MATRIC NO:18/SCI01/068

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1)

Definition

A *linear transformation* is a transformation T:Rn→Rm satisfying

T(u+v)=T(u)+T(v)T(cu)=cT(u)

for all vectors u,v in Rn and all scalars c.

Facts about linear transformations

Let T:Rn→Rm be a linear transformation. Then:

T(0)=0.

For any vectors v1,v2,...,vk in Rn and scalars c1,c2,...,ck, we have

TAc1v1+c2v2+⋯+ckvkB=c1T(v1)+c2T(v2)+⋯+ckT(vk).

Examples

i)

Define T:R→R by T(x)=x+1. Is T a linear transformation?

Solution

We have T(0)=0+1=1. Since any linear transformation necessarily takes zero to zero by the above important note, we conclude that T is *not* linear (even though its graph is a line).

Note: in this case, it was not necessary to check explicitly that T does not satisfy both defining properties: since T(0)=0 is a consequence of these properties, at least one of them must not be satisfied. (In fact, this T satisfies neither.)

<u>ii)</u>

Define T:R2→R2 by T(x)=1.5x. Verify that T is linear.

Solution

We have to check the defining properties for *all* vectors u,v and *all* scalars c. In other words, we have to treat u,v, and c as *unknowns*. The only thing we are allowed to use is the definition of T.

T(u+v)=1.5(u+v)=1.5u+1.5v=T(u)+T(v)

Since T satisfies both defining properties, T is linear.

Note: we know from this example in Section 3.1 that T is a matrix transformation: in fact,

$\mathrm{T}(\mathrm{x}) = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}\mathrm{x}.$

Since a matrix transformation is a linear transformation, this is another proof that T is linear.

<u>iii)</u>

Define T:R2→R3 by the formula

$\mathrm{T}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 3x-y \\ y \\ x \end{pmatrix}.$

Verify that T is linear.

Solution

We have to check the defining properties for *all* vectors u,v and *all* scalars c. In other words, we have to treat u,v, and c as *unknowns*; the only thing we are allowed to use is the definition of T. Since T is defined in terms of the

coordinates of u,v, we need to give those names as well;
say u=$\begin{pmatrix} x1 \\ y2 \end{pmatrix}$ and v=$\begin{pmatrix} x2 \\ y2 \end{pmatrix}$. For the first property, we have

$$\mathrm{T}\left(\begin{pmatrix} \mathrm{x1} \\ \mathrm{y1} \end{pmatrix}\right) + \left(\begin{pmatrix} x2 \\ y2 \end{pmatrix}\right) = \mathrm{T}\begin{pmatrix} x1 + x2 \\ y1 + y2 \end{pmatrix} = \begin{pmatrix} 3(x1 + x2) - (y1 + y2) \\ y1 + y2 \\ x1 + x2 \end{pmatrix}$$

$$= \begin{pmatrix} (3x1 - y1) + (3x2 - y2) \\ y1 + y2 \\ x1 + x2 \end{pmatrix}$$

$$= \begin{pmatrix} 3x1 - y1 \\ y1 \\ \mathrm{x1} \end{pmatrix} + \begin{pmatrix} 3x2 - y2 \\ y2 \\ x2 \end{pmatrix} = \mathrm{T}\begin{pmatrix} x1 \\ y1 \end{pmatrix} + \mathrm{T}\begin{pmatrix} x2 \\ y2 \end{pmatrix}$$

For the second property,

$$\mathrm{T}\left(\mathrm{c}\begin{pmatrix} \mathrm{x1} \\ \mathrm{y1} \end{pmatrix}\right) = \mathrm{T}\begin{pmatrix} cx1 \\ cy1 \end{pmatrix} = \begin{pmatrix} 3(cx1) - (cy1) \\ cy1 \\ cx1 \end{pmatrix}$$

$$= \begin{pmatrix} c(3x1 - y1) \\ cy1 \\ cx1 \end{pmatrix} = \mathrm{c}\begin{pmatrix} 3x1 - y1 \\ y1 \\ x1 \end{pmatrix} = \mathrm{cT}\begin{pmatrix} x1 \\ y1 \end{pmatrix}$$

Since T satisfies the defining properties, T is a linear transformation.

Note: we will see in this example below that

$$\mathrm{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Hence T is in fact a matrix transformation.

<u>**iv)**</u>

Verify that the following transformations from R2 to R2 are not linear:

$\mathrm{T1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} |x| \\ y \end{pmatrix}$ $\mathrm{T2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ y \end{pmatrix}$ $\mathrm{T3}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+1 \\ x-2y \end{pmatrix}$

Solution

In order to verify that a transformation T is *not* linear, we have to show that T does not satisfy *at least one* of the two defining properties. For the first, the negation of the statement "T(u+v)=T(u)+T(v) for all vectors u,v" is "there exists at least one pair of vectors u,v such that T(u+v)A=T(u)+T(v)." In other words, it suffices to find *one example* of a pair of vectors u,v such that T(u+v)A=T(u)+T(v). Likewise, for the second, the negation of the statement "T(cu)=cT(u) for all vectors u and all scalars c" is "there exists some vector u and some scalar c such that T(cu)A=cT(u)." In other words, it suffices to find *one* vector u and *one* scalar c such that T(cu)A=cT(u).

For the first transformation, we note that

$$\mathrm{T1}\left(-\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) = \mathrm{T1}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} |-1| \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

but that

$$-\mathrm{T1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} |1| \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Therefore, this transformation does not satisfy the second property.

For the second transformation, we note that

$$-\mathrm{T2}\left(2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \mathrm{T2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

but that

$$2\mathrm{T2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1.1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Therefore, this transformation does not satisfy the second property.

For the third transformation, we observe that

$$\mathrm{T3}\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}2(0)+1\\0-2(0)\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}\neq\begin{pmatrix}0\\0\end{pmatrix}$$

Since T3 does not take the zero vector to the zero vector, it cannot be linear.

v)

Define T:C3→C2T:C3→C2 by describing the output of the function for a generic input with the

$$T\left(\begin{bmatrix}x1\\x2\\x3\end{bmatrix}\right)=\begin{bmatrix}2\mathrm{x}1+\mathrm{x}3\\-4\mathrm{x}2\end{bmatrix}$$

and check the two defining properties.

$$\mathrm{T}(\mathrm{x}+\mathrm{y})=\mathrm{T}\left(\begin{bmatrix}x1\\x2\\x3\end{bmatrix}+\begin{bmatrix}y1\\y2\\y3\end{bmatrix}\right)$$

$$\mathrm{T}=\left(\begin{bmatrix}x1+y1\\x2+y2\\x3+y3\end{bmatrix}\right)$$

$$\begin{bmatrix}2(\mathrm{x}1+\mathrm{y}1)+(\mathrm{x}3+\mathrm{y}3)\\-4(\mathrm{x}2+\mathrm{y}2)\end{bmatrix}$$

$$\begin{bmatrix}(2\mathrm{x}1+\mathrm{x}3)+(2\mathrm{y}1+\mathrm{y}3)\\-4\mathrm{x}2+(-4)\mathrm{y}2\end{bmatrix}$$

$$\begin{bmatrix}2\mathrm{x}1+\mathrm{x}3\\-4\mathrm{x}2\end{bmatrix}+\begin{bmatrix}2\mathrm{y}1+\mathrm{y}3\\-4\mathrm{y}2\end{bmatrix}$$

$$\mathrm{T}\left(\begin{bmatrix}x1\\x2\\x3\end{bmatrix}\right)+\mathrm{T}\left(\begin{bmatrix}y1\\y2\\y3\end{bmatrix}\right)$$

T(x)+T(y)

And

$$T(\alpha x) = T\left(\alpha \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2(\alpha x1) + (\alpha x3) \\ -4(\alpha x2) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(2x1 + x3) \\ \alpha(-4x2) \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 2x1 + x3 \\ -4x2 \end{bmatrix}$$

$$= \alpha T\left(\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}\right)$$

$$= \alpha T(x)$$

T is a linear transformation.