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A **linear transformation** is a function from one vector space to another that respects the underlying (linear) structure of each vector space. A linear transformation is also known as a linear operator or map. The range of the transformation may be the same as the domain, and when that happens, the transformation is known as an endomorphism or, if invertible, an automorphism. The two vector spaces must have the same underlying field.

The defining characteristic of a linear transformation T:V→WT: V \to WT:V→W is that, for any vectors v1v_1v1 and v2v_2v2 in VVV and scalars aaa and bbb of the underlying field,

T(av1+bv2)=aT(v1)+bT(v2).T(av_1 + bv_2) = aT(v_1) + bT(v_2).T(av1+bv2
)=aT(v1)+bT(v2).

Linear transformations are useful because they preserve the structure of a vector space. So, many qualitative assessments of a vector space that is the domain of a linear transformation may, under certain conditions, automatically hold in the image of the linear transformation. For instance, the structure immediately gives that the kernel and image are both subspaces (not just subsets) of the range of the linear transformation.

Example 1:

The linear transformation from R3\{R}^3R3 to R2\{R}^2R2 defined by T(x, y, z)=(x−y, y−z)T(x,\,y,\,z) = (x - y,\, y - z)T(x,y,z)=(x−y,y−z) is given by the matrix

M=|1 -1 0|

|0 1 -1|

So, TTT can also be defined for vectors v=(v1, v2, v3)v = (v_1, \, v_2, \, v_3)v=(v1
,v2,v3) by the matrix product

T(v)= |1 -1 0| |

|0 1 -1| |

Example 2 :

Define L: R3 → R2 by L(x1,x2,x3) = (x3 −x1,x1 + x2).

a. Compute L(e e e1),L(e e e2), and L(e e e3).

b. Show L is a linear transformation.

c. Show L(x1,x2,x3) = x1L(e e e1) + x2L(e e e2) + x3L(e e e3).

Solu:

a. L[(1,0,0)] = (−1,1),L[(0,1,0)] = (0,1),L[(0,0,1)] = (1,0).

b. L(x x x +y y y) = L[(x1 + y1,x2 + y2,x3 + y3)]

= ((x3 + y3)−(x1 + y1),(x1 + y1) + (x2 + y2))

= (x3 −x1,x1 + x2) + (y3 −y1,y1 + y2) = L(x x x) + L(y y y)

L(cx) = L[(cx1,cx2,cx3)] = (cx3 −cx1,cx1 + cx2)

= c(x3 −x1,x1 + x2) = cL(x x x)

Thus L satisfies conditions 1 and 2 of Definition 3.1, and it is a linear transformation.

c. L(x1,x2,x3) = L(x1e e e1 + x2e e e2 + x3e e e3)

= L(x1e e e1) + L(x2e e e2) + L(x3e e e3)

= x1L(e e e1) + x2L(e e e2) + x3L(e e e3)

Example 3:

Let V be the vector space of a n-square real matrices ,let M be an arbitrary but fixedmatrix in V let F:V→X be defined by F(A)=AM=+MA, where A is any matrix in V. Show that F is Linear.

Solu:

For any matrices A and B in V and any scaler K, we have

F(A+B)=(A+B)M+M(A+B)=AM+BM+MA+MB

=(AM+MA)=(BM+MB)=F(A)+F(B)

And

F(KA)=(KA)M+M(KA)=K(AM)+K(MA)=K(AM+MA)=KF(A)

Thus , F is Linear

Example 4:

Let F: R4→R3 be the linear mapping defined by

F(x,y,z,t)=(x-y+z+t, 2x-2y+3x+4t, 3x_3y+4z+5t)

(a) Find a basis and the dimesion of the image of F

(b)Find then image of the usual basis vectors of R4

F(1,0,0,0)=(1,2,3) F(0,0,1,0)=(1,3,4)
F(0,1,0,0)=(-1,-2,-3), F(0,0,0,1)=(1,4,5)

Solu:

(a)Matix M whose rows are these image vectors and row reduce to echelon form

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, (1,2,3) and (0,1,1) form a basis of im F. Hence, dim (Im F) = 2 and rank (F) =2

(b) Find a basis and the dimesions of the kernel of the map F

Set F(v) =0, where v = ((x,y,z,t)

F(x,y,z,)=(x-y+z+t, 2x-2y+3x+4t, 3x-3y+4x+5t) = (0,0,0)

Set corresponding , components equal to each other to form the following homogenous system whose solution space is Ker F:

x- y + Z + t + = 0 X- y + Z + t = 0

2x- 2y + 3x +4z = 0 or z + 2t= 0 or x-y+z+t=0

3x-3y+4x+5t=0 z+2t = 0 z+2t =0

The free variables are y and t. Hence , dim (Ker F) =2 or nullity (F) =2

(i) Set y =1, t=0 to obtain the solution (1,1,0,0).
(ii) Set y=0, t =1 to obtain the solution(1,0,2,1)

Thus, (1,1,0,0) and (1, 0, 2) form a basis for Ker F

An expected form theorem 5, 6, dim (Im F) dim(Ker F) = 4 dim R4

Example 5:

Find a linear map F:R3→R4 whose image is spanned by (1,2,0,-4) and (2,0,-1,-3)

Solu:

Form a 4×3 matrix whose colums consists only of the given vectors , say

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 0 \\ 0 & -1 & -1 \\ -4 & -3 & -3 \end{bmatrix}$$

Recall that A determines a linear map A: R3→R4 whose image is spanned by the columns of A. Thus, A satisfies the required condition.