PTE 519 Lecture Note 3

3.0 Finite Difference Approximation (Model)

In this section of the lecture material, the focus is to define the terminology and to summarize the basic facts. The basic idea of any approximate method is to replace the original problem by another problem which is easier to solve and whose solution is in some sense close to the solution of the original problem. A simple example is adopted from Aziz Khaled: Petroleum Reservoir Simulation, 1979; considering the equation below:

$$\boldsymbol{AU} = \frac{d^2\ U}{dx^2} - \mathbf{q}(\mathbf{x}) = 0 \qquad 0 < x < L \qquad \text{- Equation 3.1}$$

$$\boldsymbol{U}(\mathbf{0}) = \boldsymbol{U}(\boldsymbol{l}) = \mathbf{0} \qquad \text{- Equation 3.2}$$

In the finite difference approach, instead of trying to find a continuous sufficiently smooth function (Ux) which satisfies equation 3.1, we seek only approximate values of the solution denoted by u on a finite set of distinct points $x_1, x_2, ..., x_N$ inside the interval (O, L). The points x_i are called grid points (also known as mesh points or net points).

The differential equation is replaced by a set of algebraic equation relating values U_i at x_i for all points. These equations are called "finite difference equations" and the differential problem is thus reduced to an algebraic problem. If we can show that the discrete problem is close to the original problem then the values u_i will approximate the true solution $U_i = U(x_i)$ at grid positions x_i **. The process of obtaining finite difference equation that approximates a given differential equation is known as "discretisation".**

At this stage, three types of question may be asked:

1. How can a given differential equation be discretised
2. How can we ascertain that the finite difference solution u_i is close to U_i in some sense and what is the magnitude of the difference?
3. What is the best method of solving the resulting system of algebraic equations?

The first two questions are discussed in this section. The third question is extremely important from practical point of view, and involves two steps. First, whenever the finite-difference equations are non-linear, they must be linearised. The second step involves the solution of the resulting matrix equation.

3.1 Discretisation in Space

Let us consider equation 3.1 with boundary conditions $\boldsymbol{U}(\mathbf{0}) = \boldsymbol{U}(\boldsymbol{L}) = \mathbf{0}$

Basically, there are three methods available for discretisation of any given operator A: The Taylor method, the integral method and the variational method as highlighted in (Forsythe and Wasow, 1960; Varga, 1962).

These correspond to differential integral and variational formulations of the conservation equation 3.1.

The problem to be solved is $AU = 0$

Instead of this, we solve $Lu = 0$

Where L is a finite difference operator approximating the differential operator A. Generally, we write

$AU_i = LU_i + R_i$ Equation 3.3 where LU_i is obtained by approximating the derivatives in the differential operator A and R_i is the remainder term usually referred to as truncation error.

3.1.1 <u>Taylor Series Method</u>

Let us consider a uniform grid with grid points at $x_0, x_1, x_2, ..., x_N$

With x_0 = 0 ; x_{N+1} = L and the grid spacing h defined by

h = x_{i+1} - x_i = L/(N + 1)

Now expanding U_{i+1} and U_{i-1} into Taylor series about U_i :

$U_{i+1} = U_i + U_i'h + U_i''\frac{h^2}{2} + U_i'''\frac{h^3}{6} + U_i^{iv}\frac{h^4}{24} + U_i^{v}\frac{h^5}{120} + U_i^{vi}\frac{h^6}{720} + ...$ Equation 3.4

$U_{i-1} = U_i - U_i'h + U_i''\frac{h^2}{2} - U_i'''\frac{h^3}{6} + U_i^{iv}\frac{h^4}{24} - U_i^{v}\frac{h^5}{120} + U_i^{vi}\frac{h^6}{720} - ....$ Equation 3.5

Using the above two expansions, we can derive several difference approximations for $\boldsymbol{U_i}'$ and for $\boldsymbol{U_i}''$

Solving equation 3.1 for U_i' we have;

$U_i' = \frac{U_{i+1} - U_i}{h}$ + R{ Equation 3.6

Where R{ = - $U_i''\frac{h^2}{2}$ - $U_i'''\frac{h^3}{6}$ Equation 3.7

In Equation 3.6, $\frac{U_{i+1} - U_i}{h}$ is the forward difference approximation for the derivative U_i'. This is obtained by assuming that R{ is small

Similarly, re-arranging we have

$U_i' = \frac{U_i - U_{i-1}}{h} + R_i^b$ Equation 3.8

Where $R_i^b = U_i''\frac{h^2}{2} - U_i'''\frac{h^3}{6}$ Equation 3.9

In Equation 3.6, the term $\frac{U_i - U_{i-1}}{h}$ is the backward difference approximation for the derivative U_i' and R_i^b is the local discretisation error term for the backward-difference approximation.

The central difference approximation of U_i' is obtained by subtracting 3.5 from 3.4 and re-arranging giving

$U_i{}' = \frac{U_{i+1} - U_{i-1}}{2h} + R_i{}^c$ Equation 3.10

Where $R_i{}^c$ = - $U_i{}'''\frac{h^3}{6}$ - $U_i{}^v\frac{h^5}{120}$ - ...... Equation 3.11

Here, $\frac{U_{i+1} - U_{i-1}}{2h}$ provides an approximation for $U_i{}'$

So far, only the first derivative has been considered. An approximation for the second derivative is accomplished by adding **Equations 3.5 and 3.4** and re-arranging

$U_i{}'' = \frac{U_{i-1} - 2U_i + U_{i+1}}{2h} + R_i{}^2$ Equation 3.12

Where

$R_i{}^2$ = - $U_i{}^{iii}\frac{h^3}{12}$ - $U_i{}^{iv}\frac{h^4}{360}$ - ...... Equation 3.13

In Equation 3.12 the term

$\frac{1}{h^2}\,\Delta^2 U_i$ = $\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}$ Equation 3.14 is the central difference approximation for U''' and $R_i{}^2$ is the corresponding remainder term.

Equation 3.10 defines the linear operator Δ^2 .

$U_i{}'' = \frac{U_{i-1} - 2U_i + U_{i+1}}{2h} + R_i{}^2$ Equation 3.15

As an example, consider the differential operator, A defined by Equation 3.14 Using the central difference approximation of the derivative, we have

$AU_i = \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}$ - q_i + $R_i{}^2$ Equation 3.16

Where q_i = $q(x_i)$ Comparing the above expression with equation 3.3 yields

$LU_i \equiv \frac{1}{h^2}\,\Delta^2 U_i - q_i$ Equation 3.17

$R_i = R_i{}^2$

Generally, it is not possible for us to obtain the exact solution U_i instead we solve the problem

$\boldsymbol{Lu_i = \frac{1}{h^2}\,\Delta^2 u_i - q_i = 0}$ **Equation 3.18** Where u_i is the finite difference approximation for U_i and **Equation 3.18** is the **finite difference approximation** for **Equation 3.1**