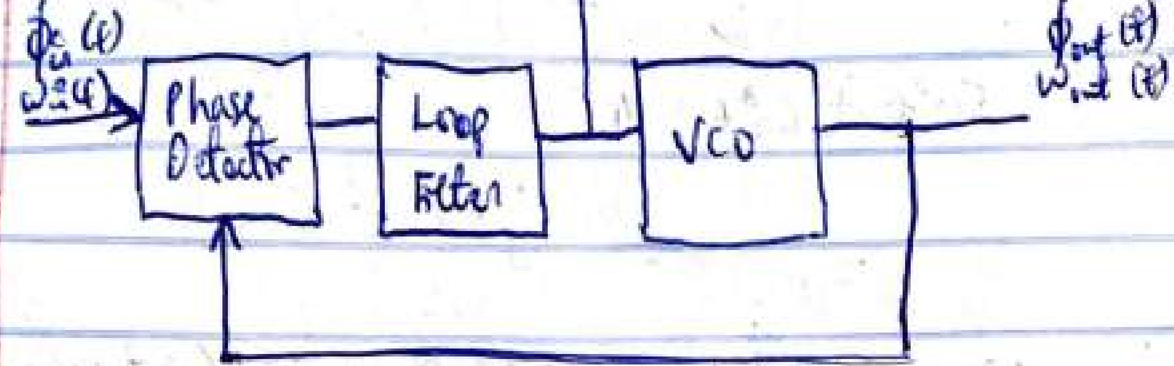
Phase Locked Loop

1) Intro



PLL output can be taken from either,

- V_{cont}
- VCO

depending upon the application.

* V_{cont} output provides a baseband output that tracks the phase variation at the input.
* VCO output can be used as a local oscillator or to generate a clock signal for a digital system.

(Either phase or frequency can be used as the input or output variables.)

Relation b/w phase and frequency =>

$$\omega(t) = \frac{d\phi}{dt}$$

$$\phi(t) = \phi(0) + \int_0^t \omega(t')\,dt'$$

3 applications for the PLL
- Clock generalization
- Frequency synthesizer
- Clock recovery in a serial data link.

(2) Phase detector

Compares the phase at each input and generates an error signal $V_e(t)$ proportional to the phase difference between the two inputs. K_D is the gain of the phase detector (V/rad)

$$V_e(t) = K_D [\phi_{out}(t) - \phi_{in}(t)]$$

An analog multiplier or mixer can be used as a phase detector.
Recall that the mixer takes the product of 2 inputs =>
$V_e(t) = A(t)B(t)$. If

$$A(t) = A\cos(\omega_1 t + \phi_A)$$
$$B(t) = B\cos(\omega_1 t + \phi_B)$$

Then $$A(t)B(t) = \frac{AB}{2}\left[\cos(\omega_1 t + \phi_A + \omega_1 t + \phi_B) + \cos(\omega_1 t + \phi_A - \omega_1 t - \phi_B)\right]$$

$$= \frac{AB}{2}\left[\cos(2\omega_1 t + \phi_A + \phi_B) + \cos(\phi_A - \phi_B)\right]$$

* The doubled freq component must be removed by the lowpass loop filter. Any phase difference then shows up as the control voltage to the VCO, a DC or slowly varying AC signal after filtering.

③ VCO

In PLL applications, VCO is treated as a linear, time-invariant system.
Excess phase of the VCO is the system output.

$$\phi_{out} = K_0 \int_{-\infty}^{t} V_{cont}\, dt'$$

The VCO oscillates at an angular frequency, ω_{out}.
- VCO oscillates at an angular frequency ω_{out}.
- Its freq is set to a nominal ω_0 when the control voltage is zero
- Frequency is assumed to be linearly proportional to the control voltage with a gain coefficient K_0 or K_{vco} (rad/s/V).

$$\omega_{out} = \omega_0 + K_0 V_{cont}$$

- Thus to obtain an arbitrary output freq (within the VCO tuning range of course) a finite V_{cont} is required.

(From the diagram)
- XOR function produces an output pulse whenever there is a phase misalignment
- Suppose output freq 'ω_1' is needed; a control voltage V_1 will be necessary to produce this output freq.
- The phase detector can produce this V_1 only by maintaining a phase offset ϕ_0 at its input.

- In order to minimize the required phase offset or error, the PLL loop gain $K_D K_O$, should be maximized since =>

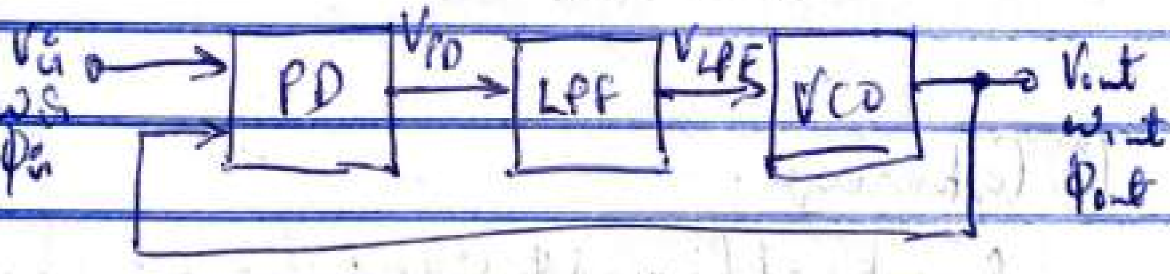
$$\phi_0 = \frac{V_1}{K_D} = \frac{\omega_1 - \omega_0}{K_D K_O}$$

* Thus, a high loop gain is beneficial for reducing phase errors.

④ **PLL dynamic response:**

=> $\phi_{in} = \omega_1 t + \phi_0 + \phi_1 u(t - t_1)$

We introduce a phase step at the input at $t = t_1$



* $V_{cont} = V_{LPF}$

= Frequency step

$\omega_2 = \omega_1 + \Delta\omega$

⑤ Lock Range

Range of input signal frequencies over which the loop remains locked once it has captured the input signal. This can be limited either by phase detector or the VCO freq range

$V_{e\text{-}max} = \pm K_D \pi/2$

When the phase detector output voltage is applied through the loop filter to the VCO.

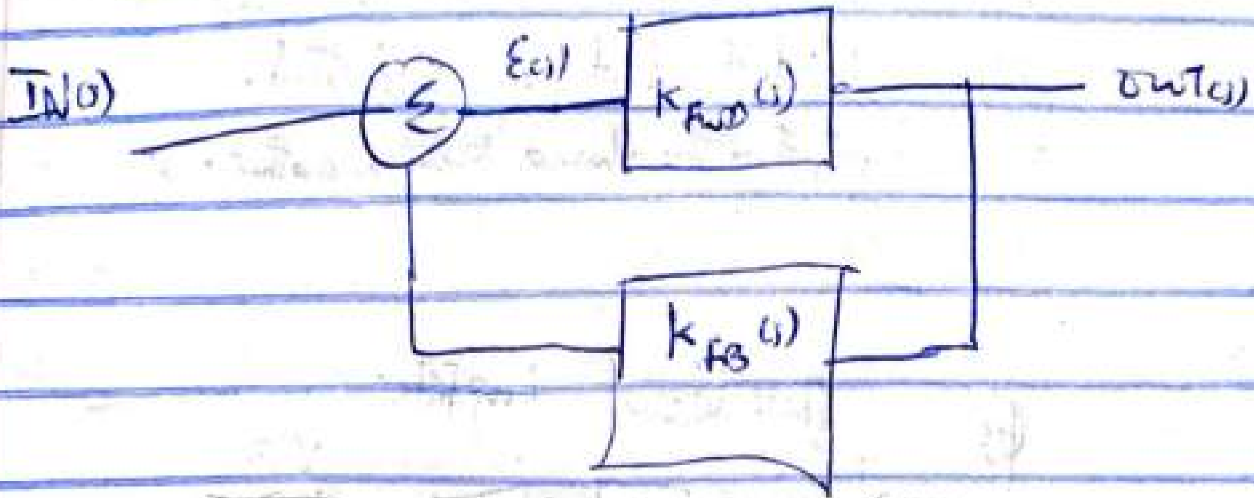
$$\Delta\omega_{out\_max} = \pm K_V \pi/2 = \omega_L \text{ (Lock range)}$$

where $K_V = K_0 K_D$

⑥ Capture range

Range of input freq around the VCO center freq onto which the loop will lock when starting from an unlocked condition.

PLL is a feedback System



Loop Gain: $T(s) = K_{FWD}(s)\, K_{FB}(s)$

Transfer function: $\dfrac{OUT(s)}{IN(s)} = H(s) = \dfrac{K_{FWD}(s)}{1 + K_{FWD}(s)\, K_{FB}(s)}$

$= \dfrac{K_{FWD}(s)}{1 + T(s)}$

The loop gain can be described as a polynomial:

$T(s) = \dfrac{K'(s+a)(s+b)\cdots}{s^n (s+\alpha)(s+\beta)\cdots}$

Phase Error $= \mathcal{E}(s) = \dfrac{IN(s)}{1 + T(s)}$

Steady State Error $= \mathcal{E}_{ss} = \lim_{s \to 0} [s\,\mathcal{E}(s)] = \lim_{t \to \infty} \mathcal{E}(t)$

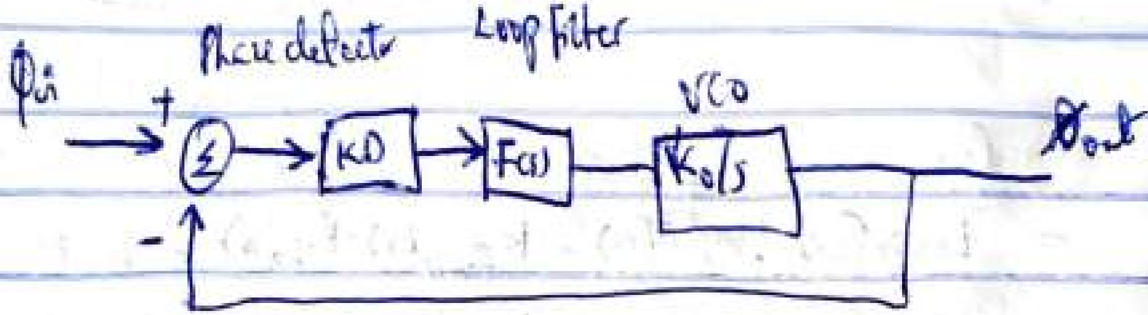
SS error is the error remaining in the loop after all transients have died out.

Frequency and phase tracking loop:

Let's consider PLL with feedback = 1

This means input & output freq are identical.

Input and output phase should track one another.



$$TF: H(s) = \frac{\text{forward path gain}}{[1+T(s)]}$$

$$H(s) = \frac{T(s)}{[1+T(s)]}$$

$$H(s) = \frac{\phi_{out}}{\phi_{in}} = \frac{K_D K_o F(s)/s}{1 + K_D K_o F(s)/s}$$

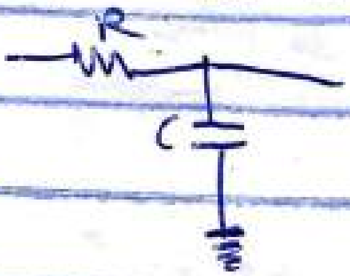
Phase error function:

$$E(s) = \phi_{in} - \phi_{out} = \frac{s\phi_{in}}{s + K_D K_o F(s)}$$

Open loop gain $T(s) \Rightarrow$

$$T(s) = K_D F(s) \, K_0/s$$

We know that the phase detector will be producing an output equal to w at twice the carrier frequency; thus some low pass filtering will be needed. We start with a simple RC lowpass network.



$V_{out} : V_i^{?}$ $\frac{R}{R + \frac{1}{sC}}$ $\frac{sRC}{sRC + 1}$ $\frac{1}{1 + \frac{1}{sRC}}$ $\frac{1}{\frac{s}{RC} + 1}$

This network has a cutoff (3dB) freq.

$$\omega_1 = 1/RC$$

$R + \frac{1}{sC}$

Thus the filter TF is a simple low pass,

$$F(s) = \frac{1}{1 + s/\omega_1}$$

Then, $T(s)$ becomes second order, type 1:

$$T(s) = \frac{K_0}{s} \frac{K_D}{1 + s/\omega_1} = \frac{K_0 K_D}{s(1 + s/\omega_1)} = \frac{K_V}{s(1 + \frac{s}{\omega_1})}$$

* From the bode plot, shows a conflict between stability and minimizing phase error.

Rootlocus

These represent the roots of the denominator of the closed loop T.F.

Set $1+G(s)=0$ and solve for $S(k)$ a function of K_v.

$$0 = 1 + \frac{K_v}{s\left(1+\frac{s}{\omega_1}\right)}$$

$$0 = 1 + \frac{K_v}{s+\frac{s^2}{\omega_1}}$$

$$0 = 1 + \frac{K_v}{\frac{\omega_1 s + s^2}{\omega_1}}$$

$$0 = 1 + \frac{K_v \omega_1}{\omega_1 s + s^2}$$

$$0 = \frac{\omega_1 s + s^2 + K_v \omega_1}{\omega_1 s + s^2}$$

$$\overset{a}{s^2} + \overset{b}{\omega_1 s} + \overset{c}{K_v \omega_1} = 0$$

$$s = \frac{-\omega_1 \pm \sqrt{\omega_1^2 - 4K_v\omega_1}}{2}$$

$$s = -\frac{\omega_1}{2} \pm \sqrt{\omega_1^2 - 4K_V\omega_1}$$

$$s = -\frac{\omega_1}{2} \pm \sqrt{\omega_1^2\left(1 - \frac{4K_V}{\omega_1}\right)}$$

$$s = -\frac{\omega_1}{2} \pm \omega_1\sqrt{1 - \frac{4K_V}{\omega_1}}$$

we see that as K_V is increased, the roots approach one another then become complex conjugates.

we can have an underdamped response when $\omega_1 << K_V$. Think about the inverse Laplace transform of the complex conjugate pole pair $e^{-\omega_1 t/2} \sin\left(\frac{\omega_1}{2}\sqrt{1 - \frac{4K_V}{\omega_1}}\right)t$

There is an exponentially decaying term determined by the real part of the root that shows how long it takes to settle after a phase or frequency step and a ringing freq dictated by the imaginary part of the pole pair.

when $\omega_1 << K_V$, we have a high ringing frequency and a long settling time, characteristic of a system that is not very useful.

It is sometimes useful to determine a natural freq ω_n and a damping factor ζ_n. This is standard terminology for 2nd order system.

The key is to put the denominator $\boxed{1+T(s)}$ in standard form:

either

$$s^2+2\zeta\omega_n s+\omega_n^2$$

or

$$\frac{s^2}{\omega_n^2}+\frac{2\zeta}{\omega_n}s+1$$

Picking the first $\Rightarrow$

$$1+T(s) = s^2+\omega_1 s+K_v\omega_1$$

$$= s^2+2\zeta\omega_n s+\omega_n^2$$

$$\omega_n=\sqrt{K_v\omega_1}$$

$$2\zeta\omega_n=\omega_1$$

$$2\zeta\sqrt{K_v\omega_1}=\omega_1$$

$$\zeta=\frac{1}{2}\frac{\omega_1}{\sqrt{K_v\omega_1}}$$

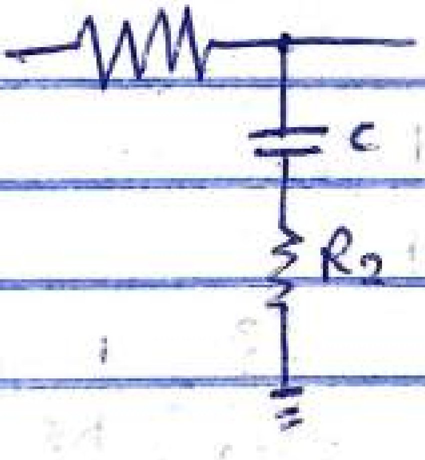
$$\zeta=\frac{1}{2}\sqrt{\frac{\omega_1^2}{K_v\omega_1}}=\frac{1}{2}\sqrt{\frac{\omega_1}{K_v}}$$

This form allows you to use standard eqn and normalized ~~plot~~ plots to describe the freq and transient response of the system.

A large K_v, which we like for reducing phase error, leads to a small ζ, which is bad for stability and settling time.

To obtain a better Transfer function that gives us more flexibility in determining the bandwidth of the filter cct and stability of the system, we add a zero to the loop filter T.F. to manipulate the root locus and ensure stability.

Adding a resistor lowpass loop filter contributes a zero to its transfer function



$$F(s) = \frac{1 + s/\omega_2}{1 + s/\omega_1}$$

where $\omega_1 = \frac{1}{(R_1+R_2)C}$

$$\omega_2 = \left(\frac{1}{R_2 C}\right)$$

Thus the zero frequency is always higher than the pole frequency.

Root Locus: Calculate the closed loop TF for this PLL with the pole zero loop filter

$$\frac{\phi_o}{\phi_i} = \frac{(1+s/\omega_2)}{\frac{s^2}{K_v\omega_1} + s\left(\frac{1}{K_v}+\frac{1}{\omega_2}\right)+1}$$

Extract ω_n and ζ from the closed loop TF.

$$\cancel{2\zeta\omega_n = \frac{1}{K_v}+\frac{1}{\omega_2}}$$

$$\omega_n^2 = K_v\omega_1$$

$$\omega_n = \sqrt{K_v\omega_1}$$

$$\frac{2\zeta}{\omega_n} = \frac{1}{K_v} + \frac{1}{\omega_2}$$

$$2\zeta = \frac{\sqrt{K_v\omega_1}}{K_v} + \frac{\sqrt{K_v\omega_1}}{\omega_2}$$

$$\zeta = \frac{1}{2}\sqrt{\frac{\omega_1}{K_v}} + \frac{1}{2}\frac{\omega_n}{\omega_2}$$

Solving for 's', the poles are

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

~~Frequency Response~~ DAY 2

Phase Error

- No frequency error when the loop is locked

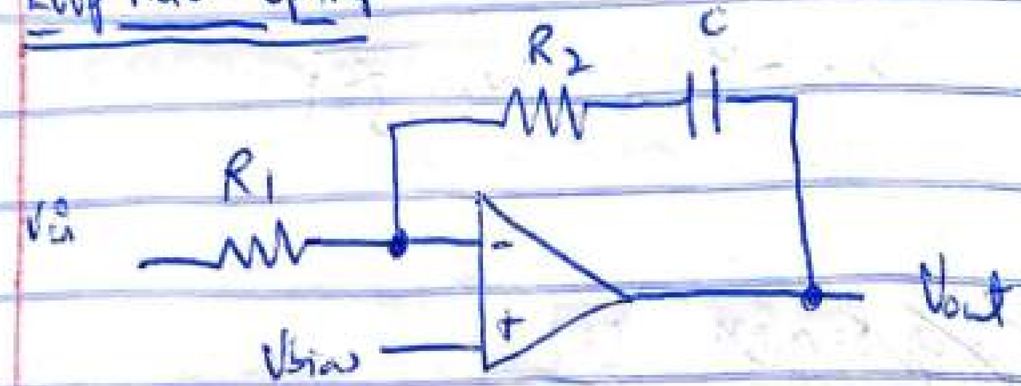
Input frequency = output frequency.

$$\text{Phase Error} = \varepsilon(s) = \frac{IN(s)}{1+T(s)}$$

$$\text{Steady state error} = \varepsilon_{ss} = \lim_{s \to 0} [s\,\varepsilon(s)] = \lim_{t \to \infty} \varepsilon(t)$$

Transient phase Error → Inverse Laplace transform of $\varepsilon(s)$.

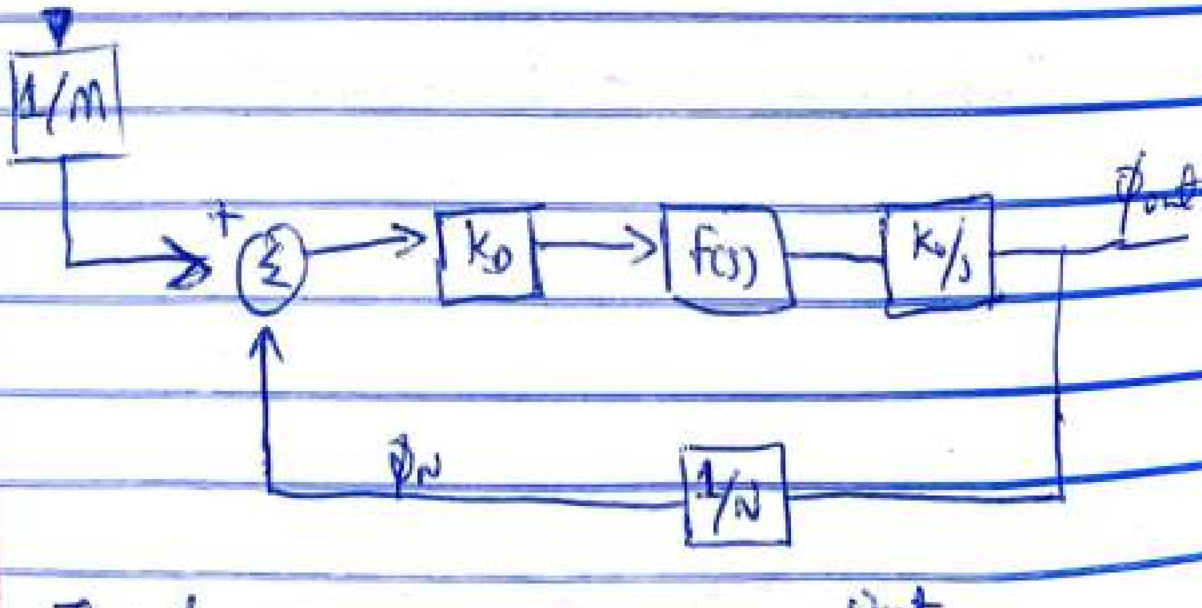
Loop Filter - OpAmp



Op-amp can be used to form a filter that includes a pole at $s=0$ and a finite zero. Virtual ground approximation can be used to analyze the circuit above to obtain $F(s)$.

$$F(s) = \frac{V_{out}}{V_{in}} = \frac{1 + sR_2C}{sR_1C}$$

Synthesizer PLL → add the divider 1/N to the feedback path.



Thus $\dfrac{\phi_N}{\phi_{out}} = \dfrac{1}{N}$ and also, $N = \dfrac{\omega_{out}}{\omega_{ref}}$

The openloop gain $T(s) = \frac{K_D K_o F(s)}{Ns}$

Loop gain is reduced by a factor of N.

$$F(s) = \frac{1 + s/\omega_2}{s/\omega_1}$$

where $\omega_1 = 1/R_1 C \quad \omega_2 = 1/R_2 C$

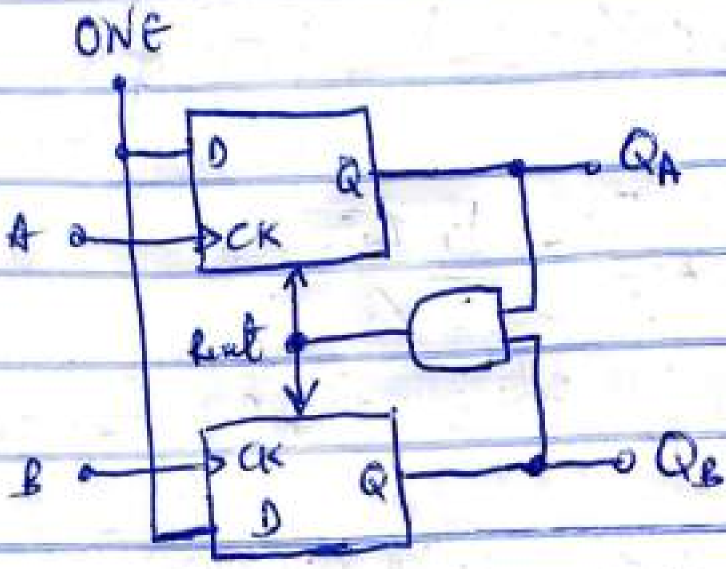
$$1 + T(s) = 1 + \frac{k_V}{Ns}\left(\frac{1 + s/\omega_2}{s/\omega_1}\right) = 0$$

$$1 + T(s) = \frac{Ns^2}{k_V \omega_1} + \frac{s}{\omega_2} + 1 = 0$$

$$\omega_n = \sqrt{\frac{k_V \omega_1}{N}} = \sqrt{\frac{k_V}{R_1 C N}}$$

$$\zeta = \frac{\omega_n}{2\omega_2} = \frac{R_2}{2}\sqrt{\frac{k_V C}{R_1 N}}$$

Phase frequency detector



Analog Multipliers

The emitter-coupled pair as a simple multiplier.

It functions only 2 quadrants so its called a two-quadrant multiplier

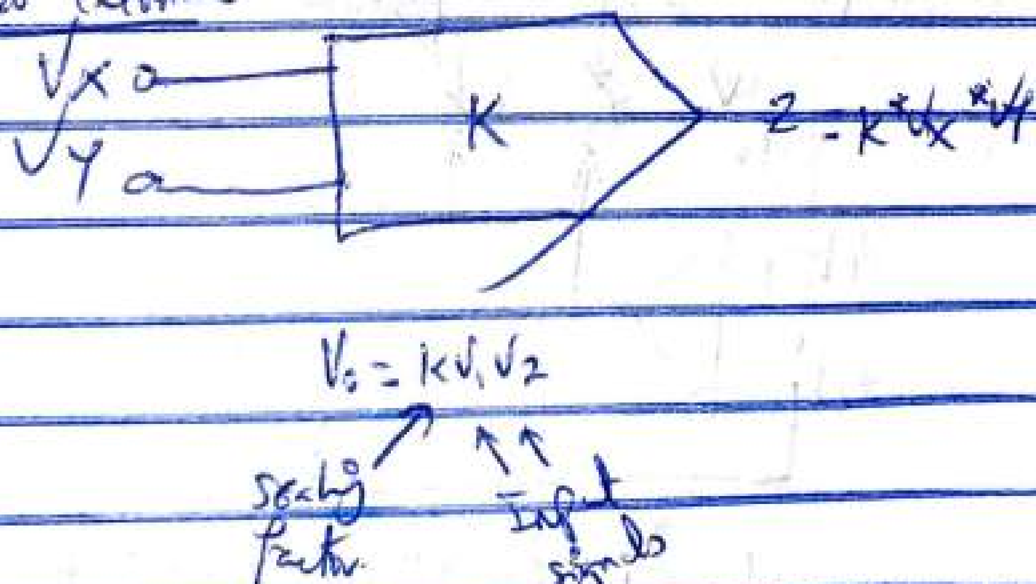
Gilbert multiplier cell allows 4-quadrant multiplication

For Gilbert Multiplier:

- If $V_1 < V_T$ and $V_2 < V_T$ then it works as a multiplier
- If one input signal is large compared to V_T, then it acts as a modulator
- If both inputs large: Non-saturating switches.

Gilbert cell as a balanced modulator

Video Tutorial



$$V_o = K V_1 V_2$$

Scaling factor

Input signals

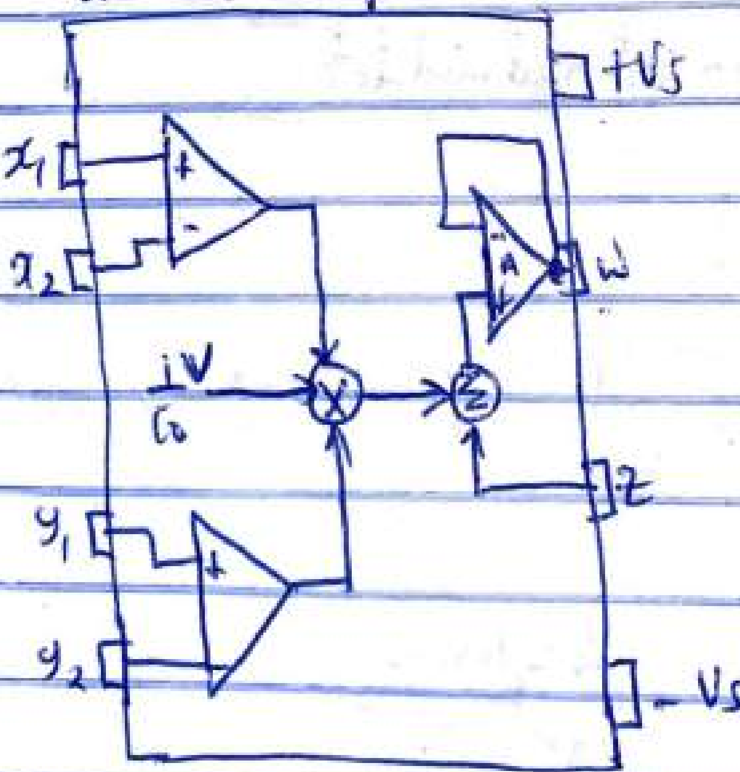
Output is proportional to product of 2 input signals

Type	V_x	V_y	V_o
Single Quadrant	Unipolar	Unipolar	Unipolar
Two Quadrant	Bipolar	Unipolar	Bipolar
	Unipolar	Bipolar	
4 Quadrant	Bipolar	Bipolar	Bipolar

Implementation: IC or discrete.

Eg AD633

low cost multiplier



Characteristics
- 4 quadrant
- FS = 10V (full scale)

$$V_o = \left[\frac{1}{10}(x_1 - x_2)(y_1 - y_2)\right] + Z$$

Important characteristics

- Accuracy = Total error (at Nominal Temperature T_{nom} and nominal voltage $V_{s,nom}$) – %FS
  - DC offset (V_o for $x = y = \phi$)
  - Nonlinearity

- Dynamic performance
  - Small-signal bandwidth
  - Slew rate (max rate of change in o/p voltage)
  - Settling time

Applications of analog multipliers

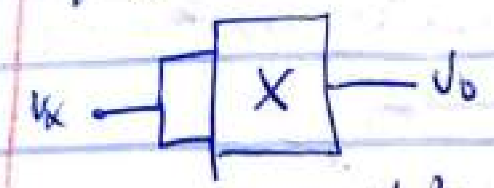
- Math operations
- Frequency doubling and shifting (i.e mixers)
- Mod/Demod.

<u>Math Op.</u>

<u>Multiplication</u>

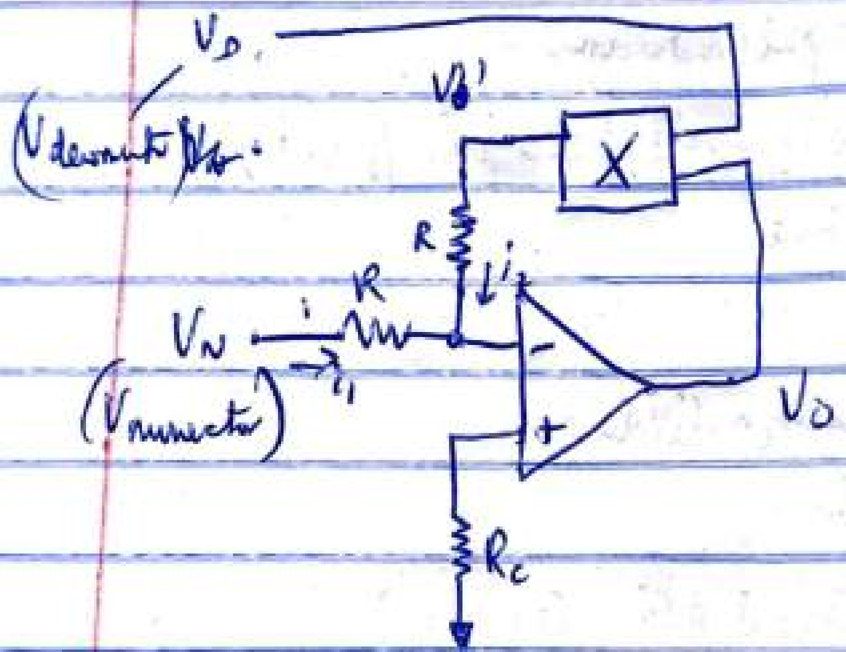
$$\underbrace{V_o = k \cdot V_x \cdot V_y}$$

Squaring



$$V_o = k . V_x^2$$

Division.



$$V_o' = k * V_o * V_D$$

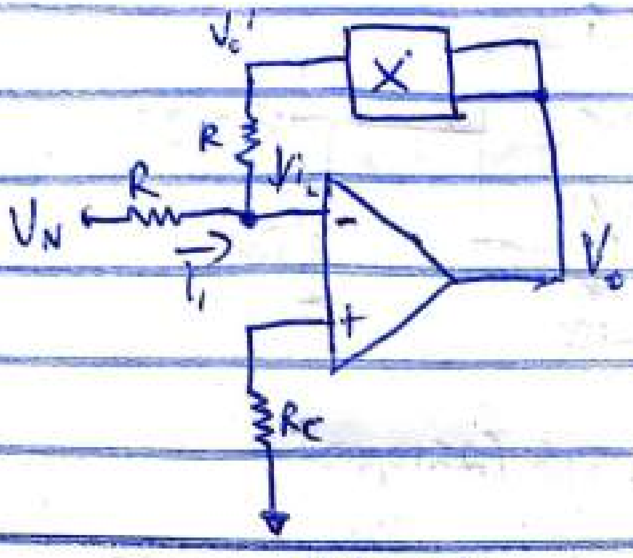
$$i_1 + i_2 = 0 \Rightarrow \frac{V_N}{R} + \frac{V_o'}{R} = 0$$

$$\Rightarrow V_N + V_o' = 0$$

$$-V_N = V_o'$$

$$V_o \Rightarrow V_o = \frac{V_o'}{k . V_D} = -\frac{1}{k} \frac{V_N}{V_D}$$

Square root



$V_o' = k \cdot V_o^2$

$i_1 + i_2 = 0$

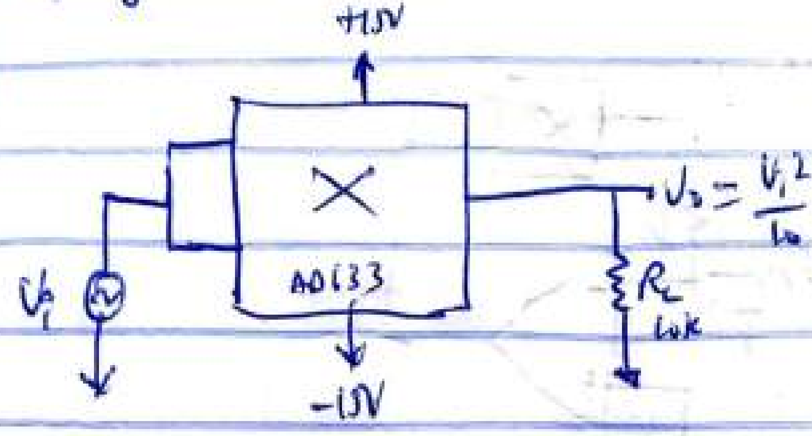
$\Rightarrow \frac{V_N}{R} + \frac{V_o'}{R} = 0$

$V_N + V_o' = 0$

$V_o' = -V_N$

$V_o = \sqrt{\frac{V_o'}{k}} = \sqrt{\frac{-V_N}{k}}$

Frequency doubler.



$$V_1 = 5\sin(2\pi 10{,}000 t)$$

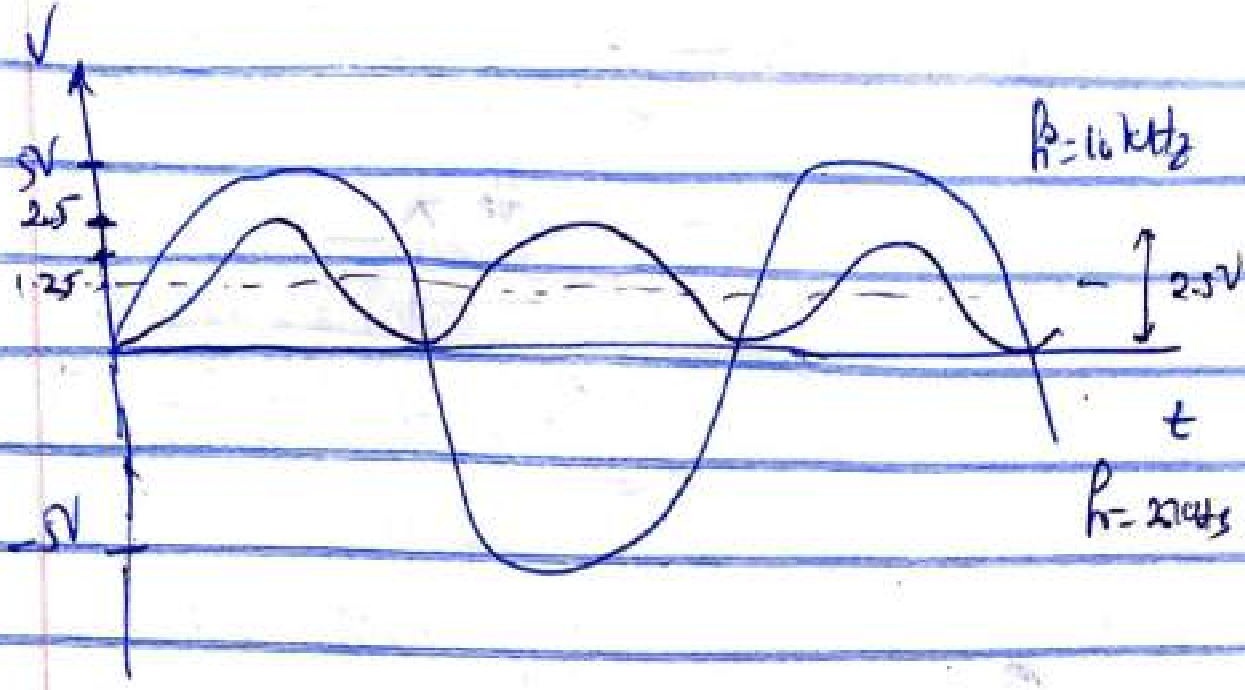
$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$V_o = \frac{V_1^2}{10}$$

$$= \frac{5^2}{10}\left[\sin(2\pi 10000 t)\right]^2$$

$$V_o = 2.5\left[\frac{1}{2}(\cos 0^\circ) - \frac{1}{2}\cos(\underbrace{2\pi 20000 t}_{\text{Twice the input frequency}})\right]$$

$$\boxed{V_o = \underbrace{\frac{2.5}{2}}_{\text{DC offset}} - \underbrace{\frac{2.5}{2}\cos(2\pi 20000 t)}_{\text{sinusoid of amplitude } 1.25V \text{ and } f = 20kHz}}$$



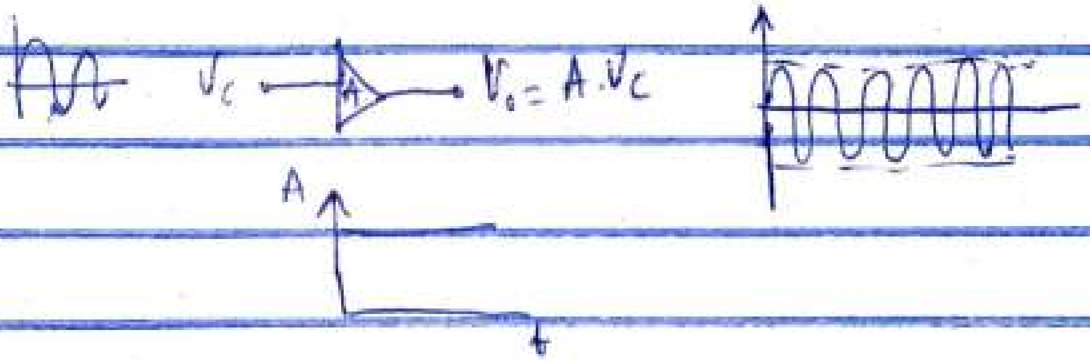
Modulators / Demodulators

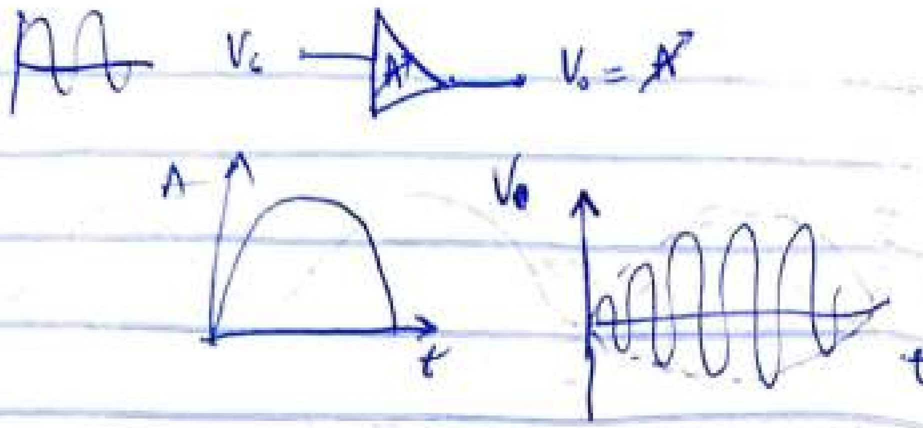
Modulators: Vary properties of a periodic signal (carrier) with an information signal (modulating signal) to facilitate transmission.

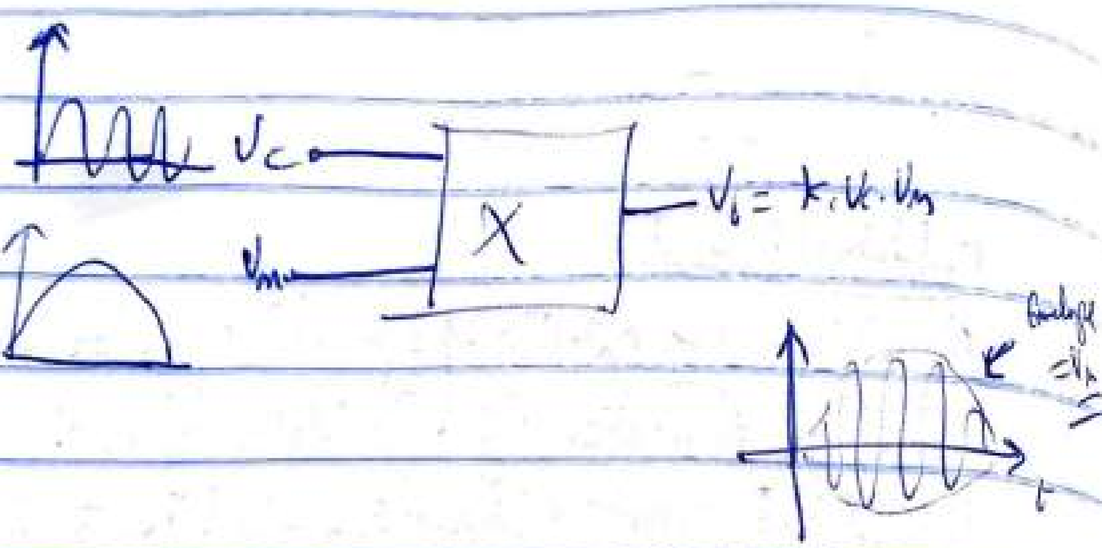
Demodulators: Extract the original info signal from the modulated signal.

Modulation schemes → AM, FM, PM.

Analog multiplier → Amplitude Modulation




Vc
A
Vo = A
A
t
Vo
t


Vc
Vm
X
Vo = K.Vc.Vm
Envelope = Vm
t