(a) $\frac{dy}{dx} = 2\sinh x - y\tanh x$

$dy/dx + y\tanh x = \sinh x$

$P = \tanh x$

$Q = 2\sinh x$

$\int P dx = \int \tanh x = \int \frac{\sinh x}{\cosh x} dx$

$\cosh x = u$

$\int \frac{\sinh x}{u} dx$

$u = \cosh x$ $\qquad dx = \frac{du}{\sinh x}$

$du/dx = \sinh x$

$\int \frac{\sinh x}{u} \cdot \frac{du}{\sinh x}$

$\int \frac{1}{u} du = \ln u = \ln \cosh x$

$IF = e^{\int P dx} = e^{\ln \cosh x}$

$IF = \cosh x$

Then $y \cdot IF = \int Q \cdot IF \, dx$

$y \cdot \cosh x = \int 2\sinh x \cdot \cosh x \, dx$

$2\sin x \cos x = \sin(2x)$

$\therefore 2\sinh x \cosh x = \sinh(2x)$

$y \cdot \cosh x = \frac{1}{2} \cdot \cancel{2} \cosh 2x + c$

$\cosh x y = \cosh 2x + c$

$y = \frac{\frac{\cosh 2x + c}{2}}{\cosh x}$

$y = \frac{\cosh 2x + 2c}{\cosh x}$

let $2c = A$

$y = \frac{\cosh 2x + A}{\cosh x}$

(b) $dy/dx + 2y = e^{3x}$

$P = 2$, $\int P dx = 2x$

$Q = e^{3x}$

$IF = e^{\int P dx} = e^{2x}$

$\therefore y \cdot IF = \int Q \cdot IF \, dx$

$y \cdot e^{2x} = \int e^{3x} \cdot e^{2x} dx$

$y \cdot e^{2x} = \int e^{3x} \cdot e^{2x} dx$

$y \cdot e^{2x} = \frac{1}{5} e^{5x} dx$

$y = \frac{\frac{1}{5}e^{5x} + c}{e^{2x}}$

(c) $dy/dx = x^2 + 2x - 3$

$dy/dx = x + 2 - 3/x$

$\therefore \int \frac{dy}{dx} = \int x + 2 - \frac{3}{x} \, dx$

$y = \frac{x^2}{2} + 2x - 3\ln x + c$

(d) $dy/dx + y/x = y^3$

$dy/dx \, y^{-3} + y^{-2}/x = 1$ $\qquad$ ...(1)

$z = y^{1-n}$, $n = 3$ $\qquad$ ...(2)

$z = y^{1-3}$ $\quad z = y^{-2}$ $\qquad$ ...(3)

Then multiply eqn 1 by $1-n$

$-2y^{-3}\frac{dy}{dx} - \frac{2y^{-2}}{x} = -2$

and $\frac{dz}{dy} = \frac{-2y^{-3} dy}{dx}$

Sub eqn 2 $ 3 into 4

$dz/dy - 2z/x = -2$

$\therefore P = -2/x$, $Q = -2$

$\int P dx = -2\ln x$

$IF = e^{-2\ln x} = x^{-2}$

$$Z \cdot IF = \int Q \cdot IF \cdot dx$$
$$Z \cdot x^{-2} = \int -2x^{-2}\, dx$$
$$= \frac{-2x^{-1}}{-1} + c$$

$$Z \cdot x^{-2} = 2x^{-1} + c$$
$$Z = \frac{2x^{-1}}{x^{-2}} + \frac{c}{x^{-2}}$$

$$Z = 2x + cx^2$$
$$Z = x(2 + cx)$$
$$Z = y^{-2}$$
$$y^{-2} = x(2 + cx)$$
$$1/y^2 = x(2 + cx)$$
$$y^2 = \frac{1}{x(2+cx)}$$
$$\therefore y = \sqrt{\frac{1}{x(2+cx)}}$$
$$y = \frac{1}{\sqrt{x(2+cx)}}$$

(e) $x^2\, dy/dx = x^3 \sin 3x + 4$

$$dy/dx = x \sin 3x + 4$$
$$\int dy/dx = \int \overset{u}{x} \overset{dv}{\sin 3x} + \int 4x^{-2}$$

$$= 1/3 \cos 3x + \int \frac{1}{3} \cos 3x + \frac{4x^{-1}}{-1}$$
$$y = \frac{\sin 3x}{9} - \frac{x \cos 3x}{3} - \frac{4}{x}$$

(f) $(x^3 + xy^2)\, dy/dx = 2y^3$

$$y = ux$$
$$dy/dx = u + x\, du/dx$$
$$u + x\, du/dx = \frac{2(ux)^3}{x^3 + u^2x^3}$$
$$u + x\, du/dx = \frac{x^3(2u^3)}{x^3(1+u^2)}$$
$$x \frac{du}{dx} = \frac{2u^3}{1+u^2}$$

$$= \frac{2v^3 - v - v^3}{1 + u^2}$$

$$x \frac{dv}{dx} = v^3$$
$$\frac{1+v^2}{v^3 - v}\, dv = 1/x\, dx$$

$$v(u-1)(v+1) = u^3 - u$$
$$\frac{1+u^2}{v^3 - v} = \frac{A}{v} + \frac{B}{u-1} + \frac{C}{v+1}$$

$$1 + u^2 = A(u-1)(u+1) + B(u)(u+1) + C(u)$$
$$(v-1),\ v = 1$$
$$1 + 1^2 = B(1)(2)$$
$$2 = B(2)$$
$$\therefore B = 1$$
$$v = -1$$
$$1 + (-1)^2 = C(-1)(-1-1)$$
$$2 = 2c$$
$$\therefore c = 1$$

$$v = 0$$
$$1 + (0)^2 = A(0-1)(0+1)$$
$$1 = A(-1)(1)$$
$$1 = A - 1$$
$$\therefore A = -1$$
$$\int \left[-1/v + \frac{1}{v-1} + \frac{1}{v+1} \right] dv = \int dx\, 1/x$$
$$\int -1/v\, dx \int \frac{1}{v-1}\, du + \int \frac{1}{v+1}\, dv = \int 1/x\, dx$$

$$-\ln v + \ln(v-1) + \ln(v+1) = \ln x + c$$
$$\ln(u-1)(u+1) - \ln v = \ln x + \ln A$$
$$\frac{u^2 - 1}{v} = Ax$$
$$y = vx \quad \therefore v = y/x$$
$$\frac{(y/x)^2 - 1}{y/x} = Ax$$

$$\frac{y^2}{x^2} - 1 = A\cancel{x} \quad \cancel{-y/x}$$

$$y^2/x^2 - 1 = Ay$$

$$\frac{y^2 - x^2}{x^2} = Ay$$

$$y^2 - x^2 = Ayx^2$$

$$y^2 = Ax^2y + x^2$$

$$y^2 = x^2(Ay + 1)$$