(a) $\frac{dy}{dx} = 2\sinh x - y\tanh x$

$dy/dx + y\tanh x = \sinh x$

$P = \tanh x$

$Q = 2\sinh x$

$\int P dx = \int \tanh x = \int \frac{\sinh x}{\cosh x} dx$

$\cosh x = u$

$\int \frac{\sinh x}{u} dx$

$u = \cosh x \qquad dx = \frac{dy}{\sinh x}$

$du/dx = \sinh x$

$\int \frac{\sinh x}{u} \cdot \frac{dy}{\sinh x}$

$\int \frac{1}{u} du = \ln u = \ln \cosh x$

$IF = e^{\int P dx} = e^{\ln \cosh x}$

$IF = \cosh x$

Then $y\ IF = \int Q \cdot IF\, dx$

$y \cdot \cosh x = \int 2\sinh x \cdot \cosh x\, dx$

$2\sin x \cos x = \sin(2x)$

$\therefore 2\sinh x \cosh x = \sinh(2x)$

$y \cdot \cosh x = \frac{1}{2} \cdot \cancel{2}\cosh 2x + c$

$\cosh x\, y = \cosh 2x + c$

$y = \frac{\frac{\cosh 2x + c}{2}}{\cosh x}$

$y = \frac{\cosh 2x + 2c}{\cosh x}$

let $2c = A$

$y = \frac{\cosh 2x + A}{\cosh x}$

(b) $dy/dx + 2y = e^{3x}$

$P = 2, \quad \int P dx = 2x$

$Q = e^{3x}$

$IF = e^{\int P dx} = e^{2x}$

$\therefore y \cdot IF = \int Q \cdot IF\, dx$

$y \cdot e^{2x} = \int e^{3x} \cdot e^{2x} dx$

$y \cdot e^{2x} = \int e^{3x} \cdot e^{2x} dx$

$y \cdot e^{2x} = \frac{1}{5} e^{5x} dx$

$y = \frac{\frac{1}{5}e^{5x} + c}{e^{2x}}$

(c) $dy/dx = x^2 + 2x - 3$

$dy/dx = x + 2 - 3/x$

$\therefore \int \frac{dy}{dx} = \int x + 2 - 3/x\, dx$

$y = \frac{x^2}{2} + 2x - 3\ln x + c$

(d) $dy/dx + y/x = y^3$

$dy/dx\ y^{-3} + y^{-2}/x = 1 \quad \ldots (1)$

$z = y^{1-n}, \quad n = 3 \quad \ldots (2)$

$z = y^{1-3}, \quad z = y^{-2} \quad \ldots (3)$

Then multiply eqn 1 by $1-n$

$-2y^{-3}\frac{dy}{dx} - \frac{2y^{-2}}{x} = -2$

and $\frac{dz}{dy} = \frac{-2y^{-3}\, dy}{dx}$

Sub eqn 2 & 3 into 4

$dz/dy - 2z/x = -2$

$\therefore P = -2/x, \ Q = -2$

$\int P dx = -2\ln x$

$IF = e^{-2\ln x} = x^{-2}$

$Z \cdot IF = \int Q \cdot IF \cdot dx$

$Z \cdot x^{-2} = \int -2x^{-2}\, dx$

$= \frac{-2x^{-1}}{-1} + C$

$Z \cdot x^{-2} = 2x^{-1} + C$

$Z = \frac{2x^{-1}}{x^{-2}} + \frac{C}{x^{-2}}$

$Z = 2x + Cx^2$

$Z = x(2 + Cx)$

$Z = y^{-2}$

$y^{-2} = x(2 + Cx)$

$1/y^2 = x(2 + Cx)$

$y^2 = \frac{1}{x(2+Cx)}$

$\therefore y = \sqrt{\frac{1}{x(2+Cx)}}$

$y = \frac{1}{\sqrt{x(2+Cx)}}$

(e) $x^2\, dy/dx = x^3 \sin 3x + 4$

$dy/dx = x \sin 3x + 4$

$\int dy/dx = \int \overset{u}{x} \overset{dv}{\sin 3x} + \int 4x^{-2}$

$= \frac{1}{3} \cos 3x + \int \frac{1}{3} \cos 3x + \frac{4x^{-1}}{-1}$

$y = \frac{\sin 3x}{9} - \frac{x \cos 3x}{3} - \frac{4}{x}$

(f) $(x^3 + xy^2)\, dy/dx = 2y^3$

$y = ux$

$dy/dx = u + x\, du/dx$

$u + x\, du/dx = \frac{2(ux)^3}{x^3 + u^2 x^3}$

$u + x\, du/dx = \frac{x^3 (2u^3)}{x^3 (1+u^2)}$

$x \frac{du}{dx} = \frac{2u^3}{1+u^2}$

$= \frac{2v^3 - v - v^3}{1 + v^2}$

$x \frac{dv}{dx} = v^3$

$\frac{1+v^2}{v^3 - v}\, dv = 1/x\, dx$

$v(v-1)(v+1) = v^3 - v$

$\frac{1+v^2}{v^3 - v} = \frac{A}{v} + \frac{B}{v-1} + \frac{C}{v+1}$

$1 + v^2 = A(v-1)(v+1) + B(v)(v+1) + C(v)$

$(v-1),\ v = 1$

$1 + 1^2 = B(1)(2)$

$2 = B(2)$

$\therefore B = 1$

$v = -1$

$1 + (-1)^2 = C(-1)(-1-1)$

$2 = 2C$

$\therefore C = 1$

$v = 0$

$1 + (0)^2 = A(0-1)(0+1)$

$1 = A(-1)(1)$

$1 = A - 1$

$\therefore A = -1$

$\int \left[-1/v + \frac{1}{v-1} + \frac{1}{v+1} \right] dv = \int dx\, 1/x$

$\int -1/v\, dv + \int \frac{1}{v-1}\, dv + \int \frac{1}{v+1}\, dv = \int 1/x\, dx$

$-\ln v + \ln(v-1) + \ln(v+1) = \ln x + C$

$\ln(v-1)(v+1) - \ln v = \ln x + \ln A$

$\frac{v^2 - 1}{v} = Ax$

$y = vx \quad \therefore \quad v = y/x$

$\frac{(y/x)^2 - 1}{y/x} = Ax$

$$\frac{y^2}{x^2} - 1 = A\cancel{x} - y/\cancel{x}$$

$$y^2/x^2 - 1 = Ay$$

$$\frac{y^2 - x^2}{\cancel{x^2}} = Ay$$

$$y^2 - x^2 = Ayx^2$$

$$y^2 = Ax^2y + x^2$$

$$y^2 = x^2(Ay + 1)$$