Question 1

(1a) $y = e^{x^2+x}$

$y' = (2x+1)e^{x^2+x}$

$y'' = 2x+1 \frac{\delta}{\delta x}(e^{x^2+x}) + e^{x^2+x} \frac{\delta}{\delta x}(2x+1)$ (Product Rule)

$y'' = (2x+1)(2x+1)e^{x^2+x} + e^{x^2+x}(2)$

$y' = (2x+1)e^{x^2+x}$

$y = e^{x^2+x}$

$y'' = y'(2x+1) + 2y \Rightarrow$ Proven

(1b) $y'' - y'(2x+1) - 2y = 0$

Using Leibnitz theorem

$w_1 = y''$

$w_2 = y'(2x+1)$

$w_3 = 2y$

degenerate eqn: $w_1 - w_2 - w_3 = 0$

w_1

$u = y''$ $u' = y'''$ hence $u^n = y^{n+2}$

$v = 1$ $v' = 0$

$w_1 = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v^2 + \ldots$

$w_1 = y^{n+2}(1) + n y^{n+1}(0)$

$w_1 = y^{n+2}$ (i)

W_2

$w_2 = y'(2x+1)$

$u = y'$ $u' = y''$ $u'' = y'''$, Hence $u^n = y^{n+1}$

$v = 2x+1$ $v' = 2$ $v'' = 0$

$w_2 = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v^2 + \frac{n(n-1)(n-2)}{3!} u^{n-3} v^3 + \ldots$

$w_2 = y^{n+1}(2x+1) + n y^n (2) + \frac{n(n-1)}{2!} y^{n-2}(0) + \ldots$

$w_2 = y^{n+1}(2x+1) + 2ny^n + 0$

$w_2 = y^{n+1}(2x+1) + 2ny^n$ ——— (ii)

w_3

$u = y \quad u' = y'$ Hence $u^n = y^n$

$v = 2 \quad v' = 0$

$$w_3 = u^n v + nu^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v^2 + \dots$$

$$w_3 = y^n(2) + ny^{n-1}(0) + \dots$$

$w_3 = 2y^n$ ——— (iii)

Putting back into the degenerate eqn.

$$w_1 - w_2 - w_3 = 0$$

$$y^{n+2} - (y^{n+1}(2x+1) + 2ny^n) - 2y^n = 0$$

$$y^{n+2} - y^{n+1}(2x+1) - 2ny^n - 2y^n = 0$$

$$y^{n+2} = y^{n+1}(2x+1) + 2ny^n + 2y^n$$

$$y^{n+2} = y^{n+1}(2x+1) + 2y^n(n+1)$$

$y^{n+2} = y^{n+1}(2x+1) + 2y^n(n+1) \implies$ Proven

Question 2

2a) $y = x^3 e^{4x}$

Using Leibnitz theorem

$u = e^{4x}$, $u' = 4e^{4x}$, $u'' = 16e^{4x}$ $u''' = 64e^{4x}$ $u^{iv} = 256e^{4x}$

$v = x^3$, $v' = 3x^2$ $v'' = 6x$ $v''' = 6$ $v^{iv} = 0$

Hence $u^n = 4^n e^{4x}$

$$y^n = u^n v + nu^{n-1}v' + \frac{n(n-1)}{2!}u^{n-2}v^2 + \frac{n(n-1)(n-2)}{3!}u^{n-3}v^3$$

$$+ \frac{n(n-1)(n-2)(n-3)}{4!}u^{n-4}v^2 + \ldots$$

$$y^n = 4^n e^{4x}(x^3) + n4^{n-1}e^{4x}(3x^2) + \frac{n(n-1)}{2!}4^{n-2}e^{4x}(6x)$$

$$+ \frac{n(n-1)(n-2)}{3!}4^{n-3}e^{4x}(6) + \frac{n(n-1)(n-2)(n-3)}{4!}4^{n-4}(0)^2$$

$y^n = 4^n x^3 e^{4x} + n3x^2 4^{n-1}e^{4x} + 3xn(n-1)4^{n-2} + n(n-1)(n-2)4^{n-3}e^{4x} + 0$

$y^n = e^{4x}4^{n-3}\left(x^3 4^3 + 3nx^2 4^2 + 3xn(n-1)4 + n(n-1)(n-2)\right)$

$y^n = e^{4x}4^{n-3}\left(64x^3 + 48x^2 n + 12x(n(n-1) + n(n-1)(n-2)\right)$

$y^5 = e^{4x}4^{5-3}\left(64x^3 + 48x^2(5) + 12x(5)(5-1) + 5(5-1)(5-2)\right)$

$y^5 = e^{4x}4^2(64x^3 + 240x^2 + 240x + 36)$

$y^5 = 16e^{4x}(64x^3 + 240x^2 + 240x + 36)$

2b) $x^2\frac{d^2y}{dx^2} + n\frac{dy}{dx} + y = 0$

$x^2 y^x + xy' + y = 0$

$w_1 + w_2 + w_3 = 0$

$\underline{W_1 = x^2 y''}$

$u = y''$ $u' = y'''$ $u'' = y^{iv}$ $u''' = y^v$ Hence, $u^n = y^{n+2}$

$v = x^2$ $v' = 2x$ $v'' = 2$ $v''' = 0$

$$w_1 = u^n(x) + nu^{n-1}v' + \frac{n(n-1)}{2!}u^{n-2}(v^{ii}) + \frac{n(n-1)(n-2)}{3!}u^{n-3}v''' + \ldots$$

$$w_1 = y^{n+2}(x^2) + ny^{n+1}(2x) + \frac{n(n-1)}{2 * 1}y^n(2) + 0$$

$$w_1 = x^2 y^{n+2} + 2xny^{n+1} + y^n n(n-1)$$

$\underline{w_2 = xy'}$

$u = y' \quad u' = y'' \quad u'' = y'''$ Hence $u^n = y^{n+1}$

$v = x \quad v' = 1 \quad v'' = 0$

$$w_2 = u^n(v) + nu^{n-1}(v') + \frac{n(n-1)}{2!} u^{n-2} v^2$$

$$w_2 = y^{n+1}(x) + ny^n(1) + 0$$

$$w_2 = \underline{xy^{n+1} + ny^n}$$

$\underline{w_3 = y}$

$u = y \quad u' = y'$ Hence $u^n = y^n$

$v = 1 \quad v' = 0$

$$w_3 = u^n(v) + nu^{n-1}(v')$$

$$= y(1) + 0 = y$$

$$w_1 + w_2 + w_3 = 0$$

$$y^{n+2}(x^2) + ny^{n+1}(2x) + n(n-1)y^n + xy^{n+1} + ny^n + y^n = 0$$

$$x^2y^{n+2} + xy^{n+1}(2n+1) + y^n(n(n-1) + n+1)$$

$$x^2y^{n+2} + xy^{n+1}(2n+1) + y^n(n^2+1) = 0$$