

$$1. \int \frac{3x-1}{(x-1)(x-2)(x-3)} dx = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

The denominators cancel out

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$$3x-1 = A(x^2-5x+6) + B(x^2-4x+3) + C(x^2-3x+2)$$

$$3x-1 = Ax^2 - 5Ax + 6A + Bx^2 - 4Bx + 3B + Cx^2 - 3Cx + 2C$$

Collect like terms

$$3x-1 = x^2(A+B+C) + x(-5A-4B-3C) + 6A+3B+2C$$

Compare the LHS with RHS

$$A+B+C=0 \quad \text{--- (1)}$$

$$-5A-4B-3C=3 \quad \text{--- (2)}$$

$$6A+3B+2C=-1 \quad \text{--- (3)}$$

$$\text{From eqn (1) } A = -B - C \quad \text{--- (4)}$$

Put eqn (4) in (2) and (3)

$$-5(-B-C) - 4B - 3C = 3$$

$$+5B + 5C - 4B - 3C = 3$$

$$+5A + (B+2C) = 3 \quad \text{--- (5)}$$

$$6(-B-C) + 3B + 2C = -1$$

$$-6B - 6C + 3B + 2C = -1$$

$$-3B - 4C = -1 \quad \text{--- (6)}$$

$$3B + 6C = 9$$

$$-3B - 4C = -1$$

$$2C = 8$$

$$C = 4$$

Also from eqn (5)

$$B + 2(4) = 3$$

$$B + 8 = 3$$

$$B = -5$$

From eqn (1) $A = -(C-5) = -4$

Hence

$$\int \frac{3x-1}{(x-1)(x-2)(x-3)} dx = \int \frac{1}{x-1} dx - \int \frac{5}{x-2} dx + \int \frac{4}{x-3} dx$$

$$= \ln|x-1| - 5\ln|x-2| + 4\ln|x-3| + C$$

2. $\int \frac{(x^2 + x + 1)}{(x+2)(x^2+1)} dx = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$

$$\frac{x^2 + x + 1}{(x+2)(x^2+1)} = \frac{A(x^2+1) + (Bx+C)(x+2)}{(x+2)(x^2+1)}$$

$$x^2 + x + 1 = A(x^2+1) + (Bx+C)(x+2)$$

$$x^2 + x + 1 = Ax^2 + A + Bx^2 + 2Bx + Cx + 2C$$

Collect like terms

$$x^2 + x + 1 = x^2(A+B) + x(2B+C) + A + 2C$$

Compare the L.H.S. with R.H.S

$$A + B = 1 \quad \text{--- (1)}$$

$$2B + C = 1 \quad \text{--- (2)}$$

$$A + 2C = 1 \quad \text{--- (3)}$$

Solve the equation

$$A = \frac{3}{5}, \quad B = \frac{2}{5}, \quad C = \frac{1}{5}$$

$$\int \frac{x^2 + x + 1}{(x+2)(x^2+1)} dx = \int \frac{3}{5(x+2)} dx + \int \frac{2x+1}{5(x^2+1)}$$

$$= \frac{3}{5} \int \frac{1}{x+2} dx + \int \frac{2x+1}{5(x^2+1)}$$

$$= \frac{3 \ln|x+2|}{5} + \frac{\ln|x^2+1| + \tan^{-1}x}{5} + C$$

$$3. \int \frac{x^2+1}{(x-3)(x-2)^2} dx = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$\frac{x^2+1}{(x-3)(x-2)^2} = \frac{A(x-2)^2 + B(x-3)(x-2) + C(x-3)}{(x-3)(x-2)^2}$$

$$x^2+1 = A(x-2)^2 + B(x-3)(x-2) + C(x-3)$$

$$x^2+1 = Ax^2 - 4Ax + 4A + Bx^2 + 5Bx + 6B + Cx - 3C$$

Collect like terms

$$x^2+1 = x^2(A+B) + x(-4A-5B+C) + 4A+6B-3C$$

Compare the L.H.S with R.H.S

$$A+B=1 \quad \text{--- (1)}$$

$$-4A-5B+C=0 \quad \text{--- (2)}$$

$$4A+6B-3C=1 \quad \text{--- (3)}$$

$$\text{From eqn (1)} \quad A=1-B \quad \text{--- (4)}$$

Put eqn (4) into 2 and 3

$$-4(1-B)-5B+C=0$$

$$-4 + 4B - 5B + C = 0$$

$$-4 - B + C = 0$$

$$-B + C = 4 \quad \text{--- (5)}$$

$$4(1-B) + 6B - 3C = 1$$

$$4 - 4B + 6B - 3C = 1$$

$$4 + 2B - 3C = 1$$

$$2B - 3C = -3 \quad \text{--- (6)}$$

$$-2B + 2C = 8$$

$$2B - 3C = -3$$

$$\hline -C = 5$$

Also for eqn (5)

$$-B + (C-5) = 4$$

$$-B - 5 = 4$$

$$-B = 9$$

$$B = -9$$

Also for eqn (1)

$$A = 1 - (C-9)$$

$$A = 1 - (-5) = 10$$

Hence

$$\int \frac{x^2 + 1}{(x-3)(x-2)^2} dx = \int \frac{10}{x-3} dx - \int \frac{9}{x-2} dx + \int \frac{5}{(x-2)^2} dx$$

$$= 10 \ln|x-3| - 9 \ln|x-2| + \frac{5}{x-2} + A + C$$

$$4 \int \frac{x^3 + x^2 + x + 1}{x-1} dx$$

$$\begin{array}{r}
 x^2 + 2x + 3 \\
 x-1 \overline{) x^3 + x^2 + x + 1} \\
 \underline{-x^3 - x^2} \\
 2x^2 + x + 1 \\
 \underline{-2x^2 - 2x} \\
 3x + 1 \\
 \underline{-3x - 3} \\
 4
 \end{array}$$

$$\frac{x^3 + x^2 + x + 1}{x-1} = x^2 + 2x + 3 + \frac{4}{x-1}$$

$$\begin{aligned}
 \int \frac{x^3 + x^2 + x + 1}{x-1} dx &= \int \left(x^2 + 2x + 3 + \frac{4}{x-1} \right) dx \\
 &= \frac{x^3}{3} + x^2 + 3x + 4 \ln|x-1| + C
 \end{aligned}$$