1. A **linear transformation** is a function from one vector space to another that respects the underlying (linear) structure of each vector space. A linear transformation is also known as a linear operator or map. The range of the transformation may be the same as the domain, and when that happens, the transformation is known as an endomorphism or, if invertible, an automorphism. The two vector spaces must have the same underlying field.

Eg1

The linear transformation from R3\mathbb{R}^3R3 to R2\mathbb{R}^2R2 defined by T(x, y, z)=(x−y, y−z)T(x,\,y,\,z) = (x - y,\, y - z)T(x,y,z)=(x−y,y−z) is given by the matrix

M=(1−1001−1).M = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.M=(10​−11​0−1​).

So, TTT can also be defined for vectors v=(v1, v2, v3)v = (v\_1, \, v\_2, \, v\_3)v=(v1​,v2​,v3​) by the matrix product

T(v)=(1−1001−1)(v1v2v3).T(v) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} v\_1 \\ v\_2 \\ v\_3 \end{pmatrix}.T(v)=(10​−11​0−1​)⎝⎛​v1​v2​v3​​⎠⎞​.

Note that the dimension of the initial vector space is the number of columns in the matrix, while the dimension of the target vector space is the number of rows in the matrix.

Eg2

Is the linear transformation T(x, y, z)=(x−y, y−z)T(x,\,y,\,z) = (x - y,\, y - z)T(x,y,z)=(x−y,y−z), from R3\mathbb{R}^3R3 to R2\mathbb{R}^2R2, injective? Is it surjective?

For a vector v=(v1, v2, v3)v = (v\_1,\,v\_2,\,v\_3)v=(v1​,v2​,v3​), this can be written as

T(v)=Mv=(1−1001−1)(v1v2v3).T(v) = M v = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} v\_1 \\ v\_2 \\ v\_3 \end{pmatrix}.T(v)=Mv=(10​−11​0−1​)⎝⎛​v1​v2​v3​​⎠⎞​.

MMM is a 2×32 \times 32×3 matrix, so **it is surjective** because the minor (1−101)\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}(10​−11​) has determinant 111 and therefore is invertible (since the determinant is nonzero). However, there are no 3×33 \times 33×3 minors, so **it is not injective**. □\_\square□​

1. Rank of a matrix A is the order of the largest square matrix or sub matrix of A whose determinant is not equal to 0
E.g

Find the rank of

3 1 2

2 0 5

5 1 7

A=

|A|=3(0-5)-1(14-25)+2(2-0)

|A|=-15+11+4=0

Hence, we need to consider a sub matrix of A

3 1

2 0

Asub =

|Asub|=0-2=-2

Hence, the rank of A is -2