Derivation of Continuity Equation:

Consider a rectangular block, with fluid flow in three directions x, y and z as shown in Figure 1 below during the time interval, Δt, the principle of mass conservation can be applied to the block to obtain the following mass balance equation:

Mass flowing into the block in time, Δt - Mass flowing out of the block in time, Δt = Mass Accumulation in the block in time, Δt .... **Equation 1.1**

Let mass flow rate per unit cross-sectional area normal to the direction of flow be denoted as F, Using this designation, fluid flow into the block in the x direction at x is denoted as $(F_x)_x$ and flow out of the block at $x + \Delta x$ is denoted as $(F_x)_{x+\Delta x}$. Similarly, flow into the block in the y and z direction are designated as $(F_y)_y$ and $(F_z)_z$ respectively and flow out of the block in the y and z direction are designated as $(F_y)_{y+\Delta y}$ and $(F_z)_{z+\Delta z}$ as shown in Figure 1.

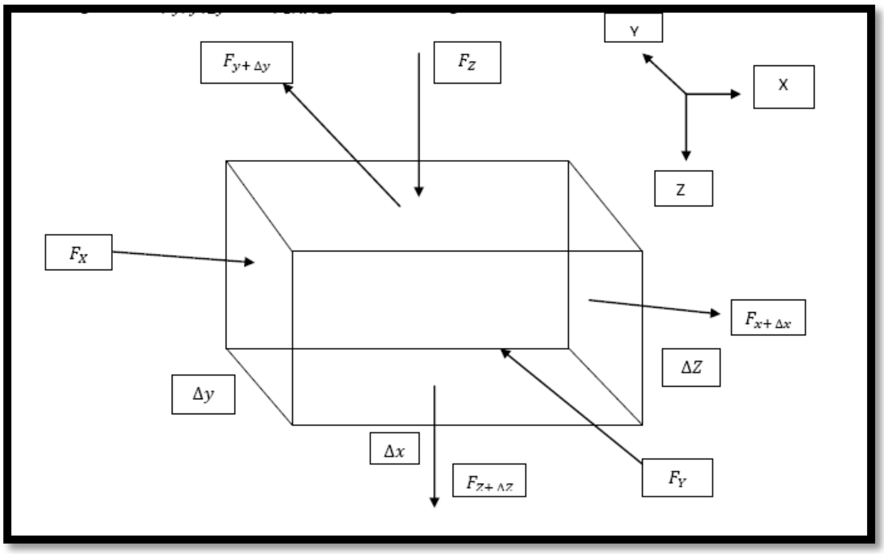

Figure 1: A Grid Block showing fluid flow in three directions

Mass flowing into the block in time, Δt $= \left[(F_x)_x\,\Delta y\,\Delta z + \left(F_y\right)_y\,\Delta x\,\Delta z + \ (F_z)_z\,\Delta x\,\Delta y\right]\Delta t$ ... **Equation 1.2**

Mass flowing out of block in time, Δt =

$\left[(F_x)_{x+\Delta x}\,\Delta y\,\Delta z + \left(F_y\right)_{y+\Delta y}\,\Delta x\,\Delta z + \ (F_z)_{z+\Delta z}\,\Delta x\,\Delta y\right]\Delta t + q\Delta x\Delta y\Delta z\Delta t$ .... **Equation 1.3**

Mass accumulation in block in time, Δt $= \left[\left(C_p\right)_{t+\Delta t} - \ \left(C_p\right)_t\right]\,\Delta x\Delta y\Delta z$ .... **Equation 1.4**

In Equation 1.3, q is a source/sink term that represents mass flow into or out of the block primarily through a well. For a well that is a producer, $q > 0\ (or\ positive)$ and for a well that is an injector, $q < 0\ (or\ negative)$ by convention. In Equation 1.4, C_p is the concentration of any phase (oil, water and gas) in the block defined in mass per unit volume.

Substituting Equations 1.2, 1.3 and 1.4 into Equation 1.1 gives:

$\left[(F_x)_x \,\Delta y\, \Delta z + (F_y)_y \,\Delta x\, \Delta z + \ (F_z)_z \,\Delta x\, \Delta y\right] \Delta t$ - $\left[(F_x)_{x+\Delta x} \,\Delta y\, \Delta z + (F_y)_{y+\Delta y} \,\Delta x\, \Delta z + (F_z)_{z+\Delta z} \,\Delta x\, \Delta y\right] \Delta t - q\Delta x\Delta y\Delta z\Delta t$ .... **Equation 1.5**

Dividing Equation 1.5 by $\Delta x\Delta y\Delta z\Delta t$ and collecting like terms yields:

$$\frac{(F_x)_{x+\Delta x} - (F_x)_x}{\Delta x} - \frac{(F_y)_{y+\Delta y} - (F_y)_y}{\Delta y} - \frac{(F_z)_{z+\Delta z} - (F_z)_z}{\Delta z} - q = \frac{\left[(C_p)_{t+\Delta t} - (C_p)_t\right]}{\Delta t}$$ .... **Equation 1.6**

Taking limits as $\Delta x, \Delta y, \Delta z, and\ \Delta t$ approach zero gives:

$$-\frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} - \frac{\partial F_z}{\partial z} - q = \frac{\partial C_p}{\partial t}$$ .... **Equation 1.7**

Equation 1.7 is the continuity equation for fluid flow in rectangular coordinates

Flow Equations for Three-Phase Flow of Oil, Water and Gas

Based on extracts from Petroleum Reservoir Engineering Practice by Nnaemeka Ezekwe, the mass flow rate per unit area in a given direction for any fluid is the product of the density of the fluid and its velocity in that direction. If oil, water and gas are denoted with the subscripts $o, w\ and\ g,$ respectively, then $\overrightarrow{(F)_o} = \frac{\rho_{osc}}{B_o} \overrightarrow{v_o}$ .... **Equation 1.8**

$$\overrightarrow{(F)_w} = \frac{\rho_{wsc}}{B_w} \overrightarrow{v_w}$$ .... **Equation 1.9**

$$\overrightarrow{(F)_g} = \frac{\rho_{gsc}}{B_g} \overrightarrow{v_g} + \frac{R_{so}\rho_{gsc}}{B_o} \overrightarrow{v_o} + \frac{R_{sw}\rho_{gsc}}{B_w} \overrightarrow{v_w}$$ .... **Equation 1.10**

It is important to note that in Equations 1.8, 1.9 and 1.10, $\vec{F}$ is the mass flow rate in $, y\ and\ z\ directions$; ρ_{osc} , ρ_{wsc} and ρ_{gsc} are the densities of oil, water and gas at standard conditions respectively. R_{so} and R_{sw} are gas solubilities in oil and water respectively, expressed in scf/STB; and B_o , B_w and B_g are formation volume factors of oil, water and gas respectively, expressed in RB/STB. For oil, water and gas, the concentration of each phase, C_p , is defined respectively, as

$$C_o = \frac{\emptyset\rho_{osc} S_o}{B_o}$$ .. .. **Equation 1.11**

$$C_w = \frac{\emptyset\rho_{wsc} S_w}{B_w}$$ .... **Equation 1.12**

$$C_g = \emptyset\rho_{gsc} \left[\frac{S_g}{B_g} + R_{so} \frac{S_o}{B_o} + R_{sw} \frac{S_w}{B_w}\right]$$ .... **Equation 1.13**

In Equations 1.11, 1.12 and 1.13, $\emptyset$ is the porosity and S_o , S_w and S_g are oil, water and gas saturations respectively.

$S_o + S_w + S_g = 1$ .... **Equation 1.14**

Substituting 1.8 to 1.13 into 1.7 with a vector expansions gives mass balance equations for each phase as :

For Oil:

$$-\left[\frac{\partial}{\partial x}\left(\frac{\rho_{osc}}{B_o}v_{xo}\right)+\frac{\partial}{\partial y}\left(\frac{\rho_{osc}}{B_o}v_{yo}\right)+\frac{\partial}{\partial z}\left(\frac{\rho_{osc}}{B_o}v_{zo}\right)\right]-q_o=\frac{\partial}{\partial t}\left(\frac{\phi\rho_{osc}S_o}{B_o}\right)$$

.... **Equation 1.15**

For Water:

$$-\left[\frac{\partial}{\partial x}\left(\frac{\rho_{wsc}}{B_w}v_{xw}\right)+\frac{\partial}{\partial y}\left(\frac{\rho_{wsc}}{B_w}v_{yw}\right)+\frac{\partial}{\partial z}\left(\frac{\rho_{wsc}}{B_w}v_{zw}\right)\right]-q_w=\frac{\partial}{\partial t}\left(\frac{\phi\rho_{wsc}S_w}{B_w}\right)$$

..... **Equation 1.16**

For Gas:

$$-\frac{\partial}{\partial x}\left(\frac{\rho_{gsc}}{B_g}v_{xg}+\frac{R_{so}\rho_{gsc}}{B_o}v_{xo}+\frac{R_{sw}\rho_{gsc}}{B_w}v_{xw}\right)-\frac{\partial}{\partial y}\left(\frac{\rho_{gsc}}{B_g}v_{yg}+\frac{R_{so}\rho_{gsc}}{B_o}v_{yo}+\frac{R_{sw}\rho_{gsc}}{B_w}v_{yw}\right)$$
$$-\frac{\partial}{\partial z}\left(\frac{\rho_{gsc}}{B_g}v_{zg}+\frac{R_{so}\rho_{gsc}}{B_o}v_{zo}+\frac{R_{sw}\rho_{gsc}}{B_w}v_{zw}\right)-q_g=\frac{\partial}{\partial t}\left[\phi\rho_{gsc}\left(\frac{S_g}{B_g}+\frac{R_{so}S_o}{B_o}+\frac{R_{sw}S_w}{B_w}\right)\right]$$

... **Equation 1.17**

Cancelling out ρ_{osc} , ρ_{wsc} and ρ_{gsc} gives:

For Oil:

$$-\left[\frac{\partial}{\partial x}\left(\frac{v_{xo}}{B_o}\right)+\frac{\partial}{\partial y}\left(\frac{v_{yo}}{B_o}\right)+\frac{\partial}{\partial z}\left(\frac{v_{zo}}{B_o}\right)\right]-\frac{q_o}{\rho_{osc}}=\frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right)$$

.... **Equation 1.18**

For Water:

$$-\left[\frac{\partial}{\partial x}\left(\frac{v_{xw}}{B_w}\right)+\frac{\partial}{\partial y}\left(\frac{v_{yw}}{B_w}\right)+\frac{\partial}{\partial z}\left(\frac{v_{zw}}{B_w}\right)\right]-\frac{q_w}{\rho_{wsc}}=\frac{\partial}{\partial t}\left(\frac{\phi S_w}{B_w}\right)$$

... **Equation 1.19**

For Gas:

$$-\frac{\partial}{\partial x}\left(\frac{v_{xg}}{B_g}+\frac{R_{so}v_{xo}}{B_o}+\frac{R_{sw}v_{xw}}{B_w}\right)-\frac{\partial}{\partial y}\left(\frac{v_{yg}}{B_g}+\frac{R_{so}v_{yo}}{B_o}+\frac{R_{sw}v_{yw}}{B_w}\right)$$
$$-\frac{\partial}{\partial z}\left(\frac{v_{zg}}{B_g}+\frac{R_{so}v_{zo}}{B_o}+\frac{R_{sw}v_{zw}}{B_w}\right)-\frac{q_g}{\rho_{gsc}}=\frac{\partial}{\partial t}\left[\phi\left(\frac{S_g}{B_g}+\frac{R_{so}S_o}{B_o}+\frac{R_{sw}S_w}{B_w}\right)\right]$$

..... **Equation 1.20**

In vector notation, Equations 1.18, 1.19 and 1.20 can be represented as:

For Oil:

$$-\nabla \cdot \frac{\vec{v}_o}{B_o} - \frac{q_o}{\rho_{osc}} = \frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right)$$

.... **Equation 1.21**

For Water:

$$-\nabla \cdot \frac{\vec{v}_w}{B_w} - \frac{q_w}{\rho_{wsc}} = \frac{\partial}{\partial t}\left(\frac{\phi S_w}{B_w}\right)$$

.... **Equation 1.22**

For Gas :

$$-\nabla \cdot \left(\frac{\vec{v}_g}{B_g} + \frac{R_{so}\vec{v}_o}{B_o} + \frac{R_{sw}\vec{v}_w}{B_w}\right) - \frac{q_g}{\rho_{gsc}} = \frac{\partial}{\partial t}\left[\phi\left(\frac{S_g}{B_g} + \frac{R_{so}S_o}{B_o} + \frac{R_{sw}S_w}{B_w}\right)\right]$$

.... **Equation 1.23**

In vector notation,

$$\nabla \cdot \vec{v} = \frac{\partial}{\partial x}v_x + \frac{\partial}{\partial y}v_y + \frac{\partial}{\partial z}v_z$$

.... **Equation 1.24**

If fluid velocities in 1.21, 1.22 and 1.24 are based on Darcy's law, then fluid velocities can be represented in terms of potential as

$$\vec{v}_p = -\overleftrightarrow{K}\lambda_p \nabla \Phi_p$$

.... **Equation 1.25**

In Equation 1.25, $\vec{v}$ is the fluid velocity of the phase, p in vector form; $\overleftrightarrow{K}$ represents permeability as a tensor; λ_p is the mobility of the phase where $\lambda_p = \frac{k_{rp}}{\mu_p}$ and Φ_p is the potential of the phase where $\Phi_p = P_p - (\rho_p z)/144$ assuming $g = g_c$ Substituting 1.25 into 1.21, 1.22 and 1.23 gives:

For Oil:

$$\nabla \cdot \frac{\overleftrightarrow{K}\lambda_o}{B_o} \cdot \nabla \Phi_o - \frac{q_o}{\rho_{osc}} = \frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right)$$

.... **Equation 1.26**

For Water:

$$\nabla \cdot \frac{\overleftrightarrow{K}\lambda_w}{B_w} \cdot \nabla\Phi_w - \frac{q_w}{\rho_{wsc}} = \frac{\partial}{\partial t}\left(\frac{\phi S_w}{B_w}\right)$$

.... **Equation 1.27**

For Gas:

$$\nabla \cdot \overleftrightarrow{K}\left(\frac{\lambda_g}{B_g}\nabla\Phi_g + \frac{R_{so}\lambda_o}{B_o}\nabla\Phi_o + \frac{R_{sw}\lambda_w}{B_w}\nabla\Phi_w\right) - \frac{q_g}{\rho_{gsc}} = \frac{\partial}{\partial t}\left[\phi\left(\frac{S_g}{B_g} + \frac{R_{so}S_o}{B_o} + \frac{R_{sw}S_w}{B_w}\right)\right]$$

..... **Equation 1.28**

Equations 1.26, 1.27 and 1.28 are the basic equations for immiscible oil, water and gas flow.