① Find the derivative of the following using first principle ⓐ $y = \sin(3/x^2)$ ⓑ $y = 4/x^3$

② Find the integral of the following ⓐ $dx/(6x^2+36)$ ⓑ $dx/(6x^2+13)$

Solution

1a. $y = \sin(3/x^2)$

$y = \sin {}^{3}/{x^2}$

~~$y = 3x^{-2}$~~ $y = \sin 3x^{-2}$

let $u = 3x^{-2}$

$\therefore \; y = \sin u$

~~$\frac{dy}{du} = \cos u$~~ $\Delta y + y = \sin(u + \Delta u)$

$\Delta y = \sin(u + \Delta u) - y$

$\Delta y = \sin(u + \Delta u) - \sin u$

recall $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$A = u + \Delta u \qquad B = u$

$\frac{A+B}{2} = \frac{u + \Delta u + u}{2} = \frac{2u + \Delta u}{2} = u + \frac{\Delta u}{2}$

$\frac{A-B}{2} = \frac{u + \Delta u - u}{2} = \frac{\Delta u}{2}$

Hence $\frac{\Delta y}{\Delta u} = \frac{2\cos\left(u + \frac{\Delta u}{2}\right)\sin\left(\frac{\Delta u}{2}\right)}{\Delta u}$

$\frac{\Delta y}{\Delta u} = \frac{2\cos\left(u + \frac{\Delta u}{2}\right)\sin\left(\frac{\Delta u}{2}\right) \times \frac{1}{2}}{\Delta u \times \frac{1}{2}}$

$\frac{\Delta y}{\Delta u}$ ~~if~~ Since $\lim_{\theta \to 0} \sin\theta/\theta = 1$

$\frac{\Delta y}{\Delta u} = \cos(u + 0) \lim_{\Delta u \to 0} \frac{\sin\left(\frac{\Delta u}{2}\right)}{\frac{\Delta u}{2}}$

$\frac{dy}{du} = \cos u \times 1$

$\frac{dy}{du} = \cos u$

Also $u = \frac{3}{x^2}$

$$\Delta u + u = \frac{3}{(x+\Delta x)^2}$$

$$\Delta u = \frac{3}{(x+\Delta x)^2} - u$$

$$\Delta u = \frac{3}{(x+\Delta x)^2} - \frac{3}{x^2}$$

$$\Delta u = \frac{3x^2 - 3(x+\Delta x)^2}{x^2(x+\Delta x)^2}$$

$$\Delta u = \frac{3x^2 - 3(x^2+2x\Delta x + (\Delta x)^2)}{x^2(x+\Delta x)^2}$$

$$\Delta u = \frac{3x^2 - 3x^2 - 6x\Delta x - 3(\Delta x)^2}{x^2(x+\Delta x)^2}$$

$$\Delta u = \frac{-6x\Delta x - 3(\Delta x)^2}{x^2(x+\Delta x)^2}$$

$$\frac{\Delta u}{\Delta x} = \frac{-6x\cancel{\Delta x}}{x^2(x+\Delta x)^2} \times \frac{1}{\cancel{\Delta x}} - \frac{3(\Delta x)^{\cancel{2}}}{x^2(x+\Delta x)^2} \times \frac{1}{\cancel{\Delta x}}$$

$$\frac{\Delta u}{\Delta x} = \frac{-6x}{x^2(x+\Delta x)^2} - \frac{3\Delta x}{x^2(x+\Delta x)^2}$$

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{-6x}{x^2(x+0)^2} - \frac{3(0)}{x^2(x+0)^2}$$

$$\frac{du}{dx} = \frac{-6x}{x^4} - 0$$

$$\frac{du}{dx} = -6x^{-3}$$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$= -6x^{-3} \times \cos u$$

recall $u = 3/x^2$

$$\therefore \frac{dy}{dx} = -\frac{6}{x^3} \times \cos\left(\frac{3}{x^2}\right)$$

$$= \frac{-6}{x^3} \cos \frac{3}{x^2}$$

16. $y = 4/x^3$

$$y + \Delta y = \frac{4}{(\Delta x + x)^3}$$

$$\Delta y = \frac{4}{(\Delta x + x)^3} - y$$

$$\Delta y = \frac{4}{(\Delta x + x)^3} - \frac{4}{x^3}$$

$$\Delta y = \frac{4x^3 - 4(\Delta x + x)^3}{x^3(\Delta x + x)^3}$$

$$\Delta y = \frac{4x^3 - 4[x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3]}{x^3[x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3]}$$

$$\Delta y = \frac{\cancel{4x^3} - \cancel{4x^3} - 12x^2(\Delta x) - 12x(\Delta x)^2 - 4(\Delta x)^3}{x^3(\Delta x + x)^3}$$

$$\frac{\Delta y}{\Delta x} = \frac{-12x^2(\cancel{\Delta x})}{x^3(\Delta x + x)^3} \times \frac{1}{\cancel{\Delta x}} - \frac{12x^2(\Delta x)^2}{x^3(\Delta x + x)^3} \times \frac{1}{\Delta x} - \frac{4(\Delta x)^3}{x^3(\Delta x + x)^3} \times \frac{1}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{-12x^2}{x^3(0 + x)^3} - 0 - 0$$

$$\frac{dy}{dx} = \frac{-12x^2}{x^6}$$

$$\frac{dy}{dx} = -12x^{-4}$$

(2) (a) $\int \frac{1}{x^2 + 36}\, dx$

$$u = x/6 \rightarrow \frac{du}{dx} = 1/6$$

$$dx = 6\,du$$

$$= \int \frac{6}{36u^2 + 36}\, du$$

$$= \frac{1}{6} \int \frac{1}{u^2 + 1}\, du$$

also $\int \frac{1}{u^2+1}\,du = \arctan(u)$

$\therefore \frac{1}{6}\int \frac{1}{u^2+1}\,du = \frac{\arctan(u)}{6}$

recall $u = \frac{x}{6}$

thus $\int \frac{1}{x^2+36}\,dx = \frac{\arctan\left(\frac{x}{6}\right)}{6} + C$

(b) $\int \frac{1}{x^2+13}\,dx$

$u = \frac{x}{\sqrt{13}} \rightarrow \frac{du}{dx} = \frac{1}{\sqrt{13}}$

$dx = \sqrt{13}\,du$

$= \int \frac{\sqrt{13}}{13u^2+13}\,du$

$= \frac{1}{\sqrt{13}} \int \frac{1}{u^2+1}\,du$

also $\int \frac{1}{u^2+1}\,du = \arctan(u)$

$\therefore \frac{1}{\sqrt{13}}\int \frac{1}{u^2+1}\,du = \frac{\arctan(u)}{\sqrt{13}}$

recall $u = \frac{x}{\sqrt{13}}$

thus $\int \frac{1}{x^2+13}\,dx = \frac{\arctan\left(\frac{x}{\sqrt{13}}\right)}{\sqrt{13}} + C$