A Linear transformation ($T: U \to V$) is a function that carries elements of vectors space U (Domain) to vector space V (Co-domain) and which has two properties;

(i) $T(u_1 + u_2) = T(u_1) + T(u_2) \ \forall \ u_1, u_2 \in U$.

(ii) $T(\alpha u) = \alpha T(u) \ \forall \ u \in U$ and $\alpha \in F$ where F is a scalar field

NB: As every vector space property derived from vector addition & scalar multiplication so to every property of linear transformation derives from any of those two properties. Below are two diagrams that convey the essence of the two defining properties of a linear transformation.

$$
\begin{tikzcd}
u_1, u_2 \arrow[r, "T"] \arrow[d, "+"] & \arrow[d, "+"] \\
u_1 + u_2 \arrow[r] & T(u_1) + T(u_2)
\end{tikzcd}
$$

Fig 1: Definition of LT, Additive.

$$
\begin{tikzcd}
u \arrow[r, "T"] \arrow[d, "\alpha"] & T(u) \arrow[d, "\alpha"] \\
\alpha u \arrow[r, "T"] & T(\alpha u) = \alpha T(u)
\end{tikzcd}
$$

NB: T is the name of the linear transformation and should be used when we want to discuss the function as a whole.
$T(u)$ is how we talk about the output of the function of vector

Example

Given the matrix $B = \begin{pmatrix} 4 & 5 & -3 \\ 2 & 5 & 1 \\ 3 & 2 & -1 \end{pmatrix}$, transform the vector $\begin{pmatrix} 9 \\ -2 \\ 0 \end{pmatrix}$

Sol

$T(x) = Bx$

$T(x) = \begin{pmatrix} 4 & 5 & -3 \\ 2 & 5 & 1 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ -2 \\ 0 \end{pmatrix}$

$T(x) = 9\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} + 0\begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$

$T(x) = \begin{pmatrix} 36 \\ 18 \\ 27 \end{pmatrix} + \begin{pmatrix} -10 \\ -10 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$T(x) = \begin{pmatrix} 26 \\ 8 \\ 23 \end{pmatrix}$. Hence, the transformation of $\begin{pmatrix} 9 \\ -2 \\ 0 \end{pmatrix}$ gives $\begin{pmatrix} 26 \\ 8 \\ 23 \end{pmatrix}$.

Example

Given that matrix $A = \begin{pmatrix} 2 & -3 & 5 \\ -1 & 4 & 1 \\ 6 & 8 & 2 \end{pmatrix}$

transform the following vectors (i) $\begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$

Sol

(i) $Tx = Ax$.

$Tx = \begin{pmatrix} 2 & -3 & 5 \\ -1 & 4 & 1 \\ 6 & 8 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}$

$Tx = 4\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} + 5\begin{pmatrix} -3 \\ 4 \\ 8 \end{pmatrix} - 1\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ -4 \\ 24 \end{pmatrix} + \begin{pmatrix} -15 \\ 20 \\ 40 \end{pmatrix} + \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} -12 \\ 15 \\ 62 \end{pmatrix}$. Hence, the transformation of $\begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}$ gives $\begin{pmatrix} -12 \\ 15 \\ 62 \end{pmatrix}$

(ii) $Tx = Ax$

$Tx = \begin{pmatrix} 2 & -3 & 5 \\ -1 & 4 & 1 \\ 6 & 8 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

$Tx = 2\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} - 1\begin{pmatrix} -3 \\ 4 \\ 8 \end{pmatrix} + 4\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ -2 \\ 12 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \\ -8 \end{pmatrix} + \begin{pmatrix} 20 \\ 4 \\ 8 \end{pmatrix}$

$= \begin{pmatrix} 27 \\ -2 \\ 12 \end{pmatrix}$. Hence, the transformation of $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ gives $\begin{pmatrix} 27 \\ -2 \\ 12 \end{pmatrix}$

(iii) $Tx = Ax$

$Tx = \begin{pmatrix} 2 & -3 & 5 \\ -1 & 4 & 1 \\ 6 & 8 & 2 \end{pmatrix} \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix}$

$Tx = 8\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} + 3\begin{pmatrix} -3 \\ 4 \\ 8 \end{pmatrix} + 1\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 16 \\ -8 \\ 48 \end{pmatrix} + \begin{pmatrix} -9 \\ 12 \\ 24 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 12 \\ 5 \\ 74 \end{pmatrix}$. Hence, the transformation of $\begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix}$ gives $\begin{pmatrix} 12 \\ 5 \\ 74 \end{pmatrix}$

(iv) $Tx = Ax$

$Tx = \begin{pmatrix} 2 & -3 & 5 \\ -1 & 4 & 1 \\ 6 & 8 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$

$Tx = 6\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} - 3\begin{pmatrix} -3 \\ 4 \\ 8 \end{pmatrix} + 2\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 12 \\ -6 \\ 36 \end{pmatrix} + \begin{pmatrix} 9 \\ -12 \\ -24 \end{pmatrix} + \begin{pmatrix} 10 \\ 2 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 31 \\ -16 \\ 16 \end{pmatrix}$. Hence the transformation of $\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$ gives $\begin{pmatrix} 31 \\ -16 \\ 16 \end{pmatrix}$