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Chemical Engineering
ENG 381 Assignment

1 If $y = e^{x^2+x}$, Show that $y'' = y'(2x+1) + 2y$ and Hence prove that
$y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^n$

Solution

$$y = e^{x^2+x}$$

Let $u = x^2+x$, $y = e^u$

$$\frac{du}{dx} = 2x+1$$

$$\frac{dy}{du} = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{(Chain Rule)}$$

$$\frac{dy}{dx} = e^u \cdot 2x+1$$

& $u = x^2+x$

$$\frac{dy}{dx} = (2x+1)e^{x^2+x} \Rightarrow y'$$

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = (2x+1)e^{x^2+x}$$

$\frac{dy}{dx}$ = let $u = (2x+1)$ $\quad du/dx = 2$

let $v = e^{x^2+x}$ $\quad dv/dx = (2x+1)e^{x^2+x}$

$$y'' = (2x+1)(2x+1)e^{x^2+x} + (e^{x^2+x}) \cdot 2$$

Recall $y = e^{x^2+x}$

$$y' = (2x+1)e^{x^2+x}$$

$$\therefore y'' = y'(2x+1) + 2y \text{ - - - - (ii)}$$

To prove $y^{(n+2)} = (2x+1)y^{n+1} + 2(n+1)y^{n}$

$(2x+1)y^{n+1} + 2(n+1)y^{n} - y^{(n+2)} = 0$

From eqn (ii) $y'' = (2x+1)y' + 2y \Rightarrow (2x+1)y' + 2y - y'' = 0$

Let $w_1 = (2x+1)y'$

$u = y'$ $\quad$ $v = 2x+1$

$u^{n} = y^{(n+1)}$ $\quad$ $v^{(1)} = 2$ $\quad \Rightarrow w_1 y^{(n)} = y^{(n+1)}(2x+1) + ny^{(n)} \cdot 2 + 0$

$u^{n-1} = y^{(n)}$ $\quad$ $v^{2} = 0$

Let $w_2 = 2y$

$u = y$ $\quad$ $v = 2$ $\quad \Rightarrow w_2 y^{(n)} = 2y^{n}$

$u^{(n)} = y^{(n)}$ $\quad$ $v^{(1)} = 0$

Let $w_3 = y''$

$u = y''$ $\quad$ $v = 1$ $\quad \Rightarrow w_3 y^{(n)} = y^{(n+2)}$

$u^{(n)} = y^{(n+2)}$ $\quad$ $v' = 0$

$u^{(n-1)} = y^{(n+1)}$

$u^{(n-2)} = y^{(n)}$

Add them together

$y^{(n+1)}(2x+1) + 2ny^{(n)} + 2y^{(n)} - y^{(n+2)} = 0$

$y^{(n+1)}(2x+1) + 2(ny^{(n)} + y^{(n)}) - y^{(n+2)} = 0$

$y^{(n+1)}(2x+1) + 2(n+1)y^{(n)} - y^{(n+2)} = 0$

$y^{(n+2)} = y^{(n+1)}(2x+1) + 2(n+1)y^{(n)}$

2i) $y = x^3 e^{4x}$

Determine $y^{(5)}$, Using Leibnitz theorem

$u = e^{4x}$ $\quad$ $v = x^3$

$u' = 4e^{4x}$ $\quad$ $v' = 3x^2$

$u'' = 16e^{4x}$ $\quad$ $v'' = 6x$

$u''' = 64e^{4x}$ $\quad$ $v''' = 6$

$u^{(4)} = 256e^{4x}$ $\quad$ $v^{(4)} = 0$

$u^{(5)} = 1024e^{4x}$

$$y^{(n)} = u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{2!}u^{(n-2)}v^2 + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^3 + \dots$$

$$y^{(5)} = u^5v + 5u^4v' + 10u^3v^2 + 10u^2v^3 + 5u'v^4 + u^{(0)}v^{(5)} + \dots$$

$$y^{(5)} = 1024e^{4x}\cdot x^3 + 5(256e^{4x})\cdot 3x^2 + 10(64e^{4x})\cdot 6x + 10(16e^{4x})\cdot 6 + 0$$

$$y^5 = 1024e^{4x}x^3 + 3840e^{4x}x^2 + 3840e^{4x}x + 960e^{4x}$$

$$y^5 = 64e^{4x}(16x^3 + 60x^2 + 60x + 15)$$

ii $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Show that $x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$

Solution: Using Leibnitz theorem

$\Rightarrow x^2y'' + xy' + y = 0$

Let $\omega_1 = x^2y''$

$\omega_2 = xy'$

$\omega_3 = y$

$\Rightarrow \omega_1 = x^2y''$

$u = y^{(2)} \quad v = x^2$

$u^{(n)} = y^{(n+2)} \quad v^{(1)} = 2x$

$u^{(n-1)} = y^{(n+1)} \quad v^{(2)} = 2$

$u^{(n-2)} = y^{(n)} \quad v^{(3)} = 0$

$\Rightarrow \omega_1 y^{(n)} = y^{(n+2)}x^2 + ny^{(n+1)}2x + \frac{n(n-1)}{2!}y^{(n)}2 + 0$

$\Rightarrow \omega_1 y^{(n)} = y^{(n+2)}x^2 + ny^{(n+1)}2x + (n^2-n)y^{(n)} + 0$

$\Rightarrow \omega_2 = xy'$

$u = y^{(1)} \quad v = x$

$u^{(n)} = y^{(n+1)} \quad v^{(1)} = 1$

$u^{(n-1)} = y^{(n)} \quad v^{(2)} = 0$

$\Rightarrow \omega_2 y^{(n)} = y^{(n+1)}x + ny^{(n)}\cdot 1 + 0$

$\Rightarrow \omega_3 = y$

$u = y \quad v = 1$

$u^{(n)} = y^{(n)} \quad v^{(1)} = 0$

$\Rightarrow \omega_3 y^{(n)} = y^{(n)}$

Add them together & Rearrange

$$\Rightarrow x^2y^{(n+2)} + 2xny^{(n+1)} + \cancel{xy^{(n+1)}} + \cancel{x}(n^2-n)y^{(n)} + xy^{(n+1)} + ny^{(n)} + y^{(n)} = 0$$

$$x^2y^{(n+2)} + 2xny^{(n+1)} + xy^{(n+1)} + (n^2-n)y^{(n)} + ny^{(n)} + y^{(n)} = 0$$

$$\cancel{x^2y^{(n+2)} + (2n+1)xy^{(n+1)}}$$

$$\Rightarrow x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + [n^2 - \cancel{n} + \cancel{n} + 1]y^{(n)} = 0$$

$$= x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$$