(5)}$

(ii) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$, show that $x^2 y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$.

Solution

(i) $y = x^3 e^{4x}$

$V = x^3 \qquad U = e^{4x}$

$V' = 3x^2. \qquad U^n = 4^n e^{4x}$

$V'' = 6x, \qquad U^{(n-1)} = 4^{(n-1)} e^{4x}$

$V''' = 6. \qquad U^{(n-2)} = 4^{(n-2)} e^{4x}$

$V'''' = 0. \qquad U^{(n-3)} = 4^{(n-3)} e^{4x}$

Using the Leibnitz theorem ...

$y^n = U^n V + nU^{(n-1)}V' + \frac{n(n-1)U^{(n-2)}V''}{2!} + \frac{n(n-1)(n-2)U^{n-3}V'''}{3!} + \ldots$

$y^n = 4^n e^{4x} \cdot x^3 + n\,4^{(n-1)} e^{4x}\,3x^2 + \frac{n(n-1)\,4^{(n-2)} e^{4x} \cdot 6x}{2!} + \frac{n(n-1)(n-2)\,4^{(n-3)} e^{4x} \cdot 6}{3!} + 0$

$y^n = 4^n e^{4x} \cdot x^3 + n\,4^{n-1} e^{4x}\,3x^2 + 3(n-1)n\,4^{n-2} e^{4x} + 2n(n-1)(n-2)\,4^{n-3} e^{4x}.$

$\therefore y^5 = 4^5 e^{4x} \cdot x^3 + 5(4^4 e^{4x}) \cdot 3x^2 + 3x(4)(5) \cdot 4^3 e^{4x} + 2(5)(4)(3)\,4^2 e^{4x}$

$\therefore y^5 = 1024 e^{4x} \cdot x^3 + 1280 e^{4x} \cdot 3x^2 + \cancel{128}\, e^{4x} \cdot 3x + 960 e^{4x} \cdot 2$

$\therefore y^5 = 1024 e^{4x} \cdot x^3 + 1280 e^{4x} \cdot 3x^2 + 1280 e^{4x} \cdot 3x + 1920 e^{4x}.$

(ii) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \qquad$ Recalling $\frac{d^2y}{dx^2} = y^2$ and $\frac{dy}{dx} = y'$

$x^2 y^2 + xy' + y = 0$

$x^2 y^{2+n}_{(n)}$ assuming $x^2y^2 = w_1$, $xy'_{(n)} = w_2$ and $y = w_3$

for $w_1 \Rightarrow U = y^2 \quad V = x^2$ | for $w_2 \Rightarrow U = y' \quad V = x$

$U^{(n)} = y^{2+n} \quad V' = 2x$ | $U^{(n)} = y^{n+1} \quad V' = 1$

$U^{(n-1)} = y^{1+n} \quad V'' = 2$ | $U^{(n-1)} = y^n \quad V'' = 0$

$U^{(n-2)} = y^n \quad V''' = 0$

(n)

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For $w_3^{(n)} = y$
$= y^n$

$\therefore$ Using $y^n = u^n v + n u^{(n-1)} v' + \frac{n(n-1) u^{(n-2)}}{2!} v'' + \frac{n(n-1)(n-2)}{3!} u^{(n-3)} v'''$
$+ \ldots$

$\therefore w_1^{(n)} = y^{(n+2)} x^2 + n y^{(n+1)} \cdot 2x + \frac{n(n-1)}{2!} y^{(n)} \cdot 2 + 0$

$w_2^{(n)} = y^{(n+1)} x + n y^n \cdot 1 + 0$

$w_3^{(n)} = y^n$

Summing this altogether we would have

$w_{(1)}^{(n)} + w_{(2)}^{(n)} + w_{(3)}^{(n)}$

$y^{(n+2)} x^2 + n y^{(n+1)} \cdot 2x + n(n-1) y^{(n)} + y^{(n+1)} x + n y^{(n)} + y^n = 0$

$\Rightarrow x^2 y^{(n+2)} + 2x n y^{(n+1)} + n(n-1) y^{(n)} + x y^{(n+1)} + n y^{(n)} + y^{(n)} = 0$

$\Rightarrow x^2 y^{(n+2)} + x y^{(n+1)} (2n+1) + (n^2 - n + n + 1) y^n = 0$

$\therefore x^2 y^{(n+2)} + x y^{(n+1)} (2n+1) + (n^2+1) y^{(n)} = 0$