MATRIC NO:18/SCI01/068

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1)

**Definition**

A *linear transformation* is a transformation $T:Rn \to Rm$ satisfying

$T(u+v)=T(u)+T(v)T(cu)=cT(u)$

for all vectors u,v in Rn and all scalars c.

**Facts about linear transformations**

*Let* $T:Rn \to Rm$ *be a linear transformation. Then:*

*$T(0)=0$.*

*For any vectors v1,v2,...,vk in Rn and scalars c1,c2,...,ck, we have*

*$TAc1v1+c2v2+\cdots+ckvkB=c1T(v1)+c2T(v2)+\cdots+ckT(vk)$.*

**Examples**

**i)**

Define $T:R \to R$ by $T(x)=x+1$. Is T a linear transformation?

Solution

We have $T(0)=0+1=1$. Since any linear transformation necessarily takes zero to zero by the above important note, we conclude that T is *not* linear (even though its graph is a line).

*Note:* in this case, it was not necessary to check explicitly that T does not satisfy both defining properties: since $T(0)=0$ is a consequence of these properties, at least one of them must not be satisfied. (In fact, this T satisfies neither.)

**ii)**

Define $T:\mathbb{R}^2 \to \mathbb{R}^2$ by $T(x)=1.5x$. Verify that T is linear.

Solution

We have to check the defining properties for *all* vectors u,v and *all* scalars c. In other words, we have to treat u,v, and c as *unknowns*. The only thing we are allowed to use is the definition of T.

$T(u+v)=1.5(u+v)=1.5u+1.5v=T(u)+T(v)$

Since T satisfies both defining properties, T is linear.

*Note:* we know from this example in Section 3.1 that T is a matrix transformation: in fact,

$$T(x) = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} x.$$

Since a matrix transformation is a linear transformation, this is another proof that T is linear.

**iii)**

Define $T:\mathbb{R}^2 \to \mathbb{R}^3$ by the formula

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - y \\ y \\ x \end{pmatrix}.$$

Verify that T is linear.

Solution

We have to check the defining

properties for *all* vectors u,v and *all* scalars c. In other words, we have to treat u,v, and c as *unknowns*; the only thing we are allowed to use is the definition of T. Since T is defined in terms of the coordinates of u,v, we need to give those names as well; say $u=\begin{pmatrix}x1\\y2\end{pmatrix}$ and $v=\begin{pmatrix}x2\\y2\end{pmatrix}$. For the first property, we have

$$T\left(\begin{pmatrix}x1\\y1\end{pmatrix}\right)+\left(\begin{pmatrix}x2\\y2\end{pmatrix}\right)=T\begin{pmatrix}x1+x2\\y1+y2\end{pmatrix}=\begin{pmatrix}3(x1+x2)-(y1+y2)\\y1+y2\\x1+x2\end{pmatrix}$$

$$=\begin{pmatrix}(3x1-y1)+(3x2-y2)\\y1+y2\\x1+x2\end{pmatrix}$$

$$=\begin{pmatrix}3x1-y1\\y1\\x1\end{pmatrix}+\begin{pmatrix}3x2-y2\\y2\\x2\end{pmatrix}=T\begin{pmatrix}x1\\y1\end{pmatrix}+T\begin{pmatrix}x2\\y2\end{pmatrix}$$

For the second property,

$$T\left(c\begin{pmatrix}x1\\y1\end{pmatrix}\right)=T\begin{pmatrix}cx1\\cy1\end{pmatrix}=\begin{pmatrix}3(cx1)-(cy1)\\cy1\\cx1\end{pmatrix}$$

$$=\begin{pmatrix}c(3x1-y1)\\cy1\\cx1\end{pmatrix}=c\begin{pmatrix}3x1-y1\\y1\\x1\end{pmatrix}=cT\begin{pmatrix}x1\\y1\end{pmatrix}$$

Since T satisfies the defining properties, T is a linear transformation.

*Note:* we will see in this example below that

$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3&-1\\0&1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Hence T is in fact a matrix transformation.

**iv)**

Verify that the following transformations from R2 to R2 are not linear:

$$T1\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}|x|\\y\end{pmatrix} \quad T2\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}xy\\y\end{pmatrix} \quad T3\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x \ +1\\x -2y\end{pmatrix}$$

Solution

In order to verify that a transformation T is *not* linear, we have to show that T does not satisfy *at least one* of the two defining properties. For the first, the negation of the statement "T(u+v)=T(u)+T(v) for all vectors u,v" is "there exists at least one pair of vectors u,v such that T(u+v)A=T(u)+T(v)." In other words, it suffices to find *one example* of a pair of vectors u,v such that T(u+v)A=T(u)+T(v). Likewise, for the second, the negation of the statement "T(cu)=cT(u) for all vectors u and all scalars c" is "there exists some vector u and some scalar c such that T(cu)A=cT(u)." In other words, it suffices to find *one* vector u and *one* scalar c such that T(cu)A=cT(u).

For the first transformation, we note that

$$T1\left(-\begin{pmatrix}-1\\0\end{pmatrix}\right) = T1\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}|-1|\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

but that

$$-T1\begin{pmatrix}1\\0\end{pmatrix} = -\begin{pmatrix}|1|\\0\end{pmatrix} = -\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$$

Therefore, this transformation does not satisfy the second property.

For the second transformation, we note that

$$-T2\left(2\begin{pmatrix}1\\1\end{pmatrix}\right) = T2\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}2.2\\2\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$$

but that

$$2T2\begin{pmatrix}1\\1\end{pmatrix} = 2\begin{pmatrix}1.1\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$$

Therefore, this transformation does not satisfy the second property.

For the third transformation, we observe that

$$T3\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}2(0) \;\; +1\\0 \, -2(0)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$$

Since T3 does not take the zero vector to the zero vector, it cannot be linear.

**v)**

Define T:C3→C2T:C3→C2 by describing the output of the function for a generic input with the

$$T\left(\begin{bmatrix}x1\\x2\\x3\end{bmatrix}\right) = \begin{bmatrix}2x1 \;\; +x3\\-4x2\end{bmatrix}$$

and check the two defining properties.

$$T(x \;\; +y) \;\; = T\left(\begin{bmatrix}x1\\x2\\x3\end{bmatrix} + \begin{bmatrix}y1\\y2\\y3\end{bmatrix}\right)$$

$$T \;\; = \left(\begin{bmatrix}x1 \;\; +y1\\x2 \;\; +y2\\x3 \;\; +y3\end{bmatrix}\right)$$

$$\begin{bmatrix}2(x1 \;\; +y1) \;\; +(x3 \;\; +y3)\\-4(x2 \;\; +y2)\end{bmatrix}$$

$$\begin{bmatrix}(2x1 \;\; +x3) \;\; +(2y1 \;\; +y3)\\-4x2 \;\; +(\,-4)y2\end{bmatrix}$$

$$\begin{bmatrix} 2x1 \ +x3 \\ -4x2 \end{bmatrix} + \begin{bmatrix} 2y1 \ +y3 \\ -4y2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}\right) + T\left(\begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix}\right)$$

T(x)+T(y)

And

$$T(\alpha x) = T\left(\alpha \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2(\alpha x1) \ +(\alpha x3) \\ -4(\alpha x2) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(2x1 \ +x3) \\ \alpha(-4x2) \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 2x1 \ +x3 \\ -4x2 \end{bmatrix}$$

$$= \alpha T\left(\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}\right)$$

$$= \alpha T(x)$$

T is a linear transformation.