1. $\int \frac{3x-1}{(x-1)(x-2)(x-3)}\,dx$

2. First,

Partial fraction decomposition

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Multiply both sides $(x-1)(x-2)(x-3)$

$$3x-1 = (x-2)(x-3)A + B(x-1)(x-3) + C(x-1)(x-2)$$

$$3x-1 = Ax^2 - 5A + 6A + Bx^2 - 4Bx + 3B + Cx^2 - 3C + 2C$$

$$3x-1 = Ax^2 + Bx^2 + Cx^2 - 5Ax - 4Bx - 3Cx + 6A + 3B + C$$

$$3x-1 = (A+B+C)x^2 + (-5A-4B-3C)x + (6A+3B+2C)$$

$-1 = 6A + 3B + 2C$ ... (1)

$3 = -5A - 4B - 3C$ ... (2)

$0 = A + B + C$ (3)

In eqn (3)

$A = -B - C$

Sub in eqn (1) and (2)

$6(-B-C) + 3B + 2C = -1$

$-5(-B-C) - 4B - 3C = 3$

$-3B - 4C = -1$ ... (4)

$B + 2C = 3$ (5)

$B = 3 - 2c$

Sub B in eqn 4

$-3(3 - 2c) - 4c = -1$

$-9 + 6c - 4c = -1$

$-9 + 2c = -1$

$c = 4$

Sub c in eqn 4

$-3B - 16 = -1$

$B = -5$

Sub B and c in eqn (1)

$-1 = 6A + -15 + 8$

$-1 = 6A - 15 + 8$

$6A = 7 - 1$

$A = 1$

$A, B, C = 1, -5, 4$

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{(x-2)} + \frac{4}{(x-3)}$$

$$\int \frac{3x-1}{(x-1)(x-2)(x-3)}\,dx = \int \frac{1}{x-1}\,dx - \int \frac{5}{x-2}\,dx + \int \frac{4}{x-3}$$

$= \ln(|x-1|) - 5\ln(|x-2|) + 4\ln(|x-3|)$

$= \ln(|x-1|) - 5\ln(|x-2|) + 4\ln(|x-3|) + c$

7. $\int \frac{x^2+x+1}{(x+2)(x^2+1)}\,dx$

Using partial fraction decomposition

$$\frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$$x^2+x+1 = (x^2+1)A + (x+2)(Bx+C)$$

$$x^2+x+1 = Ax^2 + A + Bx^2 + Cx + 2Bx + 2C$$

$$x^2+x+1 = Ax^2 + Bx^2 + Cx + 2Bx + A + 2C$$

$$x^2+x+1 = (A+B)x^2 + (C+2B)x + (A+2C)$$

$$1 = A + 2C$$
$$1 = C + 2B$$
$$1 = A + B$$

$$(A, B, C) = \left(\frac{3}{5}\right), \left(\frac{2}{5}\right), \left(\frac{1}{5}\right)$$

$$\therefore \frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$$= \frac{\frac{3}{5}}{x+2} + \frac{\frac{2}{5}x + \frac{1}{5}}{x^2+1}$$

$$= \frac{3}{5(x+2)} + \frac{2x+1}{5(x^2+1)}$$

$$= \int \frac{3}{5(x+2)}\,dx + \int \frac{2x+1}{5(x^2+1)}\,dx$$

$$= \frac{3}{5} \times \ln(|x+2|) + \frac{1}{5} \times \ln(x^2+1) + \frac{\tan^{-1}x}{5}$$

$$= 3/5 \times \ln(|x+2|) + 1/5 \times \ln(x^2+1) + \frac{\tan^{-1}(x)}{5} + C$$

$$\int \frac{x^2+1}{(x-3)(x-2)^2}\,dx$$

$$\frac{x^2+1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$x^2+1 = (x-2)^2 \times A + (x-3) \times (x-2)B + (x-3)C$$

$$x^2+1 = Ax^2 - 4Ax + 4A + Bx^2 - 5Bx + 6B + Cx - 3C$$

$$= Ax^2 + Bx^2 - 4Ax - 5Bx + Cx + 4A + 6B - 3C$$

$$x^2+1 = (A+B)x^2 + (-4A - 5B + C)x + (4A + 6B - 3C)$$

$$1 = 4A + 6B - 3C$$
$$0 = -4A - 5B + C$$
$$1 = A + B$$

$$(A, B, C) = (10, -9, -5)$$

$$\frac{x^2+1}{(x-3)(x-2)^2} = \frac{10}{x-3} - \frac{9}{x-2} - \frac{5}{(x-2)^2}$$

$$\therefore \int \frac{10}{x-3}\,dx - \int \frac{9}{x-2}\,dx - \int \frac{5}{(x-2)^2}\,dx$$

$$10\ln(|x-3|) - 9\ln(|x-2|) + \frac{5}{x-2}$$

$$= 10\ln(|x-3|) - 9\ln(|x-2|) + \frac{5}{x-2} + C$$

4) $\int \frac{x^3 + x^2 + x + 1}{x-1}$

Separate the fractions

$$\int \frac{x^3}{x-1} + \frac{x^2}{x-1} + \frac{x}{x-1} + \frac{1}{x-1}\,dx$$

$$\int \frac{x^3}{x-1}\,dx + \int \frac{x^2}{x-1}\,dx + \int \frac{x}{x-1}\,dx + \int \frac{1}{x-1}\,dx$$

$$\cancel{\frac{2x^3 + 3x^2 + 6x - 11}{6}} + \ln(|x-1|) + \cancel{\frac{x^2}{2} + x}$$

$$\frac{2x^3 + 3x^2 + 6x - 11}{6} + \ln(|x-1|) + \frac{x^2}{2} + x + \ln(|x-1|)$$

$$+ x - 1 + \ln(|x-1|) + \ln(|x-1|) + C$$

$$= \frac{2x^3 + 3x^2 + 6x - 11}{6} + 4\ln(|x-1|) + C$$