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Assignment 2

(1) If $y = e^{x^2+x}$, show that $y'' = y'(2x+1) + 2y$ and hence prove that $y^{(n+2)} = (2x+1)y^{n+1} + 2(n+1)y^n$.

Solution:

$$y = e^{x^2+x} \quad \text{--- (1)}$$

using chain rule; let $u = x^2+x$ $\therefore y = e^u$

$dy/dx = dy/du \times du/dx$; $du/dx = 2x+1$

$dy/du = e^u = e^{x^2+x}$

$dy/dx = (2x+1)e^{x^2+x} = y'$

Finding y'', using product rule

$y' = (2x+1)(e^{x^2+x})$

let $u = 2x+1$; $du/dx = 2$

let $v = e^{x+x}$; $dv/dx = (2x+1)e^{x^2+x}$

$d^2y/dx^2 = y'' = u\,dv/dx + v\,du/dx$

$y'' = (2x+1)(2x+1)(e^{x^2+x}) + e^{x^2+x}(2)$

$y = e^{x^2+x}$ and $y' = (2x+1)e^{x^2+x}$

Substitute the value of y and y' in y''

$y'' = (2x+1)(2x+1)(e^{x^2+x})$

$$y'' = (2x+1)y' + 2y \text{ ——— (ii)}$$

To prove $y^{(n+2)} = (2x+1)y^{n+1} + 2(n+1)y^n$ ——— (iii)

$(2x+1)y^{n+1} + 2(n+1)y^n - y^{(n+2)} = 0$

Relating eqn (ii) and eqn (i)

$y^{n+1} = y'$, $y^{n+2} = y''$

let $w' = y'(2x+1)$

$v = 2x+1$ $\quad u = y'$

$v' = 2$ $\quad u^n = y^{n+1}$

$u^{n-1} = y^{(n+1)} = y^n$

Applying Leibnitz theorem

$$y^n = u^n v + n u^{(n-1)} v^{(1)} + \frac{n(n-1)}{2!} v^{(2)} u^{(n-2)} + \frac{n(n-1)(n-2)}{3!} u^{(n-3)} v^3 + \ldots$$

$y^n = y^{(n+1)}(2x+1) + n y^n (2) + 0$

$y^n = (2x+1) y^{(n+1)} + 2(n) y^n$

Let $W_2 = 2y$

$u = y$ $\quad v = 2$

$u^n = y^n$ $\quad v' = 0$

$y^n = y^n(2) + 0 = 2y^n$

let $2u_3 = y''$

$\cdot\ y'' = -(y^{(0)})$

$u = y^2 \qquad v = -1$

$u^n = y^{n+2} \qquad v' = 0$

$y^n = y^{n+2}(-1) + 0 = -y^{n+2}$

$y^{n+1}(2x+1) + 2ny^n + 2y^n - y^{n+2} = 0$

$y^{n+1}(2x+1) + 2y^n(n+1) - y^{n+2} = 0$

$\therefore\ y^{n+1}(2x+1) + 2y^n(n+1) = y^{n+2}$

(2) Using the Leibnitz theorem, given that (i) $y = x^3 e^{4x}$, determine y^5,
(ii) $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$, show that $x^2 y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)}$
$= 0$

Solution

(i) $y = x^3 e^{4x}$

Using Leibnitz theorem

$u = e^{4x},\ u' = 4e^{4x};\ u'' = 16e^{4x},\ u''' = 64e^{4x};\ u^{iv} = 256e^{4x}$

$v = x^3,\ v' = 3x^2,\ v'' = 6x,\ v^{iv} = 0$

$\therefore\ u^n = 4^n e^{4x}$

$y^n = u^n v + nu^{n-1}v' + \frac{n(n-1)u^{n-2}v^2}{2!} + \frac{n(n-1)(n-2)u^{n-3}v^3}{3!} + \frac{n(n-1)(n-2)(n-3)u^{n-4}v^4}{4!} +$

$\frac{n(n-1)(n-2)(n-3)(n-4)}{5!}$

$u^{n-5} v^5$

$= 4^n e^{4x}(x^3) e^{4x} + n 4^{n-1} e^{4x}(3x^2) + \frac{n(n-1)}{2!} 4^{n-2} e^{4x} 6x + \frac{(n)(n-1)(n-2)}{3!} 4^{n-3} e^{4x} 6$

$\frac{n(n-1)(n-2)(n-3)}{4!} 4^{n-4}(e^{4x})(0)$

since $n=5$

$y^5 = 4^5 e^{4x} x^3 e^{4x} + 5(4^4 e^{4x}) 3x^2 + 5\frac{(5-1)}{2} 4^3 e^{4x} 6x + \frac{5(5-1)(5-2)}{3!} 4^2 e^{4x} 6 + 0$

$y^5 = 4^5 e^{4x} x^3 + 5(4^4 e^{4x}) 3x^2 + \frac{5(4)}{2!} 4^3 e^{4x} 6x + 5(4)(3) 4^2 e^{4x} 6 + 0$

$y^5 = 1024 e^{4x} x^3 + 3840 e^{4x} x^2 + 3840 e^{4x} x + 960 e^{4x}$

$y^5 = 64 e^{4x}(16x^3 + 60x^2 + 60x + 15$

(b) $x^2 \, d^2y/dx^2 + x \, dy/dx + y = 0$

$x^2 y'' + xy' + y = 0$

let $y_1 = x^2 y''$

$v = x^2$ $\quad$ $u^n = y'' = y^{(n+2)}$

$v' = 2x$ $\quad$ $u^n = y^{n+2}$

$v'' = 2$ $\quad$ $u^{n-1} = y^{(n+2)-1} = y^{n+1}$

$v''' = 0$ $\quad$ $u^{n-2} = y^{(n+2)-2} = y^2$

Using Leibnitz theorem

$y_1^n = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{(n-2)} v'' + 0$

$y_1^n = y^{(n+2)} x^2 + n y^{n+1}(2x) + \frac{n(n-1)}{2!} y^n (2) + 0$

$y_1^n = x^2 y^{n+2} + (2x)(n) y^{n+1} + n(n-1) y^n$

$y_1^n = x^2 y^{n+2} + 2nx y^{n-1} + n^2 - n y^n$

let $y_2 = xy'$

$u = y', u^n = y^{(n+1)}$

$v' = 1 \qquad u^{n+1} = y^{n+1-1} = y^n$

$v^2 = 0 \qquad u^{n+2} \; y^{(n+1)-2} = y^{n-1}$

$y_2^n = u^n v + n u^{n-1} v' + 0$

$y_2^n = y^{(n+1)} x + n(y^n)(1)$

$y_2^n = x y^{n+1} + n y^n$

let $y_3 = y$

$v = 1 \qquad u = y, u^n = y^n$

$v' = 0 \qquad u^{n-1} = y^{n-1}$

$y_3^n = u^n v + 0$

$y^n (1)$

$y_3^n = y^n$

$y_1^n + y_2^n + y_3^n = 0$

$$x^2y^{n+2} + 2nxy^{n+1} + xy^{n+1} + (n^2-n)y^n + ny^n + y^n = 0$$

$$\therefore x^2y^{n+2} + (2n+1)xy^{(n+1)} + (n^2+1)y^n = 0$$