PTE 519 Lecture Note 3

3.0 Finite Difference Approximation (Model)

In this section of the lecture material, the focus is to define the terminology and to summarize the basic facts. The basic idea of any approximate method is to replace the original problem by another problem which is easier to solve and whose solution is in some sense close to the solution of the original problem. A simple example is adopted from Aziz Khaled: Petroleum Reservoir Simulation, 1979; considering the equation below:

$$AU = \frac{d^2 U}{dx^2} - q(x) = 0 \qquad 0 < x < L \qquad \text{- Equation 3.1}$$

$$U(0) = U(l) = 0 \qquad \text{- Equation 3.2}$$

In the finite difference approach, instead of trying to find a continuous sufficiently smooth function (Ux) which satisfies equation 3.1, we seek only approximate values of the solution denoted by u on a finite set of distinct points $x_1, x_2, \ldots, x_N$ inside the interval (O, L). The points x_i are called grid points (also known as mesh points or net points).

The differential equation is replaced by a set of algebraic equation relating values U_i at x_i for all points. These equations are called "finite difference equations" and the differential problem is thus reduced to an algebraic problem. If we can show that the discrete problem is close to the original problem then the values u_i will approximate the true solution $U_i = U(x_i)$ at grid positions x_i **. The process of obtaining finite difference equation that approximates a given differential equation is known as "discretisation".**

At this stage, three types of question may be asked:

1. How can a given differential equation be discretised
2. How can we ascertain that the finite difference solution u_i is close to U_i in some sense and what is the magnitude of the difference?
3. What is the best method of solving the resulting system of algebraic equations?

The first two questions are discussed in this section. The third question is extremely important from practical point of view, and involves two steps. First, whenever the finite-difference equations are non-linear, they must be linearised. The second step involves the solution of the resulting matrix equation.

3.1 Discretisation in Space

Let us consider equation 3.1 with boundary conditions $U(0) = U(L) = 0$

Basically, there are three methods available for discretisation of any given operator A: The Taylor method, the integral method and the variational method as highlighted in (Forsythe and Wasow, 1960; Varga, 1962).

These correspond to differential integral and variational formulations of the conservation equation 3.1.

The problem to be solved is $AU = 0$

Instead of this, we solve $Lu = 0$

Where L is a finite difference operator approximating the differential operator A. Generally, we write

$AU_i = LU_i + R_i$ Equation 3.3 where LU_i is obtained by approximating the derivatives in the differential operator $A\ and\ R_i$ is the remainder term usually referred to as truncation error.

3.1.1 <u>Taylor Series Method</u>

Let us consider a uniform grid with grid points at x_0 , x_1 , x_2 , ..., x_N

With x_0 = 0 ; x_{N+1} = L and the grid spacing h defined by

h = x_{i+1} - x_i = L/(N + 1)

Now expanding U_{i+1} and U_{i-1} into Taylor series about U_i :

$U_{i+1} = U_i + U_i'h + U_i'' \frac{h^2}{2} + U_i''' \frac{h^3}{6} + U_i^{iv} \frac{h^4}{24} + U_i^{v} \frac{h^5}{120} + U_i^{vi} \frac{h^6}{720} + ...$ Equation 3.4

$U_{i-1} = U_i - U_i'h + U_i'' \frac{h^2}{2} - U_i''' \frac{h^3}{6} + U_i^{iv} \frac{h^4}{24} - U_i^{v} \frac{h^5}{120} + U_i^{vi} \frac{h^6}{720} - ....$ Equation 3.5

Using the above two expansions, we can derive several difference approximations for U_i' and for U_i''

Solving equation 3.1 for U_i' we have;

$U_i' = \frac{U_{i+1} - U_i}{h}$ + R{ Equation 3.6

Where R{ = - $U_i'' \frac{h^2}{2}$ - $U_i''' \frac{h^3}{6}$ Equation 3.7

In Equation 3.6, $\frac{U_{i+1} - U_i}{h}$ is the forward difference approximation for the derivative $U_{i'}$. This is obtained by assuming that R{ is small

Similarly, re-arranging we have

$U_i' = \frac{U_i - U_{i-1}}{h} + R_i^b$ Equation 3.8

Where $R_i^b = U_i'' \frac{h^2}{2} - U_i''' \frac{h^3}{6}$ Equation 3.9

In Equation 3.6, the term $\frac{U_i - U_{i-1}}{h}$ is the backward difference approximation for the derivative U_i' and R_i^b is the local discretisation error term for the backward-difference approximation.

The central difference approximation of U_i' is obtained by subtracting 3.5 from 3.4 and re-arranging giving

$U_i' = \frac{U_{i+1} - U_{i-1}}{2h} + R_i^c$ Equation 3.10

Where $R_i^c = -U_i''' \frac{h^3}{6} - U_i^v \frac{h^5}{120} - \ldots\ldots$ Equation 3.11

Here, $\frac{U_{i+1} - U_{i-1}}{2h}$ provides an approximation for U_i'

So far, only the first derivative has been considered. An approximation for the second derivative is accomplished by adding Equations 3.5 and 3.4 and re-arranging

$U_i''' = \frac{U_{i-1} - 2U_i + U_{i+1}}{2h} + R_i^2$ Equation 3.12

Where

$R_i^2 = -U_i^{iv} \frac{h^3}{12} - U_i^{vi} \frac{h^4}{360} - \ldots\ldots$ Equation 3.13

In Equation 3.12 the term

$\frac{1}{h^2} \Delta^2 U_i = \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}$ Equation 3.14 is the central difference approximation for U''' and R_i^2 is the corresponding remainder term.

Equation 3.10 defines the linear operator Δ^2 .

$U_i''' = \frac{U_{i-1} - 2U_i + U_{i+1}}{2h} + R_i^2$ Equation 3.15

As an example, consider the differential operator, A defined by Equation 3.14 Using the central difference approximation of the derivative, we have

$AU_i = \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} - q_i + R_i^2$ Equation 3.16

Where $q_i = q(x_i)$ Comparing the above expression with equation 3.3 yields

$LU_i \equiv \frac{1}{h^2} \Delta^2 U_i - q_i$ Equation 3.17

$R_i = R_i^2$

Generally, it is not possible for us to obtain the exact solution U_i instead we solve the problem

$Lu_i = \frac{1}{h^2} \Delta^2 u_i - q_i = 0$ Equation 3.18 Where u_i is the finite difference approximation for U_i and Equation 3.18 is the finite difference approximation for Equation 3.1

3.2 Grid Systems

Introduction:

The discretization in space of the difference forms of the flow equations (such as Eqns. 1.26 to 1.28) in reservoir simulators requires the use of grids. In the design of grid systems for reservoir simulation, the following factors are critical:

1. Resolution of geologic and petrophysical data
2. Size of the static geologic model
3. Limitations on the size of the reservoir flow model
4. Potential uses of the reservoir model
5. Upscaling (also termed upgridding or scaleup) techniques to convert the static geologic model into a reservoir flow model

In the subsequent section of this lecture material, different types of grid systems and their potential effects on reservoir simulation results are highlighted.

3.2.1 Cartesian Grids:

According to Nnaemeka Ezekwe, the most common type of grid system used in reservoir simulation are based on Cartesian coordinates of x, y and z. Cartesian grids can be based on a block-centred grid or point-distributed grid systems. The block-centred grid is better for calculating mass accumulation terms in the flow equations, while the point distributed grid is more accurate for calculating flows between the gridblocks. The differences between the two grid systems (block centred vs point-distributed) are insignificant, if the Cartesian grid systems are uniform. Most commercial simulators use block centred grids. Cartesian grid systems can be classified as one-dimensional (1D), two-dimensional (2D), and three dimensional (3D). Figure 3.2 shows a 1D grid system in the x-direction. Figure 3.3 shows a 2D grid system for an areal model (x, y), and a cross-sectional model (x, z).

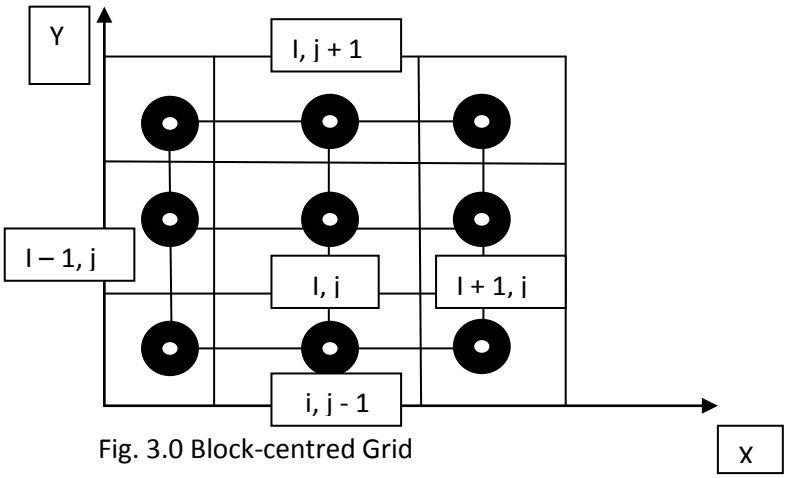


Fig. 3.0 Block-centred Grid

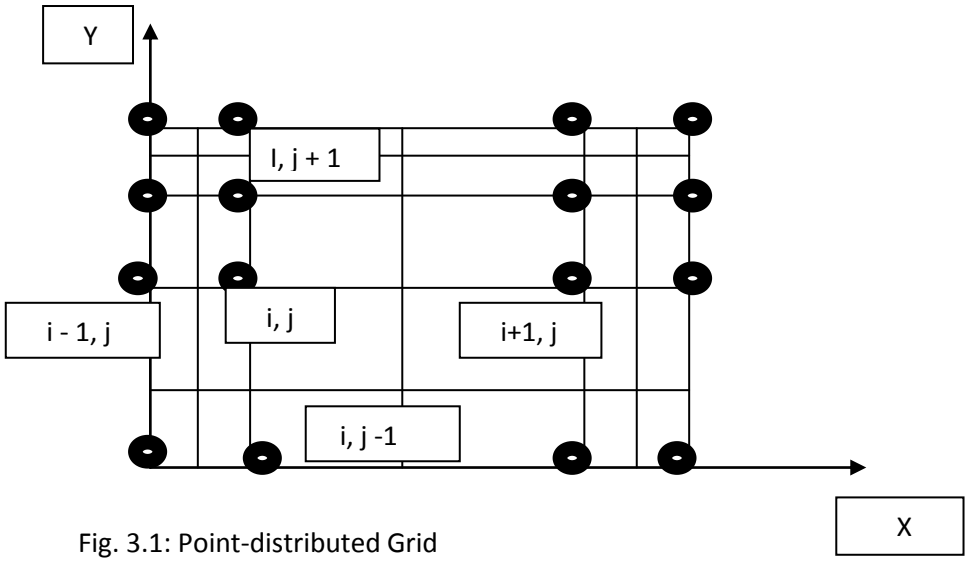


Fig. 3.1: Point-distributed Grid

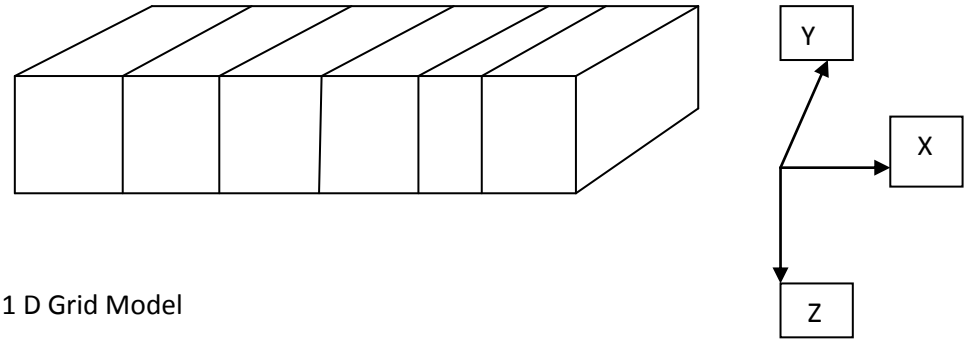


Fig. 3.2: 1 D Grid Model

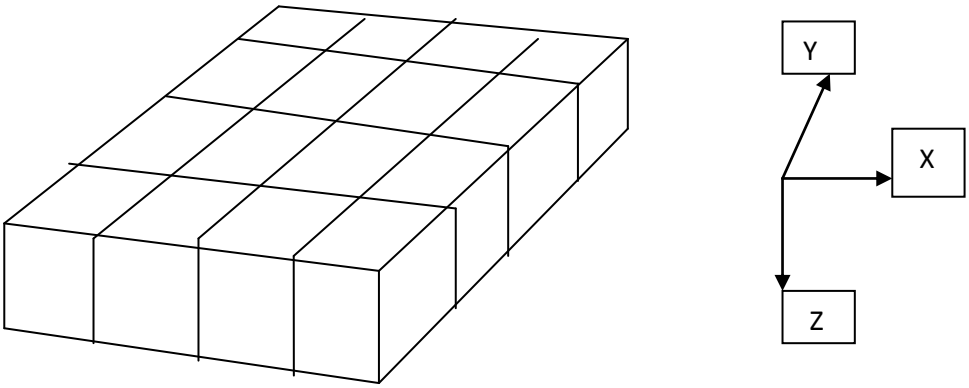
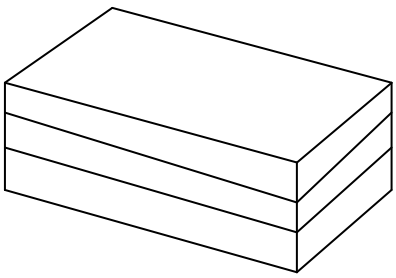


Fig. 3.3: 2D Grid Model



Cross Section

The discretization in space of the simulator flow equations assumes that the gridblocks are orthogonal. This means that each column of grid blocks is at right angles to each row of grid-blocks. In designing a Cartesian grid system for reservoir simulation, it is important to maintain the gridblocks orthogonal to each other, otherwise the results from the simulation may not be accurate if the gridblocks are severely non-orthogonal. Most commercial incorporate features to ensure orthogonality of the gridblocks for irregular grid systems, especially if the static geologic model is intended to be used in reservoir simulation.

3.2.2 Radial Grids:

Grid systems are also based on radial coordinates of r, θ, and z. An example of a grid system based on radial coordinates is shown in Fig. 3.4. Radial grid systems are used mainly in the representation of wells in models that show rapid changes in fluid saturations near the well bore as in gas cusping or water coning studies. In some applications, the wells are represented with radial grids while the remaining parts of the reservoir have Cartesian grids, thereby creating a hybrid grid system.

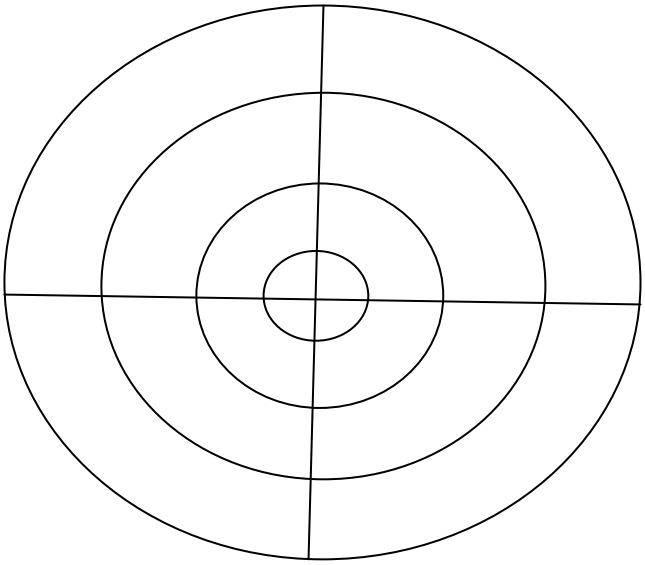


Fig. 3.4: Radial grid model

3.2.3 Corner-Point Geometry:

Corner-point geometry is based on specifying the locations of all the eight corners of each grid block as shown in Fig. 2.5. This gridding system is useful for representing complex geologic geometries such as faults, permeability barriers, irregular boundaries, sand channels etc. However, they could generate highly irregular grids that may not be orthogonal. As pointed out earlier, non-orthogonal grids, especially from corner-point geometry, could yield erroneous flow simulation results.

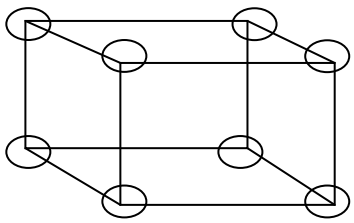


Fig. 3.5: Grid block corner point

3.2.4 Local Grid Refinement (LGR)

Local grid refinement (LGR) is a technique used to increase the grid density in regions of the reservoir where rapid changes in saturations and pressures are occurring so as to represent those changes more accurately. LGR are used mainly around well locations or to monitor saturation changes at fluid boundaries. An example of LGR around well locations is shown **in Figure 3.6 for Cartesian grid**s. LGR should be used with caution due to two main potentially adverse consequences from indiscriminate application of the technique. The first adverse result of LGR is fluid flow between a large grid block and a small grid block in an LGR area, which could cause large oscillations in computed results. The second adverse effect is a significant increase in model run times, if many LGRs are present in a flow simulation model. A reverse form of LGR is **Grid coarsening.** Grid coarsening involves reduction of the number of grid blocks in portions of the reservoir that do not have rapid changes in saturation or pressures, such as in acquifers surrounding the hydrocarbon portions of the reservoir (Fig. 3.7) Grid coarsening is beneficial in terms of reducing computation and storage memory required for a given flow simulation model.

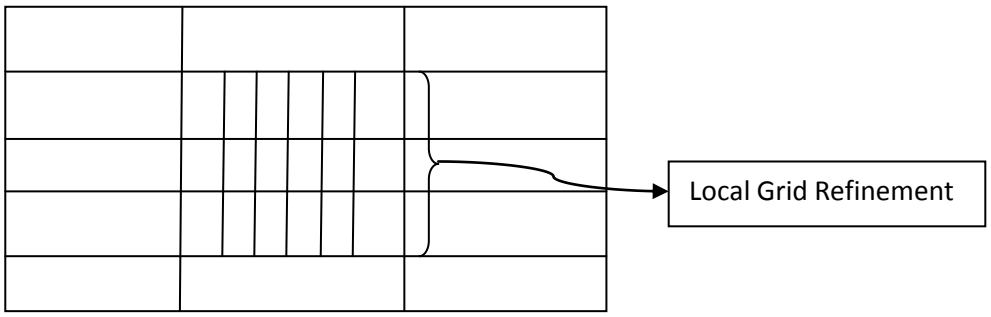


Fig. 3.6: Local grid refinement

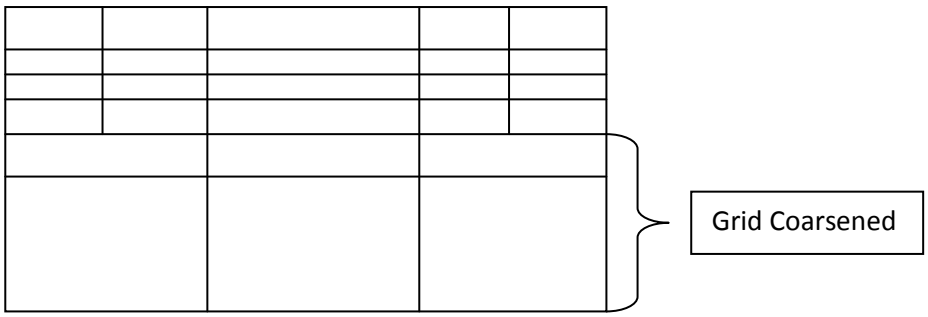


Fig. 3.7: Grid coarsening

3.3 Timesteps:

The difference equations resulting from the discretization in time of the flow equations in reservoir simulators are solved in a number of timesteps. Timesteps are progressive increments of time used to advance simulation of the process from initial time to the stated end time. As the solution progresses in time, the selection of time step sizes plays a very important role in ensuring that solutions of the difference equations are accurate, efficient and stable. Most commercial simulators feature sophisticated automatic timestep control that is based on pre-determined set of default parameters and tolerances. For many simulator applications, these automatic times step selection criteria are adequate and need no further adjustments. However, for specialized simulator applications, the user may prefer to use manual timestep control. The procedure for implementing manual timestep control is typically provided in the technical documentation of the commercial simulator.

Simulator timestep sizes are influenced by simulator startup, timing of input and output of data, and large changes in reservoir pressures, production rates and injection rates. At startup, the timestep size for most commercial simulators is automatically set at about 1 day or less depending on the application. As the simulation progress and the solution stabilize, large time steps are taken, depending on other controls applied on the simulator. One of such controls applied on the simulator is data input and output. Most simulators adjust their timestep sizes as necessary to precisely comply with data input and/or output specified by time (or date). Frequent data input or output could slow down the simulator and increase run times thereby increasing computation costs. Lareg changes in pressures, production rates, and injection rates may also cause reduction in timestep sizes. One technique that could be used to avoid sudden large changes in reservoir conditions (pressures, saturations, etc) is to incorporate additional times (or dates) for data input so as to "smooth-out" or reduce the impact of the large changes on the progress of the simulation.

Except for experienced users of commercial simulators, it is recommended that automatic time step selection should be used as provided in commercial simulators. In general, automatic timestep control in commercial simulators will yield reduced run times and computation costs.

3.4 History Matching

History matching is the process of adjusting the properties and parameters of the reservoir model to match past actual performance data measured on the reservoir. The performance data may include pressure data such as (static bottomhole pressures, flowing bottomhole pressures, tubing head pressures, average reservoir pressures, etc), production and injection data (such as oil, water and gas production rates and water and gas injection rates), well completion data (such as perforated and isolated intervals), production and injection profile data from (production logs and injection logs), etc. A large variety of performance data can be used in history matching process. What is important is to note is the reservoir type, and the methods being used to deplete or improve production from the reservoir. The history matching process is a classical example of an inverse problem which leads to a non-unique solution or response. This implies that different history matched models of a reservoir can be achieved by applying the same input data. The history matched models are non-unique solutions to the applied performance data and may yield different predictions of the future performance of the reservoir.

The process of history matching a reservoir model varies from one practitioner to the other, and depends largely on experience and knowledge of reservoir engineering principles. The history matching matching process also depends considerably on the reservoir model type, the quality and quantity of available performance data, and the objectives of the study. For instance, if the reservoir model was constructed from sparse data and limited performance data available, it is reasonable to expect that history matched model is likely to be limited in its application on the prediction of the future performance of the reservoir.

Generally, the history matching process can be divided into two phases, namely the pressure match phase and the saturation match phase. During the pressure match phase, the history matching process is focussed on matching the net voidage (in reservoir barrels) from the reservoir with the aim of matching pressure distribution in the reservoir over the specified time period. In some cases, only a match of the trend in pressure distribution can be achieved during the pressure match phase. The main goal of the pressure match phase is to obtain a reservoir model that has approximately the appropriate net volume of fluids in place over the duration of its known history. During the pressure match phase, minimal effort is applied in the validation of the distribution of the phases or saturations in the model. At the end of the pressure match phase, the history matching process proceeds to the saturation match phase. During the saturation match phase, attention is directed towards specifying the actual phases of fluids produced and/or injected into the reservoir. For instance, in history matching black oil reservoir models, oil production rates are typically specified and the model yields the corresponding water and gas production rates. The water and gas production rates generated from the model are then compared with the actual water and gas production rates. Also, pressures and other performance data generated from the model are compared with actual data. Different forms of the actual data may be used for this comparison during the process of history matching. Some practitioners use production rates, cumulative production volumes, producing gas-oil ratios and water-oil ratios, flowing bottomhole pressures, tubing head pressures, average reservoir pressures, etc The matching of data predicted to the actual can be done at the level of the field, well groups, or individual well. Again the choice of actual data to match depends on the quality and quantity of data and the objectives of the simulation study. At the end, saturation match phase generally produces a history matched model with appropriate distribution of pressures and saturations in the reservoir. Generally, the history matching process is terminated at the point where the team engaged in the process decides that an acceptable history matched model has been obtained. History matched models are ruled to be acceptable, if they are adequate to meet the objectives of the simulation study, and further adjustments to the models do not result in significant changes on the history matched variables.

The following best practices are recommended to enhance the efficiency and effectiveness of history matching process:

1.) Make a log of all activities and changes made to the model during the history matching process.
2.) The changes made on the model during history matching should be sequential/graduated and systematic.
3.) The results from the previous changes or adjustments to the model should be analysed and understood before proceeding with the next set of changes or adjustments
4.) The least reliable input data should be modified first during the history match process

5.) Recognition of the point in the history matching process when further changes on the input data cause insignificant improvement on the state of the history matched model is important.

References:

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3. Principle of Applied Reservoir Simulation by John R. Fanchi ; Published 2001 by Gulf Professional Publishers