Class Note for Signals and Systems

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The textbook used for this course is Oppenheim and Wilsky, Signals and Systems, Prentice Hall. 2nd Edition.

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Contents

Chapter 1

Fundamentals of Signals

1.1 What is a Signal?

A signal is a quantitative description of a physical phenomenon, event or process. Some common examples include:

1. Electrical current or voltage in a circuit.

2. Daily closing value of a share of stock last week.

3. Audio signal: continuous-time in its original form, or discrete-time when stored on a CD.

More precisely, a signal is a function, usually of one variable in *time*. However, in general, signals can be functions of more than one variable, e.g., image signals.

In this class we are interested in two types of signals:

1. Continuous-time signal $x(t)$, where t is a real-valued variable denoting time, i.e., $t \in \mathbb{R}$. We use parenthesis $(\cdot)$ to denote a continuous-time signal.

2. Discrete-time signal $x[n]$, where n is an integer-valued variable denoting the discrete samples of time, i.e., $n \in \mathbb{Z}$. We use square brackets $[\cdot]$ to denote a discrete-time signal. Under the definition of a discrete-time signal, $x[1.5]$ is not defined, for example.

1.2 Review on Complex Numbers

We are interested in the general complex signals:

$$x(t) \in \mathbb{C} \qquad \text{and} \qquad x[n] \in \mathbb{C},$$

where the set of complex numbers is defined as

$$\mathbb{C} = \{z \mid z = x + jy,\ x, y \in \mathbb{R},\ j = \sqrt{-1}.\}$$

A complex number z can be represented in Cartesian form as

$$z = x + jy,$$

or in polar form as

$$z = re^{j\theta}.$$

Theorem 1. *Euler's Formula*

$$e^{j\theta} = \cos\theta + j\sin\theta. \tag{1.1}$$

Using Euler's formula, the relation between x, y, r, and θ is given by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \text{and} \qquad \begin{cases} r = \sqrt{x^2 + y^2}, \\ \theta = \tan^{-1}\frac{y}{x}. \end{cases}$$

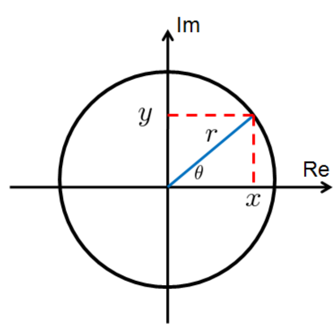


Figure 1.1: A complex number z can be expressed in its Cartesian form $z = x + jy$, or in its polar form $z = re^{j\theta}$.

A complex number can be drawn on the complex plane as shown in Fig. 1.1. The y-axis of the complex plane is known as the *imaginary* axis, and the x-axis of the complex plane is known as the *real* axis. A complex number is uniquely defined by $z = x + jy$ in the Cartesian form, or $z = re^{j\theta}$ in the polar form.

Example. Convert the following complex numbers from Cartesian form to polar form: (a) $1 + 2j$; (b) $1 - j$.

For (a), we apply Euler's formula and find that

$$r = \sqrt{1^2 + 2^2} = \sqrt{5}, \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{2}{1}\right) \approx 63.64^\circ.$$

Therefore,

$$1 + 2j = \sqrt{5}e^{j63.64^\circ}.$$

For (b), we apply Euler's formula again and find that

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}, \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{-1}{1}\right) = -45^\circ.$$

Therefore,

$$1 - j = \sqrt{2}e^{-j\pi/4}.$$

Recall that: π in radian $= 180^\circ$ in degree.

Example. Calculate the value of j^j.
$j^j = (e^{j\pi/2})^j = e^{-\pi/2} \approx 0.2078$.

1.3 Basic Operations of Signals

1.3.1 Time Shift

For any $t_0 \in \mathbb{R}$ and $n_0 \in \mathbb{Z}$, time shift is an operation defined as

$$\begin{aligned} x(t) &\longrightarrow x(t - t_0) \\ x[n] &\longrightarrow x[n - n_0]. \end{aligned} \tag{1.2}$$

If $t_0 > 0$, the time shift is known as "delay". If $t_0 < 0$, the time shift is known as "advance".

Example. In Fig. 1.2, the left image shows a continuous-time signal $x(t)$. A time-shifted version $x(t - 2)$ is shown in the right image.

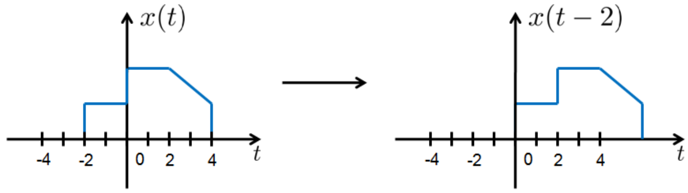


Figure 1.2: An example of time shift.

1.3.2 Time Reversal

Time reversal is defined as

$$\begin{aligned} x(t) &\longrightarrow x(-t) \\ x[n] &\longrightarrow x[-n], \end{aligned} \tag{1.3}$$

which can be interpreted as the "flip over the y-axis".

Example.

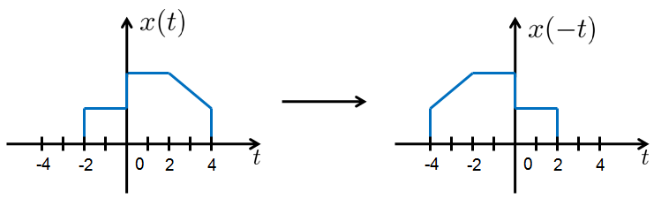


Figure 1.3: An example of time reversal.

1.3.3 Time Scaling

Time scaling is the operation where the time variable t is multiplied by a constant a:

$$x(t) \longrightarrow x(at), \quad a > 0 \tag{1.4}$$

If $a > 1$, the time scale of the resultant signal is "decimated" (speed up). If $0 < a < 1$, the time scale of the resultant signal is "expanded" (slowed down).

1.3.4 Combination of Operations

In general, linear operation (in time) on a signal $x(t)$ can be expressed as $y(t) = x(at-b), \quad a, b \in \mathbb{R}$. There are two methods to describe the output signal $y(t) = x(at-b)$.

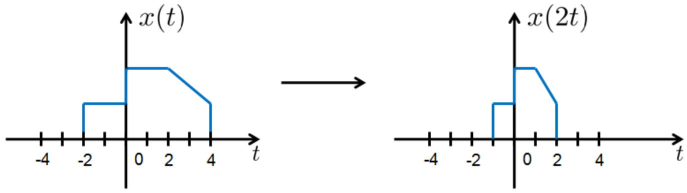


Figure 1.4: An example of time scaling.

Method A: "Shift, then Scale" (Recommended)

1. Define $v(t) = x(t - b)$,
2. Define $y(t) = v(at) = x(at - b)$.

Method B: "Scale, then Shift"

1. Define $v(t) = x(at)$,
2. Define $y(t) = v(t - \frac{b}{a}) = x(at - b)$.

Example.
For the signal $x(t)$ shown in Fig. 1.5, sketch $x(3t - 5)$.

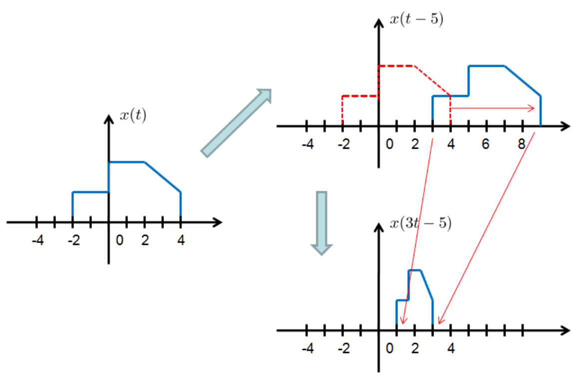


Figure 1.5: Example 1. $x(3t - 5)$.

Example.
For the signal $x(t)$ shown in Fig. 1.6, sketch $x(1 - t)$.

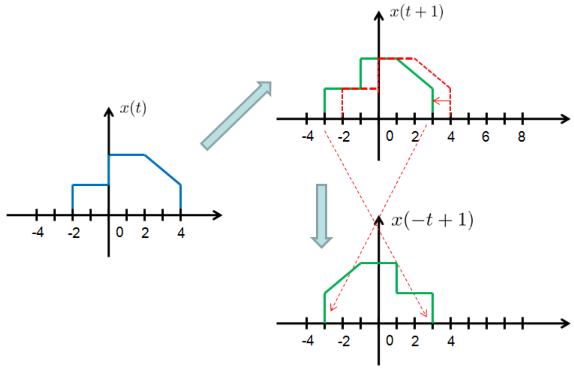


Figure 1.6: Example 2. $x(-t+1)$.

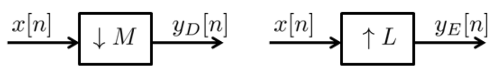


Figure 1.7: Decimation and expansion.

1.3.5 Decimation and Expansion

Decimation and expansion are standard discrete-time signal processing operations.

Decimation.

Decimation is defined as

$$y_D[n] = x[Mn], \tag{1.5}$$

for some integers M. M is called the decimation factor.

Expansion.

Expansion is defined as

$$y_E[n] = \begin{cases} x\left[\frac{n}{L}\right], & n = \text{integer multiple of } L \\ 0, & \text{otherwise.} \end{cases} \tag{1.6}$$

L is called the expansion factor.

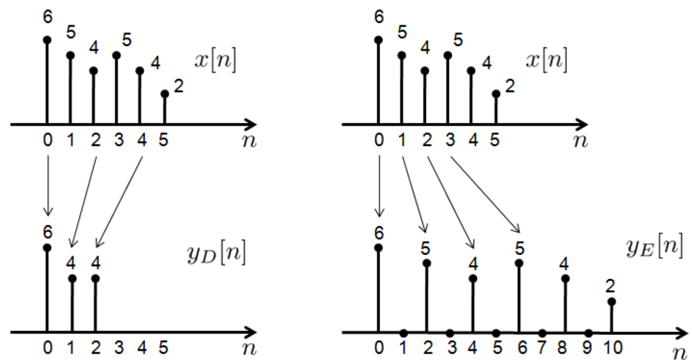


Figure 1.8: Examples of decimation and expansion for $M = 2$ and $L = 2$.

1.4 Periodicity

1.4.1 Definitions

Definition 1. *A continuous time signal $x(t)$ is periodic if there is a constant $T > 0$ such that*

$$x(t) = x(t + T), \tag{1.7}$$

for all $t \in \mathbb{R}$.

Definition 2. *A discrete time signal $x[n]$ is periodic if there is an integer constant $N > 0$ such that*

$$x[n] = x[n + N], \tag{1.8}$$

for all $n \in \mathbb{Z}$.

Signals do not satisfy the periodicity conditions are called *aperiodic* signals.

Example. Consider the signal $x(t) = \sin(\omega_0 t)$, $\omega_0 > 0$. It can be shown that $x(t) = x(t + T)$, where $T = k\frac{2\pi}{\omega_0}$ for any $k \in \mathbb{Z}^+$:

$$\begin{aligned} x(t+T) &= \sin\left(\omega_0 \left(t + k\frac{2\pi}{\omega_0}\right)\right) \\ &= \sin\left(\omega_0 t + 2\pi k\right) \\ &= \sin(\omega_0 t) = x(t). \end{aligned}$$

Therefore, $x(t)$ is a periodic signal.

Definition 3. *T_0 is called the fundamental period of $x(t)$ if it is the smallest value of $T > 0$ satisfying the periodicity condition. The number $\omega_0 = \frac{2\pi}{T_0}$ is called the fundamental frequency of $x(t)$.*

Definition 4. *N_0 is called the fundamental period of $x[n]$ if it is the smallest value of $N > 0$ where $N \in \mathbb{Z}$ satisfying the periodicity condition. The number $\Omega_0 = \frac{2\pi}{N_0}$ is called the fundamental frequency of $x[n]$.*

Example. Determine the fundamental period of the following signals: (a) $e^{j3\pi t/5}$; (b) $e^{j3\pi n/5}$.
For (a), we let $x(t) = e^{j3\pi t/5}$. If $x(t)$ is a periodic signal, then there exists $T > 0$ such that $x(t) = x(t+T)$. Therefore,

$$\begin{aligned}
& x(t) = x(t+T) \\
\Rightarrow\ & e^{j\frac{3\pi}{5}t} = e^{j\frac{3\pi}{5}(t+T)} \\
\Rightarrow\ & 1 = e^{j\frac{3\pi}{5}T} \\
\Rightarrow\ & e^{j2k\pi} = e^{j\frac{3\pi}{5}T}, \quad \text{for some} k \in \mathbb{Z}^+. \\
\Rightarrow\ & T = \tfrac{10}{3}. \quad (k=1)
\end{aligned}$$

For (b), we let $x[n] = e^{j3\pi n/5}$. If $x[n]$ is a periodic signal, then there exists an integer $N > 0$ such that $x[n] = x[n+N]$. So,

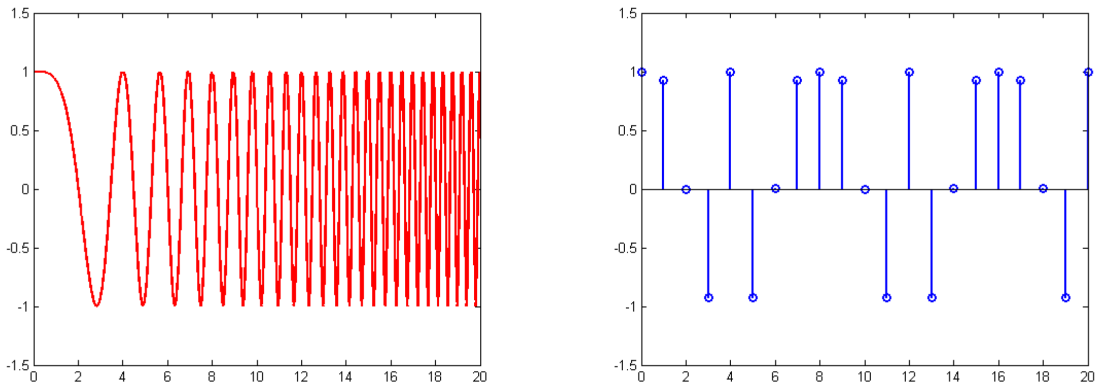
$$\begin{aligned}
& x[n] = x[n+N] \\
\Rightarrow\ & e^{j\frac{3\pi}{5}n} = e^{j\frac{3\pi}{5}(n+N)} \\
\Rightarrow\ & e^{j2k\pi} = e^{j\frac{3\pi}{5}N}, \quad \text{for some} k \in \mathbb{Z}^+ \\
\Rightarrow\ & N = \tfrac{10k}{3} \\
\Rightarrow\ & N = 10. \quad (k=3).
\end{aligned}$$

1.4.2 A More Difficult Example

Consider the following two signals

$$\begin{aligned}
x(t) &= \cos\left(\frac{\pi t^2}{8}\right), \\
x[n] &= \cos\left(\frac{\pi n^2}{8}\right).
\end{aligned}$$

We will show that $x(t)$ is aperiodic whereas $x[n]$ is periodic with fundamental period $N_0 = 8$.



(a) $\cos\left(\frac{\pi t^2}{8}\right)$ (b) $\cos\left(\frac{\pi n^2}{8}\right)$

Figure 1.9: Difference between $x(t) = \cos\left(\frac{\pi t^2}{8}\right)$ and $x[n] = \cos\left(\frac{\pi n^2}{8}\right)$. Note that $x(t)$ is aperiodic, whereas $x[n]$ is periodic.

Fig. 1.9 plots the signals

$$x(t) = \cos\left(\frac{\pi t^2}{8}\right)$$

for $-8 \leq t \leq 8$ and

$$x[n] = \cos\left(\frac{\pi n^2}{8}\right)$$

for $n = -8, -7, \ldots, 8$. It is clear that $x(t)$ is aperiodic, since the values of $t > 0$ for which $x(t) = 0$ form a sequence which the difference between consecutive elements is monotonically decreasing.

On the other hand, $x[n]$ is periodic, with fundamental period $N_0 = 8$. To see this, consider the periodicity condition $x[n] = x[n+N]$, which becomes:

$$\cos\left(\pi(n+N)^2/8\right) = \cos\left(\pi n^2/8\right),$$

for all $n \in \mathbb{Z}$. This means

$$\frac{\pi(n+N)^2}{8} = \frac{\pi n^2}{8} + 2\pi k,$$

for some $k \in \mathbb{Z}$, where k may depend on a particular time instant n. We can simplify this condition by dividing both sides of the equation by $\pi/8$ to yield

$$(n+N)^2 = n^2 + \frac{8}{\pi}(2\pi k),$$

or

$$n^2 + 2nN + N^2 = n^2 + 16k,$$

implying

$$2nN + N^2 = 16k,$$

for some $k \in \mathbb{Z}$. Next, we want to find an N such that $2nN + N^2$ is divisible by 16 for all $n \in \mathbb{Z}$. Now we claim: $N = 8$ satisfies this condition, and no smaller $N > 0$ does.

Setting $N = 8$, we get

$$2nN + N^2 = 16n + 64,$$

which, for any $n \in \mathbb{Z}$, is clearly divisible by 16. So $N = 8$ is a period of $x[n]$. You can check directly that, for any $1 \leq N < 8$, there is a value $n \in \mathbb{Z}$ such that $2nN + N^2$ is not divisible by 16. For example if we consider $N = 4$, we get

$$2nN + N^2 = 8n + 16,$$

which, for $n = 1$, is not divisible by 16. So $N = 4$ is not a period of $x[n]$.

1.4.3 Periodicity and Scaling

1. Suppose $x(t)$ is periodic, and let $a > 0$. Is $y(t) = x(at)$ periodic?
   Yes, and if T_0 is the fundamental period of $x(t)$, then T_0/a is the fundamental period of $y(t)$.

2. Suppose $x[n]$ is periodic, and let $m \in \mathbb{Z}^+$. Is $y[n] = x[mn]$ periodic?
   Yes, and if N_0 is the fundamental period of $x[n]$, then the fundamental period N of $y[n]$ is the smallest positive integer such that mN is divisible by N_0, i.e.

   $$mN \equiv 0 \pmod{N_0}.$$

   Example 1: $N_0 = 8$, $m = 2$, then $N = 4$.
   Example 2: $N_0 = 6$, $m = 4$, then $N = 3$.

1.5 Even and Odd Signals

1.5.1 Definitions

Definition 5. *A continuous-time signal $x(t)$ is* even *if*

$$x(-t) = x(t) \tag{1.9}$$

and it is odd *if*

$$x(-t) = -x(t). \tag{1.10}$$

Definition 6. *A discrete-time signal $x[n]$ is* even *if*

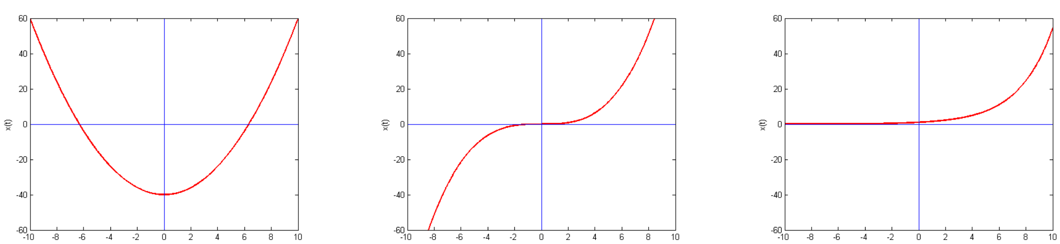
$$x[-n] = x[n] \tag{1.11}$$

and odd *if*

$$x[-n] = -x[n]. \tag{1.12}$$

Remark: The all-zero signal is both even and odd. Any other signal cannot be both even and odd, but may be neither. The following simple example illustrate these properties.

Example 1: $x(t) = t^2 - 40$ is even.
Example 2: $x(t) = 0.1t^3$ is odd.
Example 3: $x(t) = e^{0.4t}$ is neither even nor odd.



(a) $x(t) = t^2 - 40$ (b) $x(t) = 0.1t^3$ (c) $x(t) = e^{0.4t}$

Figure 1.10: Illustrations of odd and even functions. (a) Even; (b) Odd; (c) Neither.

1.5.2 Decomposition Theorem

Theorem 2. *Every continuous-time signal $x(t)$ can be expressed as:*

$$x(t) = y(t) + z(t),$$

where $y(t)$ is even, and $z(t)$ is odd.

Proof. Define

$$y(t) = \frac{x(t) + x(-t)}{2}$$

and

$$z(t) = \frac{x(t) - x(-t)}{2}.$$

Clearly $y(-t) = y(t)$ and $z(-t) = -z(t)$. We can also check that $x(t) = y(t)+z(t)$. □

Terminology: The signal $y(t)$ is called the even part of $x(t)$, denoted by $\mathcal{E}v\{x(t)\}$. The signal $z(t)$ is called the odd part of $x(t)$, denoted by $\mathcal{O}dd\{x(t)\}$.

Example: Let us consider the signal $x(t) = e^t$.

$$\mathcal{E}v\{x(t)\} = \frac{e^t + e^{-t}}{2} = \cosh(t).$$
$$\mathcal{O}dd\{x(t)\} = \frac{e^t - e^{-t}}{2} = \sinh(t).$$

Similarly, we can define even and odd parts of a discrete-time signal $x[n]$:

$$\mathcal{E}v\{x[n]\} = \frac{x[n] + x[-n]}{2}$$
$$\mathcal{O}dd\{x[n]\} = \frac{x[n] - x[-n]}{2}.$$

It is easy to check that

$$x[n] = \mathcal{E}v\{x[n]\} + \mathcal{O}dd\{x[n]\}$$

Theorem 3. *The decomposition is unique, i.e., if*

$$x[n] = y[n] + z[n],$$

then $y[n]$ is even and $z[n]$ is odd if and only if $y[n] = \mathcal{E}v\{x[n]\}$ and $z[n] = \mathcal{O}dd\{x[n]\}$.

Proof. If $y[n]$ is even and $z[n]$ is odd, then

$$x[-n] = y[-n] + z[-n] = y[n] - z[n].$$

Therefore,

$$x[n] + x[-n] = (y[n] + z[n]) + (y[n] - z[n]) = 2y[n],$$

implying $y[n] = \frac{x[n]+x[-n]}{2} = \mathcal{E}v\{x[n]\}$. Similarly $z[n] = \frac{x[n]-x[-n]}{2} = \mathcal{O}dd\{x[n]\}$.

The converse is trivial by definition, as $\mathcal{E}v\{x[n]\}$ must be even and $\mathcal{O}dd\{x[n]\}$ must be odd. □

1.6 Impulse and Step Functions

1.6.1 Discrete-time Impulse and Step Functions

Definition 7. *The discrete-time unit impulse signal $\delta[n]$ is defined as*

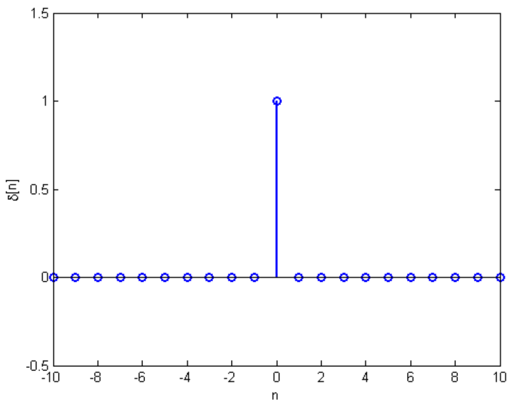
$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{1.13}$$

Definition 8. *The discrete-time unit step signal $\delta[n]$ is defined as*

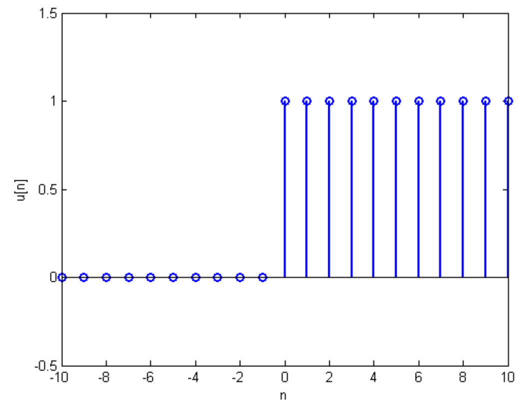
$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{1.14}$$

It can be shown that

- $\delta[n] = u[n] - u[n-1]$
- $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$
- $u[n] = \sum_{k=-\infty}^{\infty} u[k]\delta[n-k]$.



(a) $\delta[n]$



(b) $u[n]$

Figure 1.11: Definitions of impulse function and a step function.

1.6.2 Property of $\delta[n]$

Sampling Property

By the definition of $\delta[n]$, $\delta[n-n_0] = 1$ if $n = n_0$, and 0 otherwise. Therefore,

$$x[n]\delta[n-n_0] = \begin{cases} x[n], & n = n_0 \\ 0, & n \neq n_0 \end{cases}$$
$$= x[n_0]\delta[n-n_0]. \tag{1.15}$$

As a special case when $n_0 = 0$, we have $x[n]\delta[n] = x[0]\delta[n]$. Pictorially, when a signal $x[n]$ is multiplied with $\delta[n]$, the output is a unit impulse with amplitude $x[0]$.

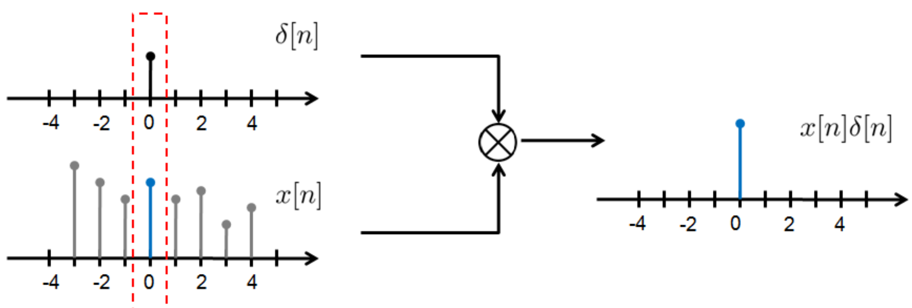


Figure 1.12: Illustration of $x[n]\delta[n] = x[0]\delta[n]$.

Shifting Property

Since $x[n]\delta[n] = x[0]\delta[n]$ and $\sum_{n=-\infty}^{\infty} \delta[n] = 1$, we have

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n] = \sum_{n=-\infty}^{\infty} x[0]\delta[n] = x[0] \sum_{n=-\infty}^{\infty} \delta[n] = x[0],$$

and similarly

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = \sum_{n=-\infty}^{\infty} x[n_0]\delta[n-n_0] = x[n_0]. \tag{1.16}$$

In general, the following result holds:

$$\sum_{n=a}^{b} x[n]\delta[n-n_0] = \begin{cases} x[n_0], & \text{if } n_0 \in [a,\ b] \\ 0, & \text{if } n_0 \notin [a,\ b] \end{cases} \tag{1.17}$$

Representation Property

Using the sampling property, it holds that

$$x[k]\delta[n-k] = x[n]\delta[n-k].$$

Summing the both sides over the index k yields

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = \sum_{k=-\infty}^{\infty} x[n]\delta[n-k] = x[n] \sum_{k=-\infty}^{\infty} \delta[n-k] = x[n].$$

This result shows that every discrete-time signal $x[n]$ can be represented as a linear combination of shifted unit impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \tag{1.18}$$

For example, the unit step function can be expressed as

$$u[n] = \sum_{k=-\infty}^{\infty} u[k]\delta[n-k].$$

Why do we use these complicated representation of $x[n]$? Because, when we consider linear time-invariant systems (Chapter 2), it will allow us to determine the system response to any signal $x[n]$ from the impulse response.

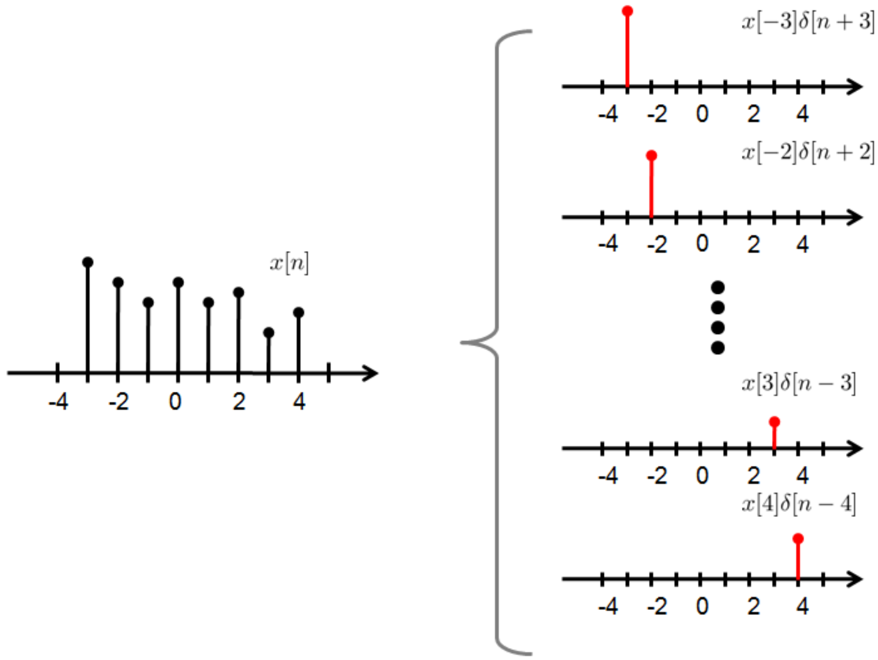


Figure 1.13: Representing of a signal $x[n]$ using a train of impulses $\delta[n-k]$.

1.6.3 Continuous-time Impulse and Step Functions

Definition 9. *The dirac delta function is defined as*

$$\delta(t) = \begin{cases} 0, & \text{if } t \neq 0 \\ \infty, & \text{if } t = 0 \end{cases},$$

where

$$\int_{-\infty}^{\infty} \delta(t)dt = 1.$$

Definition 10. *The unit step function is defined as*

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

1.6.4 Property of $\delta(t)$

The properties of $\delta(t)$ are analogous to the discrete-time case.

Sampling Property

$$x(t)\delta(t) = x(0)\delta(t). \tag{1.19}$$

To see this, note that $x(t)\delta(t) = x(0)$ when $t = 0$ and $x(t)\delta(t) = 0$ when $t \neq 0$. Similarly, we have

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0), \tag{1.20}$$

for any $t_0 \in \mathbb{R}$.

Shifting Property

The shifting property follows from the sampling property. Integrating $x(t)\delta(t)$ yields

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = \int_{-\infty}^{\infty} x(0)\delta(t)dt = x(0)\int_{-\infty}^{\infty} \delta(t)dt = x(0). \tag{1.21}$$

Similarly, one can show that

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0) \tag{1.22}$$

Representation Property

The representation property is also analogous to the discrete-time case:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau, \tag{1.23}$$

where the special case of $u(t)$ is given by

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t - \tau)d\tau.$$

As an example of the properties, let us consider $\frac{d}{dt}u(t)$.

$$\begin{aligned} u(t) &= \int_{-\infty}^{\infty} u(\tau)\delta(t - \tau)d\tau, \quad \text{(representation property)} \\ &= \int_{0}^{\infty} \delta(t - \tau)d\tau, \quad \text{because } u(\tau) = 0 \text{ if } \tau \leq 0 \\ &= \int_{-\infty}^{t} \delta(\sigma)d\sigma, \quad \text{let } \sigma = t - \tau. \end{aligned}$$

Then by fundamental theorem of calculus, we have

$$\frac{d}{dt}u(t) = \frac{d}{dt}\int_{-\infty}^{t} \delta(\sigma)d\sigma = \delta(t). \tag{1.24}$$

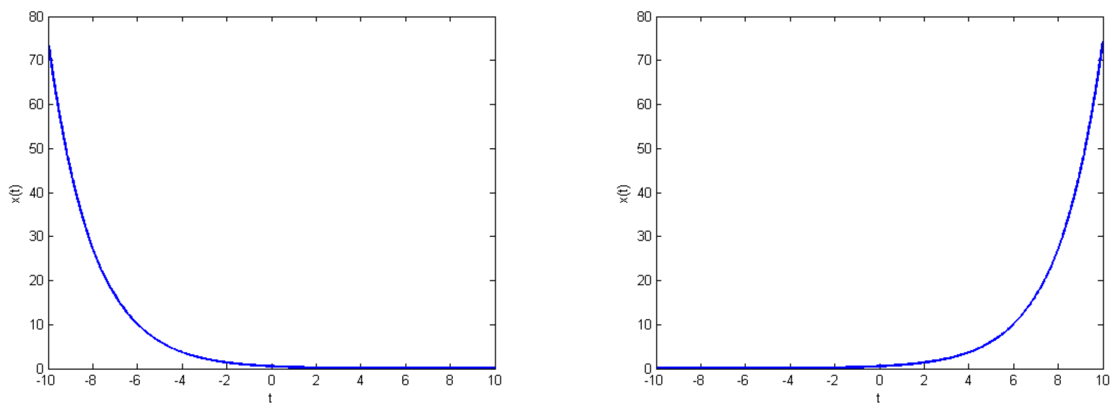
1.7 Continuous-time Complex Exponential Functions

Definition 11. *A complex exponential function is defined as*

$$x(t) = Ce^{at}, \qquad \text{where } C, a \in \mathbb{C} \tag{1.25}$$

1.7.1 Real-valued Exponential

We first consider the case of real-valued exponential functions, i.e., $C \in \mathbb{R}$ and $a \in \mathbb{R}$.



(a) $x(t) = \frac{1}{2}e^{-\frac{1}{2}t}$ (b) $x(t) = \frac{1}{2}e^{\frac{1}{2}t}$

Figure 1.14: Real exponential functions.

When $a = 0$, then $x(t) = C$, which is a constant function.

1.7.2 Periodic Complex Exponential

Let us consider the case where a is purely imaginary, i.e., $a = j\omega_0$, $\omega_0 \in \mathbb{R}$. Since C is a complex number, we have

$$C = Ae^{j\theta},$$

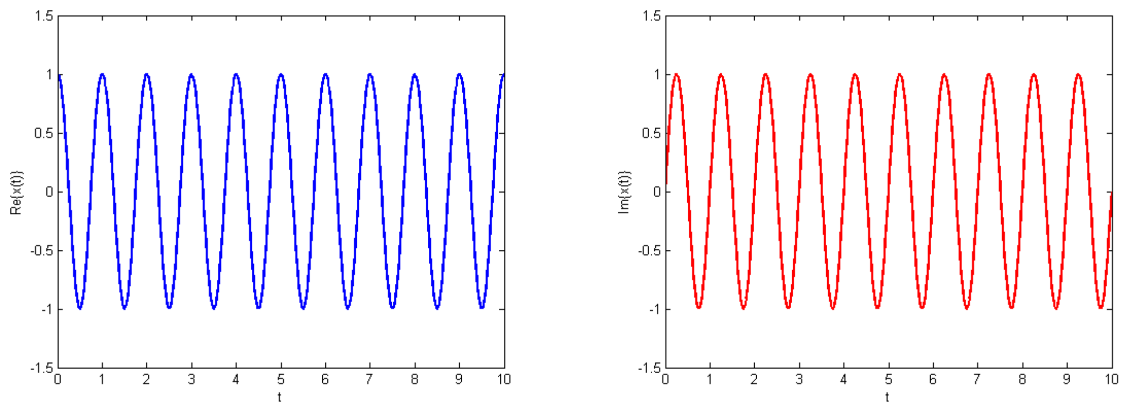
where $A, \theta \in \mathbb{R}$. Consequently,

$$\begin{aligned} x(t) = Ce^{j\omega_0 t} &= Ae^{j\theta}e^{j\omega_0 t} \\ &= Ae^{j(\omega_0 t + \theta)} = A\cos(\omega_0 t + \theta) + jA\sin(\omega_0 t + \theta). \end{aligned}$$

The real and imaginary parts of $x(t)$ are given by

$$\mathcal{R}e\{x(t)\} = A\cos(\omega_0 t + \theta)$$
$$\mathcal{I}m\{x(t)\} = A\sin(\omega_0 t + \theta).$$

We can think of $x(t)$ as a *pair* of sinusoidal signals of the same amplitude A, ω_0 and phase shift θ with one a cosine and the other a sine.



(a) $\mathcal{R}e\{Ce^{j\omega_0 t}\}$ (b) $\mathcal{I}m\{Ce^{j\omega_0 t}\}$

Figure 1.15: Periodic complex exponential function $x(t) = Ce^{j\omega_0 t}$ ($C = 1$, $\omega_0 = 2\pi$).

Claims. $x(t) = Ce^{j\omega_0 t}$ is periodic with

1. fundamental period: $T_0 = \frac{2\pi}{|\omega_0|}$.

2. fundamental frequency: $|\omega_0|$.

The second claim is the immediate result from the first claim. To show the first claim, we need to show $x(t+T_0) = x(t)$ and no smaller T_0 can satisfy the periodicity criteria.

$$\begin{aligned} x(t+T_0) &= Ce^{j\omega_0(t+\frac{2\pi}{|\omega_0|})} = Ce^{j\omega_0 t}e^{\pm j2\pi} \\ &= Ce^{j\omega_0 t} = x(t). \end{aligned}$$

It is easy to show that T_0 is the smallest period!

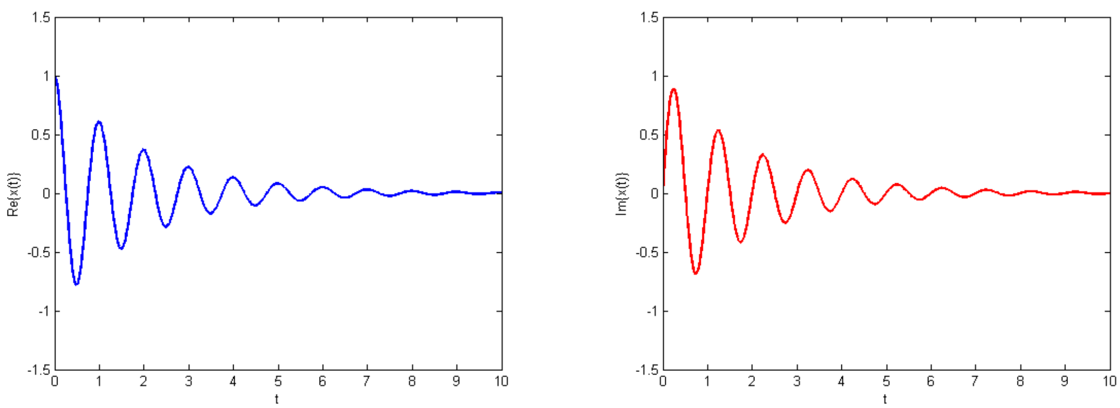
1.7.3 General Complex Exponential

In the general setting, we have $C = Ae^{j\theta}$ where $A, \theta \in \mathbb{R}$ and $a = r + j\omega_0$ where $r, \omega_0 \in \mathbb{R}$. Therefore,

$$x(t) = \left(Ae^{jr}\right) e^{(r+j\omega_0)t} = Ae^{rt}e^{j(\omega_0 t+\theta)}.$$

Rewrite this in the rectangular form:

$$x(t) = Ae^{rt}\cos(\omega_0 t + \theta) + jAe^{rt}\sin(\omega_0 t + \theta)$$



(a) $\mathcal{R}e\{Ae^{rt}e^{j\omega_0 t}\}$ (b) $\mathcal{I}m\{Ae^{rt}e^{j\omega_0 t}\}$

Figure 1.16: Periodic complex exponential function $x(t) = Ae^{rt}e^{j\omega_0 t}$ ($A = 1$, $r = -1/2$, $\omega_0 = 2\pi$).

1.8 Discrete-time Complex Exponentials

1.8.1 Definitions

A discrete-time complex exponential function has the form:

$$x[n] = Ce^{\beta n},$$

where $C, \beta \in \mathbb{C}$. Letting $\alpha = e^{\beta}$ gives

$$x[n] = C\alpha^n.$$

In the following subsections, we shall study the behavior of $x[n]$ for difference cases of C and α.

1.8.2 Real-valued Complex Exponential

$x[n]$ is a real-valued complex exponential when $C \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. In this case, $x[n] = C\alpha^n$ is a monotonic decreasing function when $0 < \alpha < 1$, and is a monotonic increasing function when $\alpha > 1$.

1.8.3 Complex-valued Complex Exponential

$x[n]$ is a complex-valued complex exponential when $C, \alpha \in \mathbb{C}$. In this case, C and α can be written as $C = |C|e^{j\theta}$, and $\alpha = |\alpha|e^{j\Omega_0}$. Consequently,

$$
\begin{aligned}
x[n] = C\alpha^n &= |C|e^{j\theta}\left(|\alpha|e^{j\Omega_0}\right)^n \\
&= |C||\alpha|^n e^{j(\Omega_0 n+\theta)} \\
&= |C||\alpha|^n \cos(\Omega_0 n + \theta) + j|C||\alpha|^n \sin(\Omega_0 n + \theta).
\end{aligned}
$$

Three cases can be considered here:

1. When $|\alpha| = 1$, then $x[n] = |C|\cos(\Omega_0 n + \theta) + j|C|\sin(\Omega_0 n + \theta)$ and it has sinusoidal real and imaginary parts (not necessarily periodic, though).
2. When $|\alpha| > 1$, then $|\alpha|^n$ is a growing exponential, so the real and imaginary parts of $x[n]$ are the product of this with sinusoids.
3. When $|\alpha| < 1$, then the real and imaginary parts of $x[n]$ are sinusoids sealed by a decaying exponential.

1.8.4 Periodic Complex Exponentials

Consider $x[n] = Ce^{j\Omega_0 n}$, $\Omega_0 \in \mathbb{R}$. We want to study the condition for $x[n]$ to be periodic. The periodicity condition requires that, for some $N > 0$,

$$x[n+N] = x[n], \qquad \forall n \in \mathbb{Z}.$$

Since $x[n] = Ce^{j\Omega_0 n}$, it holds that

$$e^{j\Omega_0(n+N)} = e^{j\Omega_0 n}e^{j\Omega_0 N} = e^{j\Omega_0 n}, \qquad \forall n \in \mathbb{Z}.$$

This is equivalent to

$$e^{j\Omega_0 N} = 1 \qquad \text{or} \qquad \Omega_0 N = 2\pi m, \qquad \text{for some } m \in \mathbb{Z}.$$

Therefore, the condition for periodicity of $x[n]$ is

$$\Omega_0 = \frac{2\pi m}{N}, \qquad \text{for some } m \in \mathbb{Z}, \text{ and some } N > 0, \ N \in \mathbb{Z}.$$

Thus $x[n] = e^{j\Omega_0 n}$ is periodic if and only if Ω_0 is a rational multiple of 2π. The fundamental period is

$$N = \frac{2\pi m}{\Omega_0},$$

where we assume that m and N are relatively prime, $\gcd(m, n) = 1$, i.e., $\frac{m}{N}$ is in reduced form.

1.8.5 Periodicity in Frequency

Suppose that $x[n] = e^{j\Omega_0 n}$, where $\Omega_0 \in \mathbb{R}$. If we increase Ω_0 by 2π, we find

$$x_1[n] = e^{j(\Omega_0 + 2\pi)n} = e^{j\Omega_0 n} e^{j2\pi n}.$$

But $n \in \mathbb{Z}$, so $e^{j2\pi n} = 1$, and we see that

$$x_1[n] = e^{j\Omega_0 n} = x[n].$$

More generally, for any $k \in \mathbb{Z}$, we have

$$x_k[n] = e^{j(\Omega_0 + 2\pi k)n} = e^{j\Omega_0 n} = x[n].$$

This means that we can limit the range of values of Ω_0 to any real interval of length 2π. The periodicity in frequency applies, of course, to the periodic complex exponential signals, so we have a different notion of low and high frequencies in the discrete-time setting.

Chapter 2

Fundamentals of Systems

A system is a quantitative description of a physical process which transforms signals (at its "input") to signals (at its "output"). More precisely, a system is a "black box" (viewed as a mathematical abstraction) that deterministically transforms input signals into output signals. In this chapter, we will study the properties of systems.

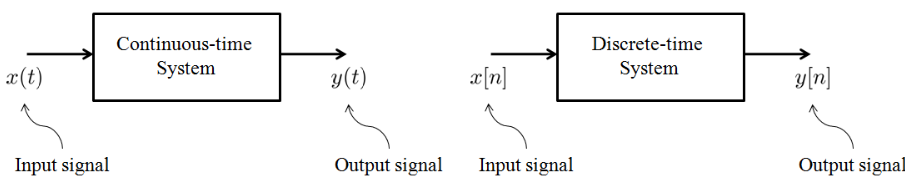


Figure 2.1: Continuous-time and discrete-time systems.

Remarks:

1. We will consider both continuous-time systems and discrete-time systems. The transformation from a continuous-time signal $x(t)$ to a discrete-time signal $x[n]$ will be discussed in Chatper 6.

2. We will focus on single-input single-output systems. Multiple-inputs to multiple-outputs (MIMO) systems are outside the scope of this course.

2.1 System Properties

2.1.1 Memoryless

Definition 12. *A system is memoryless if the output at time t (or n) depends only on the input at time t (or n).*

Examples.

1. $y(t) = (2x(t) - x^2(t))^2$ is memoryless, because $y(t)$ depends on $x(t)$ only. There is no $x(t-1)$, or $x(t+1)$ terms, for example.

2. $y[n] = x[n]$ is memoryless. In fact, this system is passing the input to output directly, without any processing.

3. $y[n] = x[n-1]$ is not memoryless, because the n-th output depends on $n-1$-th input.

4. $y[n] = x[n] + y[n-1]$ is not memoryless. To see this, we consider

$$y[n-1] = x[n-1] + y[n-2].$$

Substituting into $y[n] = x[n] + y[n-1]$ yields

$$y[n] = x[n] + (x[n-1] + y[n-2]).$$

By repeating the calculation, we have

$$\begin{aligned} y[n] &= x[n] + x[n-1] + x[n-2] + \dots \\ &= \sum_{k=-\infty}^{n} x[k]. \end{aligned}$$

Clearly, $y[n]$ depends on more than just $x[n]$.

2.1.2 Invertible

Definition 13. *A system is invertible if distinct input signals produce distinct output signals.*

In other words, a system if invertible if there exists an one-to-one mapping from the set of input signals to the set of output signals.

There are two basic rules of showing an invertible system:

1. To show that a system is invertible, one has to show the inversion formula.

2. To show that a system is not invertible, one has to give a counter example.

Example 1.
The system $y(t) = (\cos(t) + 2)x(t)$ is invertible.

Proof. To show that the system is invertible, we need to find an inversion formula. This is easy: $y(t) = (\cos(t) + 2)x(t)$ implies that (by rearranging terms)

$$x(t) = \frac{y(t)}{\cos(t) + 2},$$

which is the inversion formula. Note that the denominator is always positive, thus the division is valid. □

Example 2.
The system $y[n] = x[n] + y[n-1]$ is invertible.

Proof. $y[n] = x[n] + y[n-1]$ implies that (by rearranging terms)

$$x[n] = y[n] - y[n-1].$$

This is the inversion formula. □

Example 3.
The system $y(t) = x^2(t)$ is not invertible.

Proof. To show that a system is not invertible, we construct a counter example. Let us consider two signals

$$\begin{aligned} x_1(t) &= 1, \quad \forall t \\ x_2(t) &= -1, \quad \forall t. \end{aligned}$$

Clearly $x_1(t) \neq x_2(t)$, but $(x_1(t))^2 = (x_2(t))^2$. Therefore, we have found a counter example such that different inputs give the same output. Hence the system is not invertible. □

2.1.3 Causal

Definition 14. *A system is causal if the output at time t (or n) depends only on inputs at time $s \leq t$ (i.e., the present and past).*

Examples.

1. $y[n] = x[n-1]$ is causal, because $y[n]$ depends on the past sample $x[n-1]$.
2. $y[n] = x[n] + x[n+1]$ is not causal, because $x[n+1]$ is a future sample.
3. $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$ is causal, because the integral evaluates τ from $-\infty$ to t (which are all in the past).
4. $y[n] = x[-n]$ is not causal, because $y[-1] = x[1]$, which means the output at $n = -1$ depends an input in the future.
5. $y(t) = x(t)\cos(t+1)$ causal (and memoryless), because $\cos(t+1)$ is a constant with respect to $x(t)$.

2.1.4 Stable

To describe a stable system, we first need to define the boundedness of a signal.

Definition 15. *A signal $x(t)$ (and $x[n]$) is bounded if there exists a constant $B < \infty$ such that $|x(t)| < B$ for all t.*

Definition 16. *A system is stable if a bounded input input always produces a bounded output signal. That is, if $|x(t)| \leq B$ for some $B < \infty$, then*

$$|y(t)| < \infty.$$

Example 1.
The system $y(t) = 2x^2(t-1) + x(3t)$ is stable.

Proof. To show the system is stable, let us consider a bounded signal $x(t)$, that is, $|x(t)| \leq B$ for some $B < \infty$. Then

$$\begin{aligned}
|y(t)| &= |2x^2(t-1) + x(3t)| \\
&\leq |2x^2(t-1)| + |x(3t)| \qquad \text{, by Triangle Inequality} \\
&\leq 2|x^2(t-1)| + |x(3t)| \\
&\leq 2B^2 + B < \infty.
\end{aligned}$$

Therefore, for any bounded input $x(t)$, the output $y(t)$ is always bounded. Hence the system is stable. □

Example 2.
The system $y[n] = \sum\limits_{k=-\infty}^{n} x[k]$ is not stable.

Proof. To show that the system $y[n] = \sum_{k=-\infty}^{n} x[k]$ is not stable, we can construct a bounded input signal $x[n]$ and show that the output signal $y[n]$ is not bounded.

Let $x[n] = u[n]$. It is clear that $|x[n]| \leq 1$ (i.e., bounded). Consequently,

$$\begin{aligned} |y[n]| &= \left| \sum_{k=-\infty}^{n} u[k] \right| \\ &= \sum_{k=0}^{n} u[k] \\ &\leq \sum_{k=0}^{n} 1 = n+1, \end{aligned}$$

which approaches ∞ as $n \to \infty$. Therefore, $|y[n]|$ is not bounded. □

2.1.5 Time-invariant

Definition 17. *A system is time-invariant if a time-shift of the input signal results in the same time-shift of the output signal. That is, if*

$$x(t) \longrightarrow y(t),$$

then the system is time-invariant if

$$x(t - t_0) \longrightarrow y(t - t_0),$$

for any $t_0 \in \mathbb{R}$.

Fig. 2.2 illustrates an interpretation of a time-invariant system: If a signal $x(t)$ is input to a time-invariant system and get an output $y(t)$, then the input $x(t - t_0)$ will result an output $y(t - t_0)$.

Example 1.
The system $y(t) = \sin[x(t)]$ is time-invariant.

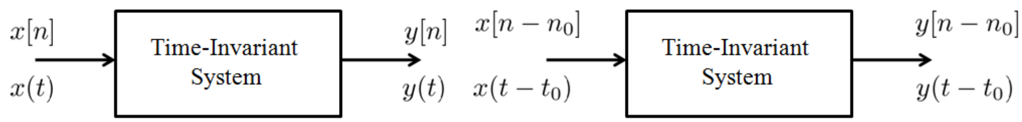


Figure 2.2: Illustration of a time-invariant system.

Proof. Let us consider a time-shifted signal $x_1(t) = x(t-t_0)$. Correspondingly, we let $y_1(t)$ be the output of $x_1(t)$. Therefore,

$$y_1(t) = \sin[x_1(t)] = \sin[x(t-t_0)].$$

Now, we have to check whether $y_1(t) = y(t-t_0)$. To show this, we note that

$$y(t-t_0) = \sin[x(t-t_0)],$$

which is the same as $y_1(t)$. Therefore, the system is time-invariant. $\square$

Example 2.
The system $y[n] = nx[n]$ is not time-invariant.

Proof. To show that the system in not time-invariant, we can construct a counter example. Let $x[n] = \delta[n]$, then $y[n] = n\delta[n] = 0$, $\forall n$ (Why?). Now, let $x_1[n] = x[n-1] = \delta[n-1]$. If $y_1[n]$ is the output produced by $x_1[n]$, it is easy to show that

$$\begin{aligned} y_1[n] &= nx_1[n] \\ &= n\delta[n-1] \\ &= \delta[n-1]. \qquad \text{(Why?)} \end{aligned}$$

However, $y[n-1] = (n-1)x[n-1] = (n-1)\delta[n-1] = 0$ for all n. So $y_1[n] \neq y[n-1]$. In other words, we have constructed an example such that $y[n-1]$ is not the output of $x[n-1]$. $\square$

2.1.6 Linear

Definition 18. *A system is linear if it is additive and scalable. That is,*

$$ax_1(t) + bx_2(t) \longrightarrow ay_1(t) + by_2(t),$$

for all $a, b \in \mathbb{C}$.

Example 1.
The system $y(t) = 2\pi x(t)$ is linear. To see this, let's consider a signal

$$x(t) = ax_1(t) + bx_2(t),$$

where $y_1(t) = 2\pi x_1(t)$ and $y_2(t) = 2\pi x_2(t)$. Then

$$\begin{aligned} ay_1(t) + by_2(t) &= a\,(2\pi x_1(t)) + b\,(2\pi x_2(t)) \\ &= 2\pi\,[ax_1(t) + bx_2(t)] = 2\pi x(t) = y(t). \end{aligned}$$

Example 2.
The system $y[n] = (x[2n])^2$ is not linear. To see this, let's consider the signal

$$x[n] = ax_1[n] + bx_2[n],$$

where $y_1[n] = (x_1[2n])^2$ and $y_2[n] = (x_2[2n])^2$. We want to see whether $y[n] = ay_1[n] + by_2[n]$. It holds that

$$ay_1[n] + by_2[n] = a\,(x_1[2n])^2 + b\,(x_2[2n])^2\,.$$

However,

$$y[n] = (x[2n])^2 = (ax_1[2n] + bx_2[2n])^2 = a^2(x_1[2n])^2 + b^2(x_2[2n])^2 + 2abx_1[n]x_2[n].$$

2.2 Convolution

2.2.1 What is Convolution?

Linear time invariant (LTI) systems are good models for many real-life systems, and they have properties that lead to a very powerful and effective theory for analyzing their behavior. In the followings, we want to study LTI systems through its characteristic function, called the impulse response.

To begin with, let us consider discrete-time signals. Denote by $h[n]$ the "impulse response" of an LTI system S. The impulse response, as it is named, is the response of the system to a unit impulse input. Recall the definition of an unit impulse:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases} \tag{2.1}$$

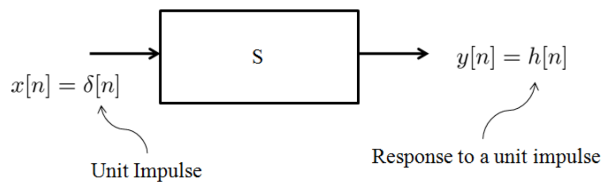


Figure 2.3: Definition of an impulse response

We have shown that

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]. \tag{2.2}$$

Using this fact, we get the following equalities:

$$\begin{array}{lcll}
x[n]\delta[n] & = & x[0]\delta[n] & (n_0 = 0) \\
x[n]\delta[n-1] & = & x[1]\delta[n-1] & (n_0 = 1) \\
x[n]\delta[n-2] & = & x[2]\delta[n-2] & (n_0 = 2) \\
\vdots & & \vdots & \\
\underbrace{\qquad\qquad}_{=x[n]\left(\sum_{k=-\infty}^{\infty}\delta[n-k]\right)} & & \underbrace{\qquad\qquad}_{=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]} &
\end{array}$$

The sum on the left hand side is

$$x[n]\left(\sum_{k=-\infty}^{\infty}\delta[n-k]\right) = x[n],$$

because $\sum_{k=-\infty}^{\infty}\delta[n-k] = 1$ for all n. The sum on the right hand side is

$$\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

Therefore, equating the left hand side and right hand side yields

$$\boxed{x[n] = \sum_{k=-\infty}^{\infty}x[k]\delta[n-k]} \tag{2.3}$$

In other words, for any signal $x[n]$, we can always express it as a sum of impulses!

Next, suppose we know that the impulse response of an LTI system is $h[n]$. We want to determine the output $y[n]$. To do so, we first express $x[n]$ as a sum of impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

For each impulse $\delta[n-k]$, we can determine its impulse response, because for an LTI system:

$$\delta[n-k] \longrightarrow h[n-k].$$

Consequently, we have

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k] = y[n].$$

This equation,

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad (2.4)}$$

is known as the convolution equation.

2.2.2 Definition and Properties of Convolution

Definition 19. *Given a signal $x[n]$ and the impulse response of an LTI system $h[n]$, the convolution between $x[n]$ and $h[n]$ is defined as*

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

We denote convolution as $y[n] = x[n] * h[n]$.

- Equivalent form: Letting $m = n - k$, we can show that

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{m=-\infty}^{\infty} x[n-m]h[m] = \sum_{k=-\infty}^{\infty} x[n-k]h[k].$$

- Convolution is true only when the system is LTI. If the system is time-varying, then
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n-k].$$
i.e., $h[n]$ is different at every time instant k.

The following "standard" properties can be proved easily:

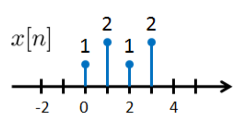
1. Commutative: $x[n] * h[n] = h[n] * x[n]$
2. Associative: $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$
3. Distributive: $x[n] * (h_1[n] + h_2[n]) = (x(t) * h_1[n]) + (x[n] * h_2[n])$

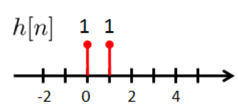
2.2.3 How to Evaluate Convolution?

To evaluate convolution, there are three basic steps:

1. Flip
2. Shift
3. Multiply and Add

Example 1. (See Class Demonstration) Consider the signal $x[n]$ and the impulse response $h[n]$ shown below.





Let's compute the output $y[n]$ one by one. First, consider $y[0]$:

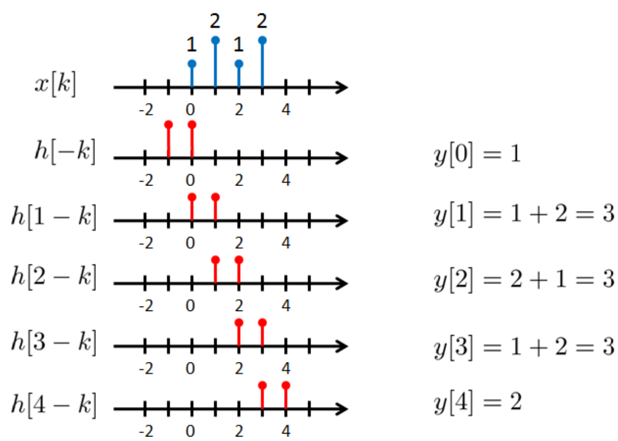
$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = \sum_{k=-\infty}^{\infty} x[k]h[-k] = 1.$$

Note that $h[-k]$ is the flipped version of $h[k]$, and $\sum_{k=-\infty}^{\infty} x[k]h[-k]$ is the multiply-add between $x[k]$ and $h[-k]$.

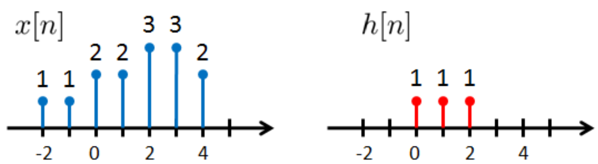
To calculate $y[1]$, we flip $h[k]$ to get $h[-k]$, shift $h[-k]$ go get $h[1-k]$, and multiply-add to get $\sum_{k=-\infty}^{\infty} x[k]h[1-k]$. Therefore,

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 1 \times 1 + 2 \times 1 = 3.$$

Pictorially, the calculation is shown in the figure below.



Example 2. (See Class Demonstration)



Example 3. (See Class Demonstration)

$$x[n] = \left(\frac{1}{2}\right)^n u[n],$$

and

$$h[n] = \delta[n] + \delta[n-1].$$

2.3 System Properties and Impulse Response

With the notion of convolution, we can now proceed to discuss the system properties in terms of impulse responses.

2.3.1 Memoryless

A system is memoryless if the output depends on the current input only. An equivalent statement using the impulse response $h[n]$ is that:

Theorem 4. *An LTI system is memoryless if and only if*

$$h[n] = a\delta[n], \textit{for some } a \in \mathbb{C}. \tag{2.5}$$

Proof. If $h[n] = a\delta[n]$, then for any input $x[n]$, the output is

$$\begin{aligned} y[n] = x[n] * h[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[k]a\delta[n-k] \\ &= ax[n]. \end{aligned}$$

So, the system is memoryless. Conversely, if the system is memoryless, then $y[n]$ cannot depend on the values $x[k]$ for $k \neq n$. Looking at the convolution sum formula

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k],$$

we conclude that

$$h[n-k] = 0, \quad \text{for all } k \neq n,$$

or equivalently,

$$h[n] = 0,, \quad \text{for all } n \neq 0.$$

This implies

$$y[n] = x[n]h[0] = ax[n],$$

where we have set $a = h[0]$. □

2.3.2 Invertible

Theorem 5. *An LTI system is invertible if and only if there exist $g[n]$ such that*

$$h[n] * g[n] = \delta[n]. \tag{2.6}$$

Proof. If a system S is invertible, then $x_1[n] \neq x_2[n]$ implies $y_1[n] \neq y_2[n]$. So there exists an injective mapping (one-to-one map) S such that $y[n] = S(x[n]) = h[n]*x[n]$. Since f is injective, there exists an inverse mapping S^{-1} such that

$$S^{-1}(S(x[n])) = x[n],$$

for any $x[n]$. Therefore, there exists $g[n]$ such that

$$g[n] * (h[n] * x[n]) = x[n].$$

By associativity of convolution, we have $(g[n]*h[n])*x[n] = x[n]$, implying $g[n]*h[n] = \delta[n]$.

Conversely, if there exist $g[n]$ such that $h[n] * g[n] = \delta[n]$, then for any $x_1[n] \neq x_2[n]$, we have

$$\begin{aligned} y_1[n] &= h[n] * x_1[n] \\ y_2[n] &= h[n] * x_2[n], \end{aligned}$$

and $y_1[n] \neq y_2[n]$. Taking the difference between $y_1[n]$ and $y_2[n]$, we have

$$y_1[n] - y_2[n] = h[n] * \{x_1[n] - x_2[n]\}.$$

Convolving both sides by $g[n]$ yields

$$g[n] * (y_1[n] - y_2[n]) = \delta[n] * (x_1[n] - x_2[n]).$$

Since $x_1[n] \neq x_2[n]$, and $g[n] \neq 0$ for all n, we must have $y_1[n] \neq y_2[n]$. Therefore, the system is invertible. □

2.3.3 Causal

Theorem 6. *An LTI system is causal if and only if*

$$h[n] = 0, \quad \textit{for all } n < 0. \tag{2.7}$$

Proof. If S is causal, then the output $y[n]$ cannot depend on $x[k]$ for $k > n$. From the convolution equation,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k],$$

we must have

$$h[n-k] = 0, \quad \text{for } k > n, \quad \text{or equivalently} \quad h[n-k] = 0, \quad \text{for } n-k < 0.$$

Setting $m = n - k$, we see that

$$h[m] = 0, \text{for } m < 0.$$

Conversely, if $h[k] = 0$ for $k < 0$, then for input $x[n]$,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k].$$

Therefore, $y[n]$ depends only upon $x[m]$ for $m \le n$. $\square$

2.3.4 Stable

Theorem 7. *An LTI system is stable if and only if*

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty.$$

Proof. Suppose that $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$. For any bounded signal $|x[n]| \le B$, the output is

$$\begin{aligned} |y[n]| &\le \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right| \\ &= \sum_{k=-\infty}^{\infty} |x[k]| \cdot |h[n-k]| \\ &\le B \cdot \sum_{k=-\infty}^{\infty} |h[n-k]| . \end{aligned}$$

Therefore, $y[n]$ is bounded.

Conversely, suppose that $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$. We want to show that $y[n]$ is not bounded. Define a signal

$$x[n] = \begin{cases} 1, & h[-n] > 0, \\ -1, & h[-n] < 0. \end{cases}$$

Clearly, $|x[n]| \leq 1$ for all n. The output $y[0]$ is given by

$$\begin{aligned}
|y[0]| &= \left| \sum_{k=-\infty}^{\infty} x[k]h[-k] \right| \\
&= \left| \sum_{k=-\infty}^{\infty} |h[-k]| \right| \\
&= \sum_{k=-\infty}^{\infty} |h[-k]| = \infty.
\end{aligned}$$

Therefore, $y[n]$ is not bounded. □

2.4 Continuous-time Convolution

Thus far we have been focusing on the discrete-time case. The continuous-time case, in fact, is analogous to the discrete-time case. In continuous-time signals, the signal decomposition is

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau, \tag{2.8}$$

and consequently, the continuous time convolution is defined as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \tag{2.9}$$

Example.
The continuous-time convolution also follows the three step rule: flip, shift, multiply-add. To see an example, let us consider the signal $x(t) = e^{-at}u(t)$ for $a > 0$, and impulse response $h(t) = u(t)$. The output $y(t)$ is
Case A: $t > 0$:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau) \\
&= \int_{0}^{t} e^{-a\tau}d\tau \\
&= \frac{1}{-a}\left[1-e^{-at}\right].
\end{aligned}$$

Case B: $t \leq 0$:

$$y(t) = 0.$$

Therefore,

$$y(t) = \frac{1}{a}\left[1 - e^{-at}\right] u(t).$$

2.4.1 Properties of CT Convolution

The following properties can be proved easily:

1. Commutative: $x(t) * h(t) = h(t) * x(t)$
2. Associative: $x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$
3. Distributive: $x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$

2.4.2 Continuous-time System Properties

The following results are analogous to the discrete-time case.

Memoryless.
An LTI system is memoryless if and only if

$$h(t) = a\delta(t), \text{for some } a \in \mathbb{C}.$$

Invertible.
An LTI system is invertible if and only if there exist $g(t)$ such that

$$h(t) * g(t) = \delta(t).$$

Causal.
A system is causal if and only if

$$h(t) = 0, \qquad \text{for all } t < 0.$$

Stable.
A system is stable if and only if

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty.$$

Chapter 3

Fourier Series

The objective of this chapter is to identify a family of signals $\{x_k(t)\}$ such that:

1. Every signal in the family passes through any LTI system with only a scale change (or other simply described change)

$$x_k(t) \longrightarrow \lambda_k x_k(t),$$

   where λ_k is a scale factor.

2. "Any" signal can be represented as a "linear combination" of signals in their family.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k x_k(t).$$

   This would allow us to determine the output generated by $x(t)$:

$$x(t) \longrightarrow \sum_{k=-\infty}^{\infty} a_k \lambda_k x_k(t),$$

   where the scalar a_k comes from the definition of linear combination.

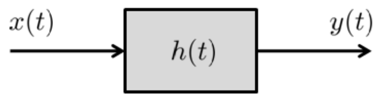
3.1 Eigenfunctions of an LTI System

To answer the first question, we need the notion of eigenfunction of an LTI system.

Definition 20. *For an LTI system, if the output is a scaled version of its input, then the input function is called an eigenfunction of the system. The scaling factor is called the eigenvalue of the system.*

3.1.1 Continuous-time Case

Consider an LTI system with impulse response $h(t)$ and input signal $x(t)$:



Suppose that $x(t) = e^{st}$ for some $s \in \mathbb{C}$, then the output is given by

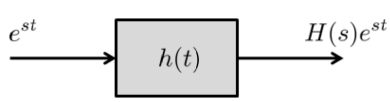
$$\begin{aligned} y(t) = h(t) * x(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \\ &= \left[\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\right] e^{st} = H(s)e^{st} = H(s)x(t), \end{aligned}$$

where $H(s)$ is defined as

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau.$$

The function $H(s)$ is known as the *transfer function* of the continuous-time LTI system. Note that $H(s)$ is defined by the impulse response $h(t)$, and is a function in s (independent of t). Therefore, $H(s)x(t)$ can be regarded as a scalar $H(s)$ multiplied to the function $x(t)$.

From the derivation above, we see that if the input is $x(t) = e^{st}$, then the output is a scaled version $y(t) = H(s)e^{st}$:



Therefore, using the definition of eigenfunction, we show that

1. e^{st} is an eigenfunction of any continuous-time LTI system, and
2. $H(s)$ is the corresponding eigenvalue.

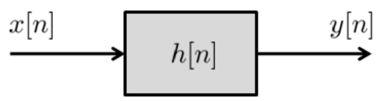
If we specialize to the subclass of periodic complex exponentials of the $e^{j\omega t}$, $\omega \in \mathbb{R}$ by setting $s = j\omega$, then

$$H(s)|_{s=j\omega} = H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau.$$

$H(j\omega)$ is called the *frequency response* of the system.

3.1.2 Discrete-time Case

Next, we consider the discrete-time case:



Suppose that the impulse response is given by $h[n]$ and the input is $x[n] = z^n$, then the output $y[n]$ is

$$\begin{aligned} y[n] = h[n] * x[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]z^{n-k} \\ &= z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = H(z)z^n, \end{aligned}$$

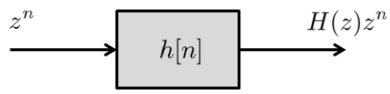
where we defined

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k},$$

and $H(z)$ is known as the transfer function of the discrete-time LTI system.

Similar to the continuous-time case, this result indicates that

1. z^n is an eigenfunction of a discrete-time LTI system, and
2. $H(z)$ is the corresponding eigenvalue.



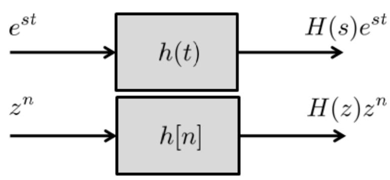
Considering the subclass of periodic complex exponentials $e^{-j(2\pi/N)n}$ by setting $z = e^{j2\pi/N}$, we have

$$H(z)|_{z=e^{j\Omega}} = H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k},$$

where $\Omega = \frac{2\pi}{N}$, and $H(e^{j\Omega})$ is called the *frequency response* of the system.

3.1.3 Summary

In summary, we have the following observations:



That is, e^{st} is an eigenfunction of a CT system, whereas z^n is an eigenfunction of a DT system. The corresponding eigenvalues are $H(s)$ and $H(z)$.

If we substitute $s = j\omega$ and $z = e^{j\Omega}$ respectively, then the eigenfunctions become $e^{j\omega t}$ and $e^{j\Omega n}$; the eigenvalues become $H(j\omega)$ and $H(e^{j\Omega})$.

3.1.4 Why is eigenfunction important?

The answer to this question is related to the second objective in the beginning. Let us consider a signal $x(t)$:

$$x(t) = a_1e^{s_1t} + a_2e^{s_2t} + a_3e^{s_3t}.$$

According the eigenfunction analysis, the output of each complex exponential is

$$\begin{aligned} e^{s_1t} &\longrightarrow H(s_1)e^{s_1t} \\ e^{s_2t} &\longrightarrow H(s_2)e^{s_2t} \\ e^{s_3t} &\longrightarrow H(s_3)e^{s_3t}. \end{aligned}$$

Therefore, the output is

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}.$$

The result implies that if the input is a linear combination of complex exponentials, the output of an LTI system is also a linear combination of complex exponentials. More generally, if $x(t)$ is an infinite sum of complex exponentials,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t},$$

then the output is again a sum of complex exponentials:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k) e^{s_k t}.$$

Similarly for discrete-time signals, if

$$x[n] = \sum_{k=-\infty}^{\infty} a_k z_k^n,$$

then

$$x[n] = \sum_{k=-\infty}^{\infty} a_k H(z_k) z_k^n.$$

This is an important observation, because as long as we can express a signal $x(t)$ as a linear combination of eigenfunctions, then the output $y(t)$ can be easily determined by looking at the transfer function (which is fixed for an LTI system!). Now, the question is : How do we express a signal $x(t)$ as a linear combination of complex exponentials?

3.2 Fourier Series Representation

Existence of Fourier Series

In general, not every signal $x(t)$ can be decomposed as a linear combination of complex exponentials. However, such decomposition is still possible for an extremely large class of signals. We want to study one class of signals that allows the decomposition. They are the periodic signals

$$x(t+T) = x(t)$$

which satisfy the square integrable condition,

$$\int_T |x(t)|^2 dt < \infty,$$

or Dirichlet conditions (You may find more discussions in OW § 3.4):

1. Over any period $x(t)$ must be *absolutely integrable*, that is,

$$\int_T |x(t)| dt < \infty.$$

2. In any finite interval of time $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

3. In any finite interval of time, there are only a finite number of discontinuities.

For this class of signals, we are able to express it as a linear combination of complex exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

Here, ω_0 is the fundamental frequency

$$\omega_0 = \frac{2\pi}{T},$$

and the coefficients a_k are known as the Fourier Series coefficients.

Given a periodic signal $x(t)$ that is square integrable, how do we determine the Fourier Series coefficients a_k? This is answered by the following theorem.

3.2.1 Continuous-time Fourier Series Coefficients

Theorem 8. *The continuous-time Fourier series coefficients a_k of the signal*

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

is given by

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt.$$

Proof. Let us consider the signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

If we multiply on both sides $e^{-jn\omega_0 t}$, then we have

$$x(t)e^{-jn\omega_0 t} = \left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right] e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}.$$

Integrating both sides from 0 to T yields

$$\begin{aligned}\int_0^T x(t)e^{-jn\omega_0 t}dt &= \int_0^T \left[\sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}\right] dt \\ &= \sum_{k=-\infty}^{\infty} \left[a_k \int_0^T e^{j(k-n)\omega_0 t} dt\right].\end{aligned}$$

The term $\int_0^T e^{j(k-n)\omega_0 t}dt$ can be evaluated as (You should check this!)

$$\frac{1}{T}\int_0^T e^{j(k-n)\omega_0 t}dt = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

This result is known as the *orthogonality* of the complex exponentials.

Using Eq. (3.1), we have

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = Ta_n,$$

which is equivalent to

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt.$$

□

Example 1. Sinusoids
Consider the signal $x(t) = 1+\frac{1}{2}\cos 2\pi t+\sin 3\pi t$. The period of $x(t)$ is $T = 2$ [Why?] so the fundamental frequency is $\omega_0 = \frac{2\pi}{T} = \pi$. Recall Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$, we have

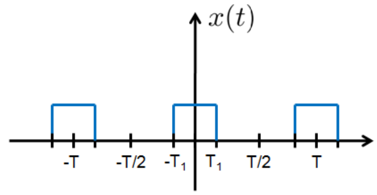
$$x(t) = 1 + \frac{1}{4}\left[e^{j2\pi t} + e^{-j2\pi t}\right] + \frac{1}{2j}\left[e^{j3\pi t} - e^{-j3\pi t}\right].$$

Therefore, the Fourier series coefficients are (just "read off" from this equation!):

$$a_0 = 1, \qquad a_1 = a_{-1} = 0, \qquad a_2 = a_{-2} = \frac{1}{4}, \qquad a_3 = \frac{1}{2j}, \qquad a_{-3} = -\frac{1}{2j},$$

and $a_k = 0$ otherwise.

Example 2. Periodic Rectangular Wave



Let us determine the Fourier series coefficients of the following signal

$$x(t) = \begin{cases} 1 & |t| < T_1, \\ 0 & T_1 < |t| < \frac{T}{2} \end{cases}.$$

The Fourier series coefficients are ($k \neq 0$):

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt \\ &= \frac{-1}{jk\omega_0 T}\left[e^{-jk\omega_0 t}\right]_{-T_1}^{T_1} \\ &= \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right] = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}. \end{aligned}$$

If $k = 0$, then

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}.$$

Example 3. Periodic Impulse Train

Consider the signal $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$. The fundamental period of $x(t)$ is T [Why?]. The F.S. coefficients are

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)dt = \frac{1}{T},$$

for any k.

3.2.2 Discrete-time Fourier Series coefficients

To construct the discrete-time Fourier series representation, we consider periodic discrete-time signal with period N

$$x[n] = x[n+N],$$

and assume that $x[n]$ is square-summable, i.e., $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$, or $x[n]$ satisfies the Dirichlet conditions. In this case, we have

Theorem 9. *The discrete-time Fourier series coefficients a_k of the signal*

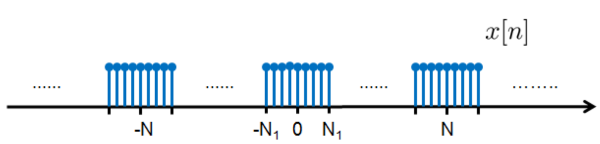
$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 n},$$

is given by

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n}.$$

Here, $\sum_{n=\langle N \rangle}$ means summing the signal within a period N. Since a periodic discrete-time signals repeats every N samples, it does not matter which sample to be picked first.

Example.
Let us consider the following signal shown below. We want to determine the discrete-time F.S. coefficient.



For $k \neq 0, \pm N, \pm 2N, \ldots$, we have

$$
\begin{aligned}
a_k &= \frac{1}{N} \sum_{n=\langle N \rangle} e^{-jk\Omega_0 n} = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk\Omega_0 n} \\
&= \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk\Omega_0 (m-N_1)}, \qquad (m = n + N_0) \\
&= \frac{1}{N} e^{jk\Omega_0 N_1} \sum_{m=0}^{2N_1} e^{-jk\Omega_0 m}.
\end{aligned}
$$

Since

$$
\sum_{m=0}^{2N_1} e^{-jk\Omega_0 m} = \frac{1 - e^{-jk\Omega_0 (2N_1+1)}}{1 - e^{-jk\Omega_0}},
$$

it follows that

$$
\begin{aligned}
a_k &= \frac{1}{N} e^{jk\Omega_0 N_1} \left(\frac{1 - e^{-jk\Omega_0 (2N_1+1)}}{1 - e^{-jk\Omega_0}} \right), \qquad (\Omega_0 = 2\pi/N) \\
&= \frac{1}{N} \frac{e^{-jk(2\pi/2N)} \left[e^{jk2\pi(N_1+1/2)/N} - e^{-jk2\pi(N_1+1/2)/N} \right]}{e^{-jk(2\pi/2N)} \left[e^{jk(2\pi/2N)} - e^{-jk(2\pi/2N)} \right]} \\
&= \frac{1}{N} \frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\frac{\pi k}{N})}.
\end{aligned}
$$

For $k = 0, \pm N, \pm 2N, \ldots$, we have

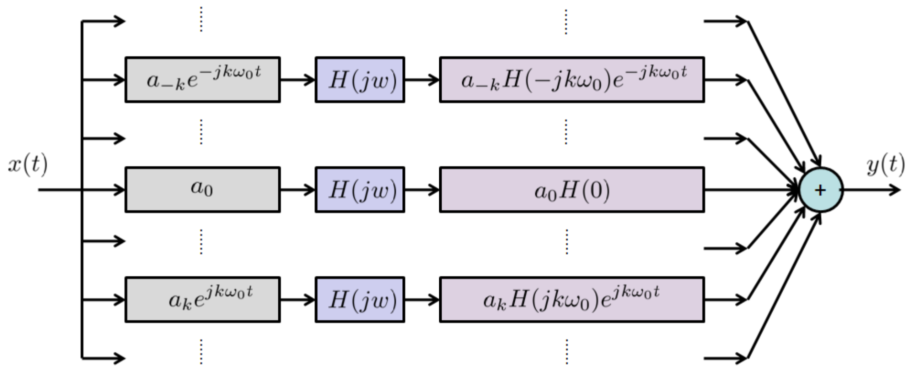
$$
a_k = \frac{2N_1 + 1}{N}.
$$

3.2.3 How do we use Fourier series representation?

Fourier series representation says that any periodic square integrable signals (or signals that satisfy Dirichlet conditions) can be expressed as a linear combination of complex exponentials. Since complex exponentials are eigenfunctions to LTI systems, the output signal must be a linear combination of complex exponentials.

That is, for any signal $x(t)$ we represent it as

$$
x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.
$$



Then, the output signal is given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}.$$

Letting $b_k = H(jk\omega_0)a_k$, we have

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}.$$

3.2.4 How many Fourier series coefficients are sufficient?

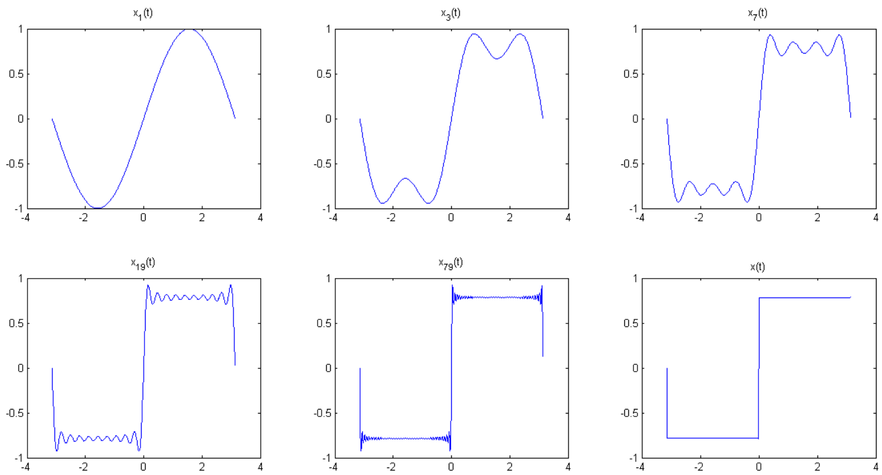
If we define

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t},$$

then $x_N(t)$ is an approximation of $x(t)$. As $N \to \infty$, we see that $x_N(t) \to x(t)$. As an illustration of $x_N(t)$ as N increases, we can see the following figure.
Therefore, the number of Fourier series coefficients depends on the accuracy that we want to achieve. Typically, the number N is chosen such that the residue of the approximation

$$\int_{-\infty}^{\infty} |x(t) - x_N(t)|^2 \, dt \le \varepsilon,$$

for some target error level ε.



3.3 Properties of Fourier Series Coefficients

There are a number of Fourier series properties that we encourage you to read the text. The following is a quick summary of these properties.

1. Linearity: If $x_1(t) \longleftrightarrow a_k$ and $x_2(t) \longleftrightarrow b_k$, then

$$Ax_1(t) + Bx_2(t) \longleftrightarrow Aa_k + Bb_k.$$

   For DT case, we have if $x_1[n] \longleftrightarrow a_k$ and $x_2[n] \longleftrightarrow b_k$, then

$$Ax_1[n] + Bx_2[n] \longleftrightarrow Aa_k + Bb_k.$$

2. Time Shift:

$$x(t - t_0) \longleftrightarrow a_k e^{-jk\omega_0 t_0}$$
$$x[n - n_0] \longleftrightarrow a_k e^{-jk\Omega_0 n_0}$$

   To show the time shifting property, let us consider the F.S. coefficient b_k of the signal $y(t) = x(t - t_0)$.

$$b_k = \frac{1}{T}\int_T x(t - t_0) e^{-j\omega_0 t} dt.$$

Letting $\tau = t - t_0$ in the integral, we obtain

$$\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau = e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0\tau}d\tau$$

where $x(t) \longleftrightarrow a_k$. Therefore,

$$x(t-t_0) \longleftrightarrow a_k e^{-jk\omega_0 t_0}.$$

3. Time Reversal:

$$x(-t) \longleftrightarrow a_{-k}$$
$$x[-n] \longleftrightarrow a_{-k}$$

The proof is simple. Consider a signal $y(t) = x(-t)$. The F.S. representation of $x(-t)$ is

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}.$$

Letting $k = -m$, we have

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}.$$

Thus, $x(-t) \longleftrightarrow a_{-k}$.

4. Conjugation:

$$x^*(t) \longleftrightarrow a^*_{-k}$$
$$x^*[n] \longleftrightarrow a^*_{-k}$$

5. Multiplication: If $x(t) \longleftrightarrow a_k$ and $y(t) \longleftrightarrow b_k$, then

$$x(t)y(t) \longleftrightarrow \sum_{l=-\infty}^{\infty} a_k b_{k-l}.$$

6. Parseval Equality:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$
$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

You are required to read Table 3.1 and 3.2.

Chapter 4

Continuous-time Fourier Transform

Let us begin our discussion by reviewing some limitations of Fourier series representation. In Fourier series analysis, two conditions on the signals are required:

1. The signal must be periodic, i.e., there exist a $T > 0$ such that $x(t+T) = x(t)$.
2. The signal must be square integrable $\int_T |x(t)|^2 dt < \infty$, or satisfies the Dirichlet conditions.

In this chapter, we want to extend the idea of Fourier Series representation to aperiodic signals. That is, we want to relax the first condition to aperiodic signals.

4.1 Insight from Fourier Series

Let's first consider the following periodic signal

$$x(t) = \begin{cases} 1 & |t| \le T_1, \\ 0 & T_1 \le |t| < \frac{T}{2}, \end{cases}$$

where $x(t) = x(t+T)$. The Fourier Series coefficients of $x(t)$ are (check yourself!)

$$x(t) \overset{F.S.}{\longleftrightarrow} a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}.$$

If we substitute $\omega = k\omega_0$, then

$$a_k = \frac{2\sin(\omega T_1)}{wT}\Big|_{\omega = k\omega_0}$$

Multiplying T on both sides yields

$$Ta_k = \frac{2\sin(\omega T_1)}{\omega}, \tag{4.1}$$

which is the normalized Fourier Series coefficient.

Pictorially, (4.1) indicates that the normalized Fourier series coefficients Ta_k are bounded by the envelop $X(\omega) = \frac{2\sin(\omega T_1)}{\omega}$, as illustrated in Fig. 4.1.

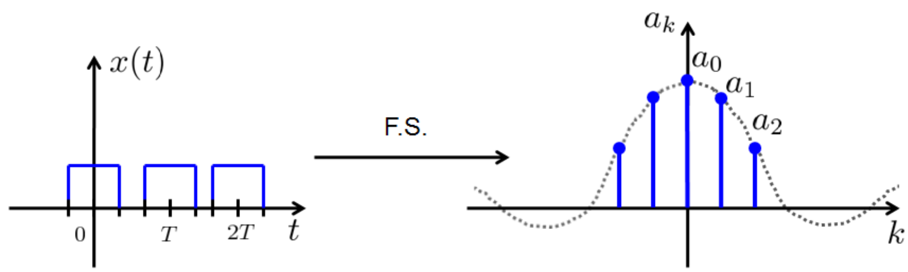


Figure 4.1: Fourier Series coefficients of $x(t)$ for some T.

When T increases, the spacing between consecutive a_k reduces. However, the shape of the envelop function $X(\omega) = \frac{2\sin(\omega T_1)}{\omega}$ remains the same. This can be seen in Fig. 4.2.

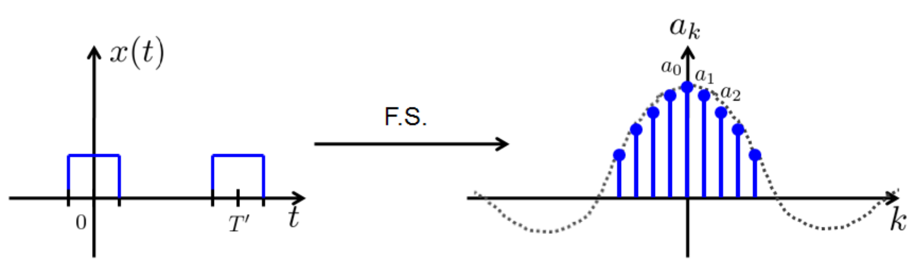


Figure 4.2: Fourier Series coefficients of $x(t)$ for some T', where $T' > T$.

In the limiting case where $T \to \infty$, then the Fourier series coefficients Ta_k approaches the envelop function $X(\omega)$. This suggests us that if we have an aperiodic signal, we can treat it as a periodic signal with $T \to \infty$. Then the corresponding Fourier series coefficients approach to the envelop function $X(\omega)$. The envelop function is called the Fourier Transform of the signal $x(t)$. Now, let us study Fourier Transform more formally.

4.2 Fourier Transform

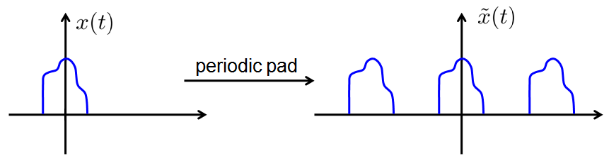
The derivation of Fourier Transform consists of three steps.

Step 1.

We assume that an aperiodic signal $x(t)$ has finite duration, i.e., $x(t) = 0$ for $|t| > T/2$, for some T. Since $x(t)$ is aperiodic, we first construct a periodic signal $\widetilde{x}(t)$:

$$\widetilde{x}(t) = x(t),$$

for $-T/2 < t < T/2$, and $\widetilde{x}(t+T) = \widetilde{x}(t)$. Pictorially, we have



Step 2.

Since $\widetilde{x}(t)$ is periodic, we may express $\widetilde{x}(t)$ using Fourier Series:

$$\widetilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{4.2}$$

where

$$a_k = \frac{1}{T}\int_T \widetilde{x}(t)e^{-jk\omega_0 t}dt.$$

The Fourier Series coefficients a_k can further be calculated as

$$\begin{aligned}
a_k &= \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} \widetilde{x}(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt, \qquad \widetilde{x}(t) = x(t), \text{ for } -T/2 < t < T/2 \\
&= \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt, \qquad x(t) = 0, \text{ for } |t| > T/2.
\end{aligned}$$

If we define

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{4.3}$$

then it holds that

$$a_k = \frac{1}{T} X(jk\omega_0). \tag{4.4}$$

Consequently, substituting (4.4) into (4.2) yields

$$\widetilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} X(jk\omega_0) e^{jk\omega_0 t} \omega_0. \tag{4.5}$$

Step 3.

Now, note that $\widetilde{x}(t)$ is the periodic padded version of $x(t)$. When the period $T \to \infty$, the periodic signal $\widetilde{x}(t)$ approaches $x(t)$. Therefore,

$$\widetilde{x}(t) \longrightarrow x(t), \tag{4.6}$$

as $T \to \infty$.

Moreover, when $T \to \infty$, or equivalently $\omega_0 \to 0$, the limit of the sum in (4.5) becomes an integral:

$$\lim_{\omega_0 \to 0} \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} X(jk\omega_0) e^{jk\omega_0 t} \omega_0 = \int_{-\infty}^{\infty} \frac{1}{2\pi} X(j\omega) e^{jwt} dw. \tag{4.7}$$

Graphically, this can be seen in Fig. 4.3.

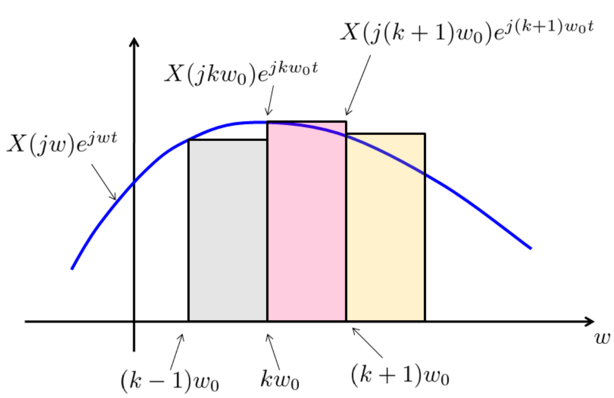


Figure 4.3: Illustration of (4.7).

Combining (4.7) and (4.6), we have

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{jwt} dw \tag{4.8}$$

The two equations (4.3) and (4.8) are known as the Fourier Transform pair. (4.3) is called the *Analysis Equation* (because we are analyzing the time signal in the Fourier domain) and (4.8) is called the *Synthesis Equation* (because we are gathering the Fourier domain information and reconstruct the time signal).

To summarize we have

Theorem 10. *The Fourier Transform $X(j\omega)$ of a signal $x(t)$ is given by*

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt,$$

and the inverse Fourier Transform is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{jwt} dw.$$

4.3 Relation to Fourier Series

At this point you may wonder: What is the difference between Fourier Series and Fourier Transform? To answer this question, let us apply Fourier Transform to the following two types of signals.

1. **Aperiodic Signal**: As we discussed in the derivation of Fourier Transform, the Fourier Transform of an aperiodic signal is the limiting case (when $\omega_0 \to 0$) of applying Fourier Series analysis on the periodically padded version of the aperiodic signal. Fourier Transform can be applied to *both* periodic and aperiodic signals, whereas Fourier Series analysis can only be applied to periodic signals. See Fig. 4.4.

2. **Periodic Signal**: If the signal $x(t)$ is periodic, then we do not need to construct $\widetilde{x}(t)$ and set $\omega_0 \to 0$. In fact, ω_0 is fixed by the period of the signal: If the period

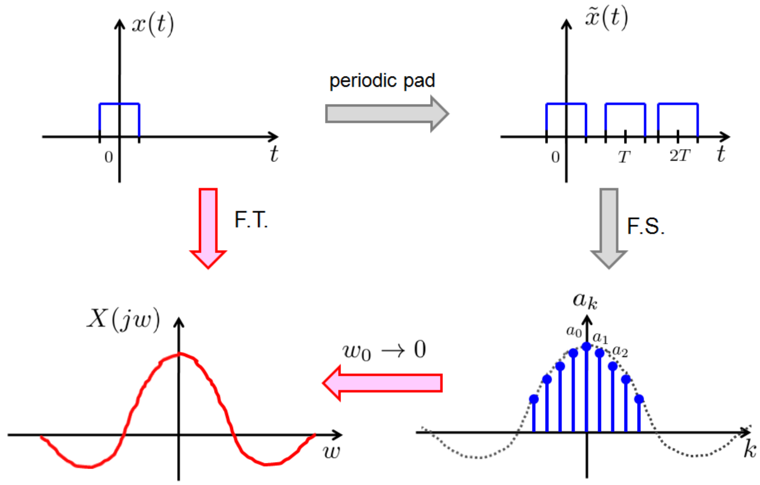


Figure 4.4: Fourier Transform on aperiodic signals is equivalent to applying Fourier series analysis on the periodically padded version of the signal, and set the limit of $\omega_0 \to 0$.

of $x(t)$ is T_0, then $\omega_0 = \frac{2\pi}{T_0}$. Now, since $x(t)$ is periodic, we can apply Fourier Series analysis to $x(t)$ and get

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{4.9}$$

where a_k is the Fourier Series coefficient. If we further apply Fourier Transform to (4.9), then we have

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right] e^{-j\omega t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k \left[\int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt \right] \\
&= \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0).
\end{aligned}$$

Here, the last equality is established by the fact that inverse Fourier Transform of $2\pi\delta(\omega - k\omega_0)$ is $e^{jk\omega_0 t}$. To show this, we have

$$\begin{aligned}\mathcal{F}^{-1}\{2\pi\delta(\omega - k\omega_0)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - k\omega_0)e^{jwt}dw \\ &= \int_{-\infty}^{\infty} \delta(\omega - k\omega_0)e^{jk\omega_0 t}dw = e^{jk\omega_0 t}.\end{aligned}$$

Therefore, we showed that the Fourier Transform of a periodic signal is a train of impulses with amplitude defined by the Fourier Series coefficients (and scaled by a factor of 2π).

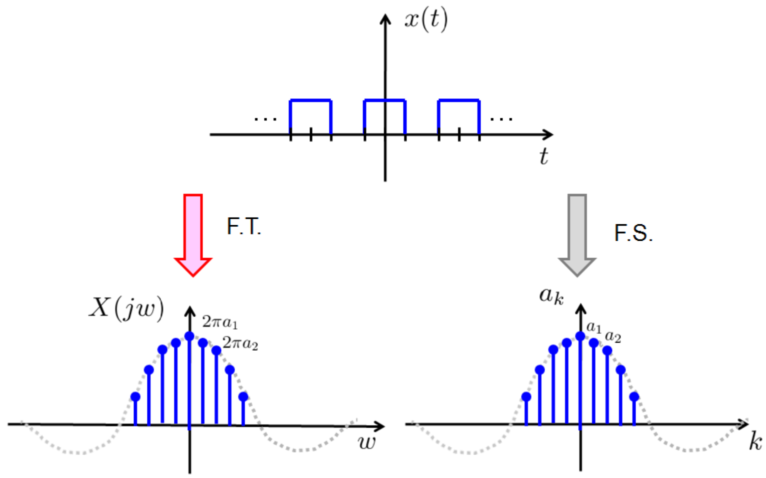


Figure 4.5: Fourier Transform and Fourier Series analysis of a periodic signal: Both yields a train of impulses. For Fourier Transform, the amplitude is multiplied by a factor of 2π. For Fourier Series coefficients, the separation between each coefficient is ω_0.

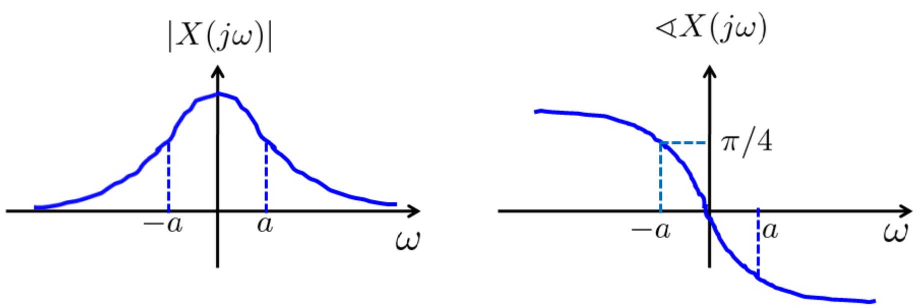
4.4 Examples

Example 1.
Consider the signal $x(t) = e^{-at}u(t)$, for $a > 0$. Determine the Fourier Transform $X(j\omega)$, its magnitude $|X(j\omega)|$ and its phase $\sphericalangle X(j\omega)$.

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}u(t)dt \\ &= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \qquad , u(t) = 0, \text{whenever } t < 0 \\ &= \frac{-1}{a+j\omega}\left[e^{-(a+j\omega)t}\right]_0^\infty \\ &= \frac{1}{a+j\omega}. \end{aligned}$$

The magnitude and phase can be calculated as

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \sphericalangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$



Example 2.
Consider the signal $x(t) = \delta(t)$. The Fourier Transform is

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1. \end{aligned}$$

Example 3.
Consider the signal $x(t) = e^{j\omega_0 t}$. We want to show that $X(j\omega) = 2\pi\delta(\omega - \omega_0)$. To

see, we take the inverse Fourier Transform:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ &= \int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{j\omega t}d\omega \\ &= e^{j\omega_0 t}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)d\omega = e^{j\omega_0 t}. \end{aligned}$$

Example 4.
Consider the aperiodic signal

$$x(t) = \begin{cases} 1 & |t| \le T_1 \\ 0 & |t| > T_1 \end{cases}$$

The Fourier Transform is

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{-T_1}^{T_1} e^{-j\omega t}dt = \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{-T_1}^{T_1} = 2\frac{\sin\omega T_1}{\omega}. \end{aligned}$$

Example 5.
Let us determine the CTFT of the unit step function $u(t)$. To do so, we apply CTFT and get

$$U(j\omega) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{j\omega t}dt = -\left[\frac{1}{j\omega}e^{j\omega t}\right]_0^{\infty}.$$

That is, we have to evaluate the function $\frac{1}{j\omega}e^{j\omega t}$ at $t=0$ and $t=\infty$. However, the evaluation at $t=\infty$ is indeterminate! Therefore, we express $y(t)$ as a limit of decaying exponentials

$$u(t) = \lim_{a\to 0} e^{-at}u(t).$$

Then applying CTFT on both sides,

$$\begin{aligned} U(j\omega) &= \lim_{a\to 0}\mathcal{F}\left\{e^{-at}u(t)\right\} = \lim_{a\to 0}\frac{1}{a+j\omega} \\ &= \lim_{a\to 0}\frac{a-j\omega}{a^2+\omega^2} \\ &= \lim_{a\to 0}\left(\frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}\right). \end{aligned}$$

Now, the second term is

$$\lim_{a\to 0} -j\frac{\omega}{a^2+\omega^2} = \frac{1}{j\omega}.$$

The first term satisfies

$$\lim_{a\to 0}\frac{a}{a^2+\omega^2} = 0, \qquad \text{for } \omega \neq 0.$$

and

$$\lim_{a\to 0}\frac{a}{a^2+\omega^2} = \lim_{a\to 0}\frac{1}{a} = \infty, \qquad \text{for } \omega = 0,$$

while

$$\int_{-\infty}^{\infty}\frac{a}{a^2+\omega^2}d\omega = \tan^{-1}\frac{\omega}{a}\Big|_{-\infty}^{\infty} = \pi, \quad \forall\, a \in \mathbb{R}.$$

Therefore,

$$\lim_{a\to 0}\frac{a}{a^2+\omega^2} = \pi\delta(\omega),$$

and so

$$U(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

4.5 Properties of Fourier Transform

The properties of Fourier Transform is very similar to those of Fourier Series.

1. Linearity If $x_1(t) \longleftrightarrow X_1(j\omega)$ and $x_2(t) \longleftrightarrow X_2(j\omega)$, then

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(j\omega) + bX_2(j\omega).$$

2. Time Shifting

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0}X(j\omega)$$

Physical interpretation of $e^{-j\omega t_0}$:

$$\begin{aligned} e^{-j\omega t_0}X(j\omega) &= |X(j\omega)|e^{j\sphericalangle X(j\omega)}e^{-j\omega t_0} \\ &= |X(j\omega)|e^{j[\sphericalangle X(j\omega)-\omega t_0]} \end{aligned}$$

So $e^{-j\omega t_0}$ is contributing to phase shift of $X(j\omega)$.

3. Conjugation

$$x^*(t) \longleftrightarrow X^*(-j\omega)$$

If $x(t)$ is real, then $x^*(t) = x(t)$, so $X(j\omega) = X^*(-j\omega)$.

4. Differentiation and Integration

$$\frac{d}{dt}x(t) \longleftrightarrow j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

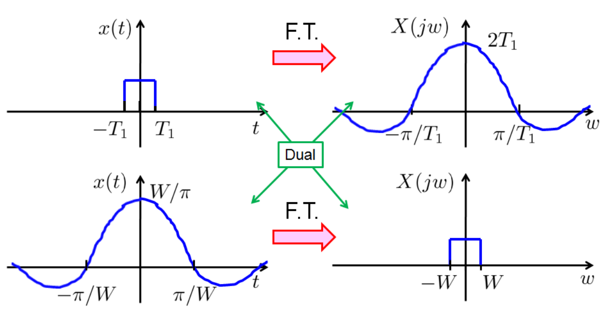
5. Time Scaling

$$x(t) \longleftrightarrow \frac{1}{|a|}X(\frac{j\omega}{a})$$

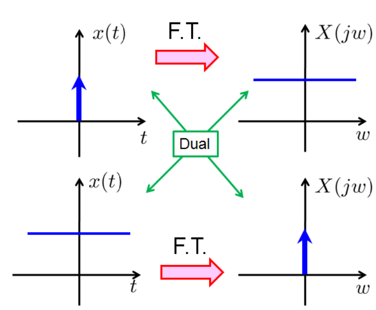
6. Parseval Equality

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

7. Duality The important message here is: If $x(t) \longleftrightarrow X(j\omega)$, then if another signal $y(t)$ has the shape of $X(j\omega)$, we can quickly deduce that $X(j\omega)$ will have the shape of $x(t)$. Here are two examples:



Duality Example 1



Duality Example 2

8. Convolution Property

$$h(t) * x(t) \longleftrightarrow H(j\omega)X(j\omega)$$

Proof: Consider the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Taking Fourier Transform on both sides yields

$$\begin{aligned}
Y(j\omega) &= \mathcal{F}\left\{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right\} \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\left[e^{-j\omega\tau}H(j\omega)\right]d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau H(j\omega) \\
&= X(j\omega)H(j\omega)
\end{aligned}$$

9. Multiplication Property (you can derive this by yourself)

$$x(t)y(t) \longleftrightarrow \frac{1}{2\pi}X(j\omega) * Y(j\omega)$$

Example.
Consider the signal $x(t) = m(t)\cos(\omega_0 t)$, where $m(t)$ is some bandlimited signal. Suppose the Fourier Transform of $m(t)$ is $M(j\omega)$. Since

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \longleftrightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)],$$

by convolution property, the CTFT of $x(t)$ is

$$\begin{aligned}
m(t)\cos(\omega_0 t) &\longleftrightarrow \frac{1}{2\pi}M(j\omega) * \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] \\
&= \frac{1}{2}\left[M(j(\omega-\omega_0)) + M(j(\omega+\omega_0))\right].
\end{aligned}$$

4.6 System Analysis using Fourier Transform

First Order System
Let us consider the first order system

$$\frac{dy(t)}{dt} + ay(t) = x(t), \tag{4.10}$$

for some $a > 0$. Applying the CTFT to both sides,

$$\mathcal{F}\left\{\frac{dy(t)}{dt} + ay(t)\right\} = \mathcal{F}\left\{x(t)\right\},$$

and use linearity property, and differentiation property of CTFT, we have

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega).$$

Rearranging the terms, we can find the frequency response of the system

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{a + j\omega}.$$

Now, recall the CTFT pair:

$$h(t) = e^{-at}u(t) \Longleftrightarrow H(j\omega) = \frac{1}{a + j\omega}, \tag{4.11}$$

$h(t)$ can be deduced. Just as quick derivation of this equation, we note that

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(j\omega+a)t}dt \\ &= -\left[\frac{1}{j\omega + a}e^{-(j\omega+a)t}\right]_0^{\infty} = \frac{1}{j\omega + a}. \end{aligned}$$

General Systems
In general, we want to study the system

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{4.12}$$

Our objective is to determine $h(t)$ and $H(j\omega)$. Applying CTFT on both sides:

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}.$$

Therefore, by linearity and differentiation property, we have

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega).$$

The convolution property gives $Y(j\omega) = X(j\omega)H(j\omega)$, so

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}. \tag{4.13}$$

Now, $H(j\omega)$ is expressed as a rational function, i.e., a ratio of polynomial. Therefore, we can apply the technique of partial fraction expansion to express $H(j\omega)$ in a form that allows us to determine $h(t)$ by inspection using the transform pair

$$h(t) = e^{-at}u(t) \Longleftrightarrow H(j\omega) = \frac{1}{a + j\omega},$$

and related transform pair, such as

$$te^{-at}u(t) \Longleftrightarrow \frac{1}{(a + j\omega)^2}.$$

Example 1.
Consider the LTI system

$$\frac{d^2 y(t)}{dt^2} + 4y(t) + 3y(t) = \frac{dx(t)}{dt} + 2x(t).$$

Taking CTFT on both sides yields

$$(j\omega)^2 Y(j\omega) + 4j\omega Y(j\omega) + 3Y(j\omega) = j\omega X(j\omega) + 2X(j\omega).$$

and by rearranging terms we have

$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4(j\omega) + 3} = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}.$$

Then, by partial fraction expansion we have

$$H(j\omega) = \frac{1}{2}\left(\frac{1}{j\omega + 1}\right) + \frac{1}{2}\left(\frac{1}{j\omega + 3}\right).$$

Thus, $h(t)$ is

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).$$

Example 2.
If the input signal is $x(t) = e^{-t}u(t)$, what should be the output $y(t)$ if the impulse response of the system is given by $h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$?

Taking CTFT, we know that $X(j\omega) = \frac{1}{j\omega+1}$, and $H(j\omega) = \frac{j\omega+2}{(j\omega+1)(j\omega+3)}$. Therefore, the output is

$$Y(j\omega) = H(j\omega)X(j\omega) = \left[\frac{j\omega+2}{(j\omega+1)(j\omega+3)}\right]\left[\frac{1}{j\omega+1}\right] = \frac{j\omega+2}{(j\omega+1)^2(j\omega+3)}.$$

By partial fraction expansion, we have

$$Y(j\omega) = \frac{\frac{1}{4}}{j\omega+1} + \frac{\frac{1}{2}}{(j\omega+1)^2} - \frac{\frac{1}{4}}{j\omega+3}.$$

Therefore, the output is

$$y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right]u(t).$$

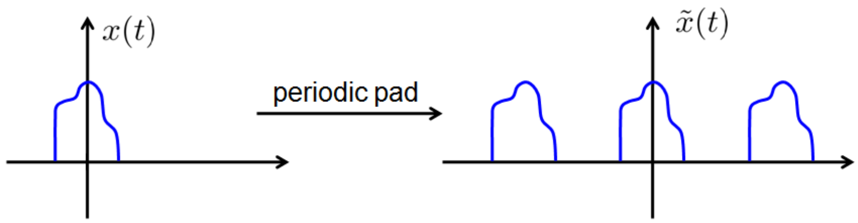
Chapter 5

Discrete-time Fourier Transform

5.1 Review on Continuous-time Fourier Transform

Before we derive the discrete-time Fourier Transform, let us recall the way we constructed continuous-time Fourier Transform from the continuous-time Fourier Series. In deriving the continuous-time Fourier Transform, we basically have the following three steps:

- **Step 1:** Pad the aperiodic signal [1] $x(t)$ to construct a periodic replicate $\tilde{x}(t)$



- **Step 2:** Since $\tilde{x}(t)$ is periodic, we find the Fourier series coefficients a_k and represent $\tilde{x}(t)$ as

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

[1] We are interested in aperiodic signals, because periodic signals can be handled using continuous-time Fourier Series!

By defining

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t} dt, \tag{5.1}$$

which is known as the continuous-time Fourier Transform, we showed

$$a_k = \frac{1}{T} X(jk\omega_0).$$

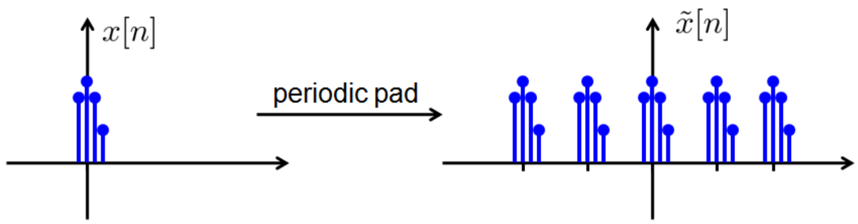
- **Step 3:** Setting $T \to \infty$, we showed $\tilde{x}(t) \to x(t)$ and

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} dt \tag{5.2}$$

5.2 Deriving Discrete-time Fourier Transform

Now, let's apply the same concept to discrete-time signals. In deriving the discrete-time Fourier Transform, we also have three key steps.

- **Step 1:** Consider an aperiodic discrete-time signal $x[n]$. We pad $x[n]$ to construct a periodic signal $\tilde{x}[n]$.



- **Step 2:** Since $\tilde{x}[n]$ is periodic, by discrete-time Fourier Series we have

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}, \tag{5.3}$$

where a_k can be computed as

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} \tilde{x}[n]\, e^{jk\omega_0 n}.$$

Here, the frequency is

$$\omega_0 = \frac{2\pi}{N}.$$

Now, note that $\tilde{x}[n]$ is a periodic signal with period N, and the non-zero entries of $\tilde{x}[n]$ in a period are the same as the non-zero entries of $x[n]$. Therefore, it holds that

$$\begin{aligned} a_k &= \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}\,[n]\, e^{jk\omega_0 n} \\ &= \frac{1}{N} \sum_{n=-\infty}^{\infty} x\,[n]\, e^{jk\omega_0 n}. \end{aligned}$$

If we define

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \qquad (5.4)}$$

then

$$a_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{jk\omega_0 n} = \frac{1}{N} X(e^{jk\omega_0}). \qquad (5.5)$$

- **Step 3:** Putting Equation (5.5) into Equation (5.3), we have

$$\begin{aligned} \tilde{x}[n] &= \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \\ &= \sum_{k=\langle N \rangle} \left[\frac{1}{N} X(e^{jk\omega_0}) \right] e^{jk\omega_0 n} \\ &= \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0, \qquad \omega_0 = \frac{2\pi}{N}. \qquad (5.6) \end{aligned}$$

As $N \to \infty$, $\omega_0 \to 0$ and $\tilde{x}[n] \to x[n]$. Also, from Equation (5.6) becomes

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0 \longrightarrow \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega.$$

Therefore,

$$\boxed{x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega. \qquad (5.7)}$$

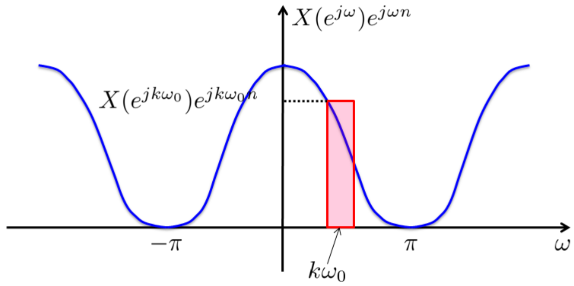


Figure 5.1: As $N \to \infty, \omega_0 \to 0$. So the area becomes infinitesimal small and sum becomes integration.

5.3 Why is $X(e^{j\omega})$ periodic ?

It is interesting to note that the continuous-time Fourier Transform $X(j\omega)$ is aperiodic in general, but the discrete-time Fourier Transform $X(e^{j\omega})$ is always periodic. To see this, let us consider the discrete-time Fourier Transform (we want to check whether $X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$!):

$$\begin{aligned} X(e^{j(\omega+2\pi)}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi)n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}(e^{-j2\pi})^n = X(e^{j\omega}), \end{aligned}$$

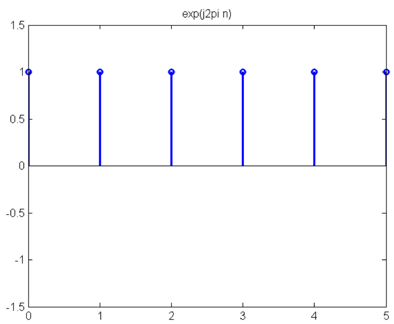
because $(e^{-j2\pi})^n = 1^n = 1$, for any integer n. Therefore, $X(e^{j\omega})$ is periodic with period 2π.

Now, let us consider the continuous-time Fourier Transform (we want to check whether $X(j\omega) = X(j(\omega + 2\pi))$!):

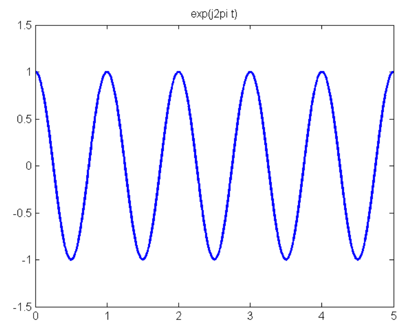
$$X(j(\omega + 2\pi)) = \int_{-\infty}^{\infty} x(t)e^{-j(\omega+2\pi)t}dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}(e^{-j2\pi})^t dt.$$

Here, $t \in \mathbb{R}$ and is running from $-\infty$ to ∞. Pay attention that $e^{-j2\pi t} \neq 1$ unless t is an integer (which different from the discrete-time case where n is always an integer!). Therefore,

$$\int_{-\infty}^{\infty} x(t)e^{-j\omega t}e^{-j2\pi t}dt \neq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt,$$



(a) $(e^{j2\pi})^n = 1$ for all n, because n is integer.



(b) $(e^{j2\pi})^t \neq 1$ unless t is an integer.

and consequently,

$$X(j(\omega + 2\pi)) \neq X(j\omega).$$

5.4 Properties of Discrete-time Fourier Transform

Discrete-time Fourier Transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Discrete-time Inverse Fourier Transform:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

1. Periodicity:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

2. Linearity:

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

3. Time Shift:

$$x[n - n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$$

4. Phase Shift:

$$e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$$

5. Conjugacy:

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$$

6. Time Reversal

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

7. Differentiation

$$nx[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

8. Parseval Equality

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

9. Convolution

$$y[n] = x[n] * h[n] \longleftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

10. Multiplication

$$y[n] = x_1[n]x_2[n] \longleftrightarrow Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})X_2(e^{j(\omega-\theta)})d\theta$$

5.5 Examples

Example 1.
Consider $x[n] = \delta[n] + \delta[n-1] + \delta[n+1]$. Then

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} (\delta[n] + \delta[n-1] + \delta[n+1])e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n-1]e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n+1]e^{-j\omega n} \\
&= 1 + e^{-j\omega} + e^{j\omega} = 1 + 2\cos\omega.
\end{aligned}$$

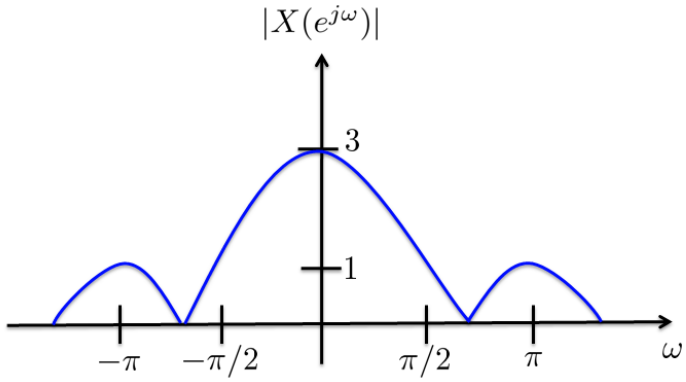


Figure 5.2: Magnitude plot of $|X(e^{j\omega})|$ in Example 1.

To sketch the magnitude $|X(e^{j\omega})|$, we note that $|X(e^{j\omega})| = |1 + 2\cos\omega|$.

Example 2.
Consider $x[n] = \delta[n] + 2\delta[n-1] + 4\delta[n-2]$. The discrete-time Fourier Transform is

$$X(e^{j\omega}) = 1 + 2e^{-j\omega} + 4e^{-j4\omega}.$$

If the impulse response is $h[n] = \delta[n] + \delta[n-1]$, then

$$H(e^{j\omega}) = 1 + e^{-j\omega}.$$

Therefore, the output is

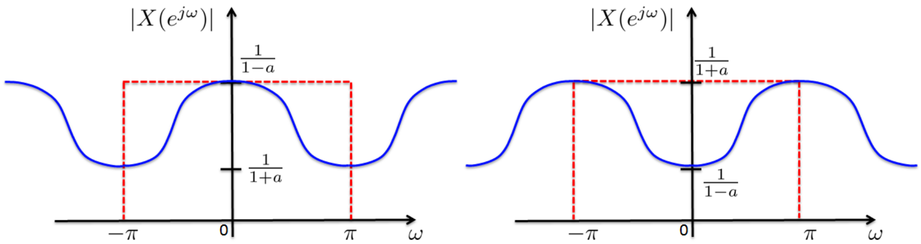
$$\begin{aligned} Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\ &= \left[1 + e^{-j\omega}\right]\left[1 + 2e^{-j\omega} + 4e^{-j2\omega}\right] \\ &= 1 + 3e^{-j\omega} + 6e^{-j2\omega} + 4e^{-j3\omega}. \end{aligned}$$

Taking the inverse discrete-time Fourier Transform, we have

$$y[n] = \delta[n] + 3\delta[n-1] + 6\delta[n-2] + 4\delta[n-3].$$

Example 3.
Consider $x[n] = a^n u[n]$, with $|a| < 1$. The discrete-time Fourier Transform is

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$



(a) $0 < a < 1$ (b) $-1 < a < 0$

Next, let us draw the magnitude $|X(e^{j\omega})|$. To do so, let's consider

$$\begin{aligned}
\left|X(e^{j\omega})\right|^2 = X(e^{j\omega})X^*(e^{j\omega}) &= \frac{1}{1-ae^{-j\omega}} \cdot \frac{1}{1-ae^{j\omega}} \\
&= \frac{1}{1-a(e^{-j\omega}e^{j\omega})+a^2} \\
&= \frac{1}{1-2a\cos\omega+a^2}
\end{aligned}$$

Case A. If $0 < a < 1$, then $\left|X(e^{j\omega})\right|^2$ achieves maximum when $\omega = 0$, and $\left|X(e^{j\omega})\right|^2$ achieves minimum when $\omega = \pi$. Thus,

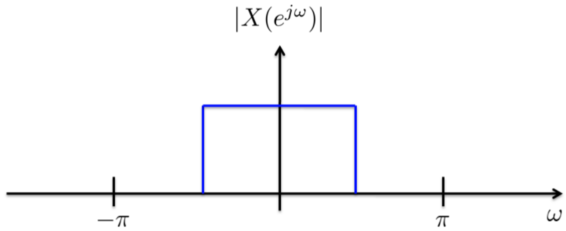
$$\begin{aligned}
\max\left\{\left|X(e^{j\omega})\right|^2\right\} &= \frac{1}{1-2a+a^2} = \frac{1}{(1-a)^2} \\
\min\left\{\left|X(e^{j\omega})\right|^2\right\} &= \frac{1}{1+2a+a^2} = \frac{1}{(1+a)^2}.
\end{aligned}$$

Case B: If $-1 < a < 0$, then $\left|X(e^{j\omega})\right|^2$ achieves maximum when $\omega = \pi$, and $\left|X(e^{j\omega})\right|^2$ achieves minimum when $\omega = 0$. Thus,

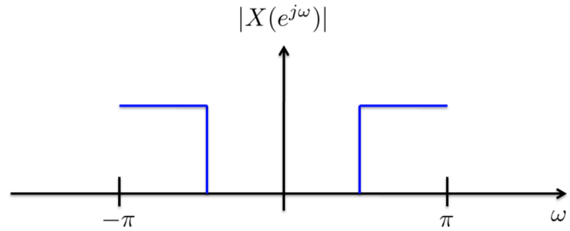
$$\begin{aligned}
\max\left\{\left|X(e^{j\omega})\right|^2\right\} &= \frac{1}{1+2a+a^2} = \frac{1}{(1+a)^2} \\
\min\left\{\left|X(e^{j\omega})\right|^2\right\} &= \frac{1}{1-2a+a^2} = \frac{1}{(1-a)^2}.
\end{aligned}$$

5.6 Discrete-time Filters

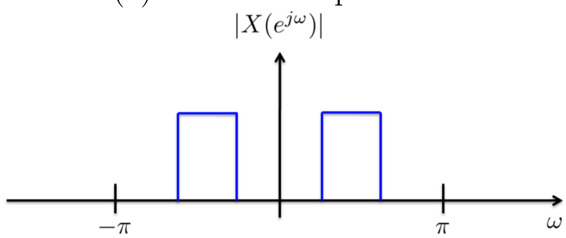
In digital signal processing, there are generally four types of filters that we often use. Namely, they are the lowpass filters, highpass filters, bandpass filters and bandstop filters. There is no precise definition for being "low" or "high". However, it is usually easy to infer the type from viewing their magnitude response.



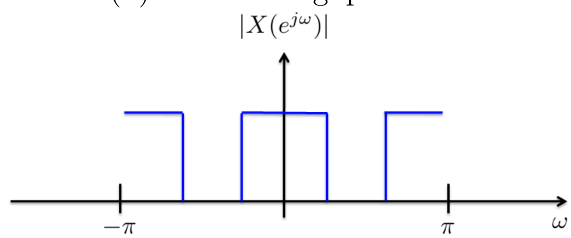
(a) An ideal lowpass filter



(b) An ideal highpass filter



(c) An ideal bandpass filter



(d) An ideal bandstop filter

5.7 Appendix

Geometric Series:

$$\sum_{n=0}^{N} x^n = 1 + x + x^2 + \ldots + x^n = \frac{1 - x^{N+1}}{1 - x},$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1 - x}, \quad \text{when } |x| < 1.$$

Chapter 6

Sampling Theorem

Sampling theorem plays a crucial role in modern digital signal processing. The theorem concerns about the minimum sampling rate required to convert a continuous time signal to a digital signal, without loss of information.

6.1 Analog to Digital Conversion

Consider the following system shown in Fig. 6.1. This system is called an analog-to-digital (A/D) conversion system. The basic idea of A/D conversion is to take a continuous-time signal, and convert it to a discrete-time signal.

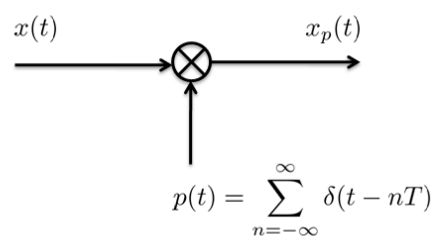


Figure 6.1: An analog to digital (A/D) conversion system.

Mathematically, if the continuous-time signal is $x(t)$, we can collect a set of samples by multiplying $x(t)$ with an impulse train $p(t)$:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT),$$

where T is the period of the impulse train. Multiplying $x(t)$ with $p(t)$ yields

$$\begin{aligned} x_p(t) &= x(t)p(t) \\ &= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT). \end{aligned}$$

Pictorially, $x_p(t)$ is a set of impulses bounded by the envelop $x(t)$ as shown in Fig. 6.2.

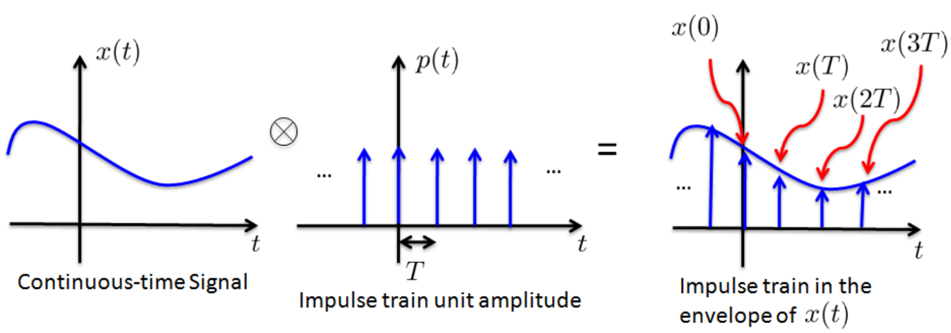


Figure 6.2: An example of A/D conversion. The output signal $x_p(t)$ represents a set of samples of the signal $x(t)$.

We may regard $x_p(t)$ as the samples of $x(t)$. Note that $x_p(t)$ is still a continuous-time signal! (We can view $x_p(t)$ as a discrete-time signal if we define $x_p[n] = x(nT)$. But this is not an important issue here.)

6.2 Frequency Analysis of A/D Conversion

Having an explanation of the A/D conversion in time domain, we now want to study the A/D conversion in the frequency domain. (Why? We need it for the development of Sampling Theorem!) So, how do the frequency responses $X(j\omega)$, $P(j\omega)$ and $X_p(j\omega)$ look like?

6.2.1 How does $P(j\omega)$ look like?

Let's start with $P(j\omega)$. From Table 4.2 of the textbook, we know that

$$\boxed{p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \overset{F.T.}{\longleftrightarrow} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T}) = P(j\omega) \qquad (6.1)}$$

This means that the frequency response of the impulse train $p(t)$ is another impulse train. The only difference is that the period of $p(t)$ is T, whereas the period of $P(j\omega)$ is $\frac{2\pi}{T}$.

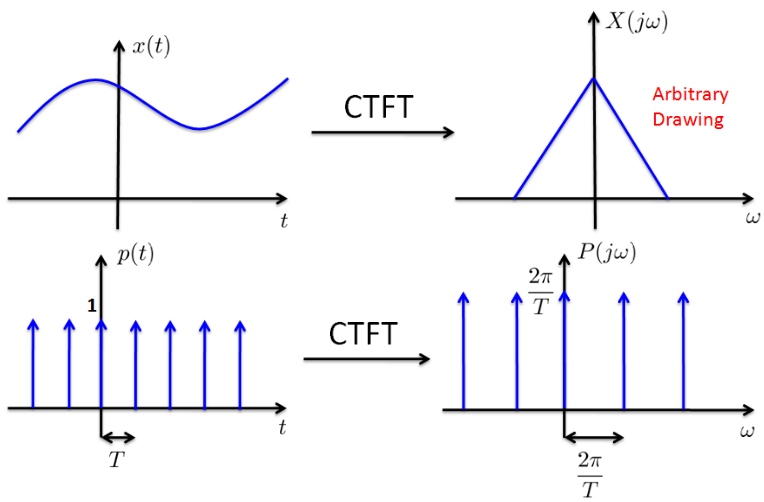


Figure 6.3: Illustration of $X(j\omega)$ and $P(j\omega)$.

6.2.2 How does $X_p(j\omega)$ look like?

Next, suppose that the signal $x(t)$ has a frequency response $X(j\omega)$. We want to know the frequency response of the output $x_p(t)$. From the definition of $x_p(t)$, we know know that

$$x_p(t) = x(t)p(t).$$

Therefore, by the multiplication property of Fourier Transform, we have

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega).$$

Shown in Fig. 6.3 are the frequency response of $X(j\omega)$ and $P(j\omega)$ respectively. To perform the convolution in frequency domain, we first note that $P(j\omega)$ is an impulse train. Therefore, convolving $X(j\omega)$ with $P(j\omega)$ is basically producing replicates at every $\frac{2\pi}{T}$. The result is shown in Fig. 6.4.

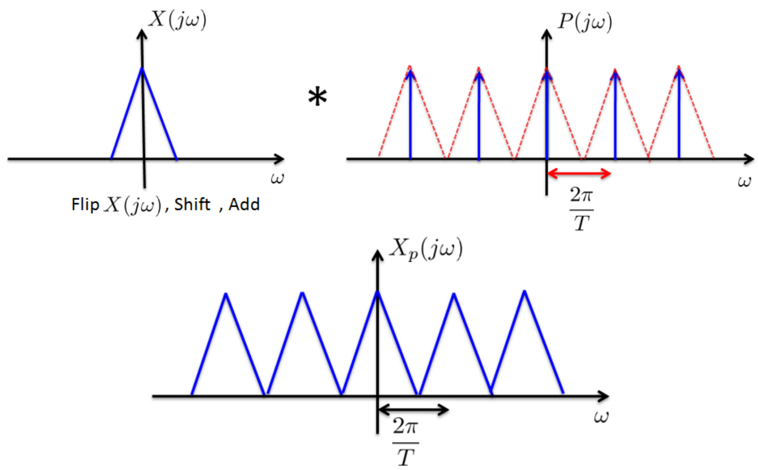


Figure 6.4: Convolution between $X(j\omega)$ and $P(j\omega)$ yields periodic replicates of $X(j\omega)$.

Mathematically, the output $X_p(j\omega)$ is given by

$$\begin{aligned}
X_p(j\omega) &= \frac{1}{2\pi} X(j\omega) * P(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\left[\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-\theta-\frac{2\pi k}{T})\right]d\theta \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\left[\int_{-\infty}^{\infty} X(j\theta)\delta(\omega-\theta-\frac{2\pi k}{T})d\theta\right] \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j(\omega-\frac{2\pi k}{T})\right).
\end{aligned}$$

The result is illustrated in Fig. 6.5.

6.2.3 What happens if T becomes larger and larger ?

If T becomes larger and larger (i.e., we take fewer and fewer samples), we know from the definition of $p(t)$ that the period (in time domain) between two consecutive

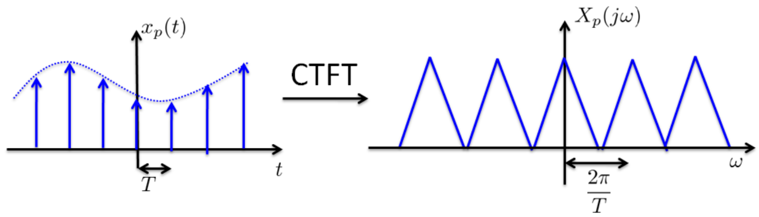


Figure 6.5: Illustration of $x_p(t)$ and $X_p(j\omega)$.

impulses increases (i.e., farther apart). In frequency domain, since

$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T}),$$

the period $\frac{2\pi}{T}$ reduces! In other words, the impulses are more packed in frequency domain when T increases. Fig. 6.6 illustrates this idea.

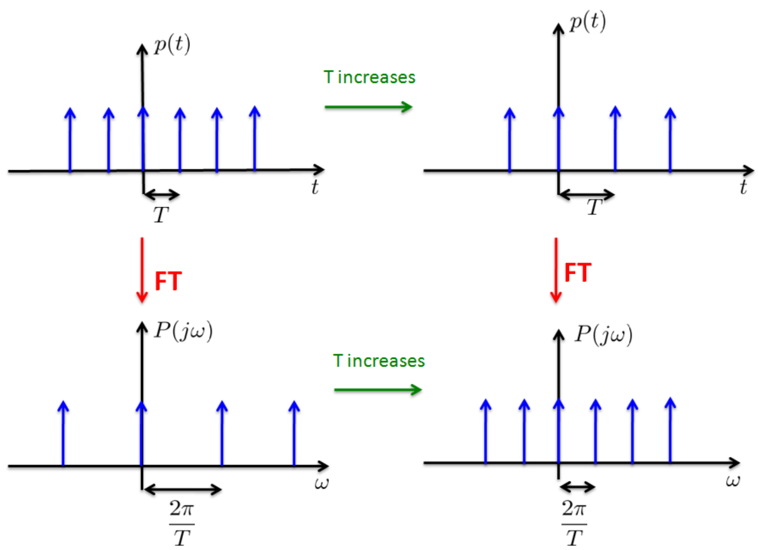


Figure 6.6: When T increases, the period in frequency domain reduces.

If we consider $X_p(j\omega)$, which is a periodic replicate of $X(j\omega)$ at the impulses given by $P(j\omega)$, we see that the separation between replicates reduces. When T hits certain limit, the separation becomes zero; and beyond that limit, the replicates start to overlap! When the frequency replicates overlap, we say that there is *aliasing*.

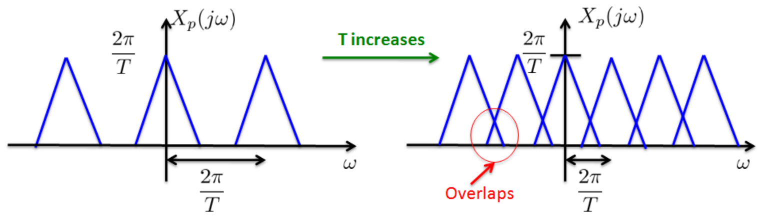


Figure 6.7: When T is sufficiently large, there will be overlap between consecutive replicates.

Therefore, in order to avoid aliasing, T cannot be too large. If we define the sampling rate to be

$$\omega_s = \frac{2\pi}{T},$$

then smaller T implies higher ω_s. In other words, there is a minimum sampling rate such that no aliasing occurs.

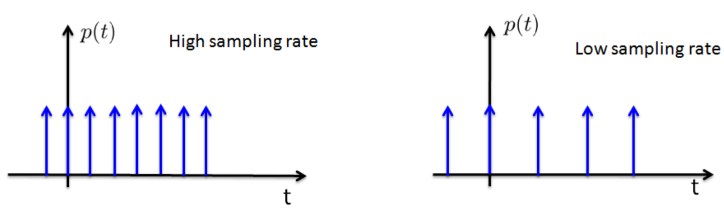


Figure 6.8: Meanings of high sampling rate v.s. low sampling rate.

6.2.4 What is the minimum sampling rate such that there is no aliasing?

Here, let us assume that the signal $x(t)$ is band-limited. That is, we assume $X(j\omega) = 0$ for all $|\omega| > W$, where W is known as the band-width.

To answer this question, we need the **Sampling Theorem**.

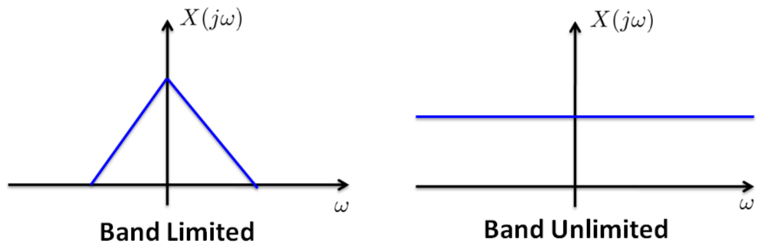


Figure 6.9: Left: A band limited signal (since $X(j\omega) = 0$ for all $\omega > |W|$.) Right: A band non-limited signal.

6.3 Sampling Theorem

Theorem 11 (Sampling Theorem)**.** *Let $x(t)$ be a band limited signal with $X(j\omega) = 0$ for all $|\omega| > W$. Then the minimum sampling rate such that no aliasing occurs in $X_p(j\omega)$ is*

$$\omega_s > 2W,$$

where $\omega_s = \frac{2\pi}{T}$.

6.3.1 Explanation

Suppose $x(t)$ has bandwidth W. The tightest arrangement that no aliasing occurs is shown in Fig. 6.10

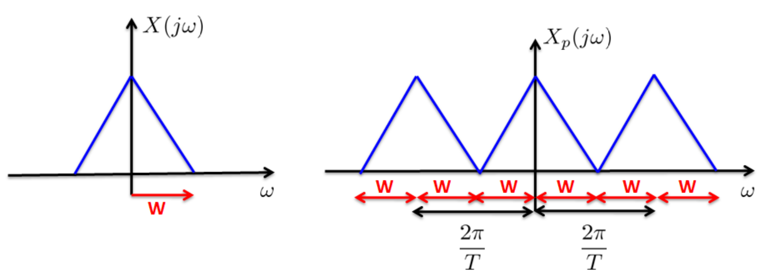


Figure 6.10: Minimum sampling rate that there is no aliasing.

In this case, we see that the sampling rate ω_s $(= \frac{2\pi}{T})$ is

$$\omega_s = 2W.$$

If T is larger (or ω_s is smaller), then $\frac{2\pi}{T}$ becomes less than $2W$, and aliasing occurs. Therefore, the minimum sampling rate to ensure no aliasing is

$$\omega_s > 2W.$$

6.3.2 Example

Suppose there is a signal with maximum frequency 40kHz. What is the minimum sampling rate ?

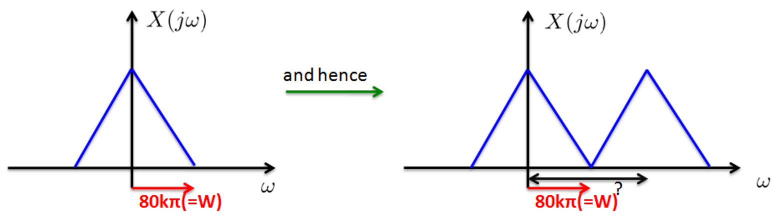


Figure 6.11: Example: Minimum sampling frequency.

Answer :
Since $\omega = 2\pi f$, we know that the max frequency (in rad) is $\omega = 2\pi(40 \times 10^3) = 80 \times 10^3\pi$ (rad). Therefore, the minimum Sampling rate is: $2 \times (80 \times 10^3\pi)$, which is $160 \times 10^3\pi$ (rad) $=$ 80kHz.

6.4 Digital to Analog Conversion

In the previous sections, we studied A/D conversion. Now, given a discrete-time signal (assume no aliasing), we would like to construct the continuous time signal.

6.4.1 Given $X_p(t)$ (no aliasing), how do I recover $x(t)$?

If no aliasing occurs during the sampling processing (i.e., multiply $x(t)$ with $p(t)$), then we can apply a lowpass filter $H(j\omega)$ to extract the $x(t)$ from $x_p(t)$. Fig. 6.12 shows a schematic diagram of how this is performed.
To see how an ideal lowpass filter can extract $x(t)$ from $x_p(t)$, we first look at the frequency response of $X_p(j\omega)$. Suppose that $p(t)$ has a period of T (so that $\omega_s = \frac{2\pi}{T}$).

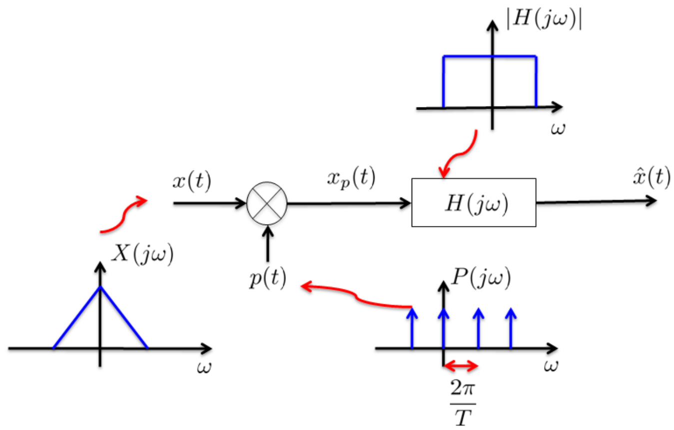


Figure 6.12: Schematic diagram of recovering $x(t)$ from $x_p(t)$. The filter $H(j\omega)$ is assumed to be an ideal lowpass filter.

Then

$$X_p(j\omega) = \frac{1}{T}\sum_{-\infty}^{\infty} X(j(\omega - k\omega_s)).$$

As shown in the top left of Fig. 6.13, $X_p(j\omega)$ is a periodic replicate of $X(j\omega)$. Since we assume that there is no aliasing, the replicate covering the y-axis is identical to $X(j\omega)$. That is, for $|\omega| < \frac{\omega_s}{2}$,

$$X_p(j\omega) = X(j\omega).$$

Now, if we apply an ideal lowpass filter (shown in bottom left of Fig. 6.13):

$$H(j\omega) = \begin{cases} 1, & |\omega| < \frac{\omega_s}{2}, \\ 0, & \text{otherwise}, \end{cases}$$

then

$$X_p(j\omega)H(j\omega) = X(j\omega),$$

for all ω. Taking the inverse continuous-time Fourier transform, we can obtain $x(t)$.

6.4.2 If $X_p(t)$ has aliasing, can I still recover $x(t)$ from $x_p(t)$?

The answer is NO. If aliasing occurs, then the condition

$$X_p(j\omega) = X(j\omega)$$

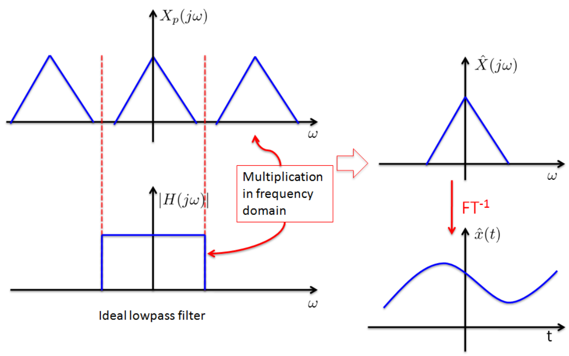


Figure 6.13: Left: Multiplication between $X_p(j\omega)$ and the lowpass filter $H(j\omega)$. The extracted output $\hat{X}(j\omega)$ is identical to $X(j\omega)$ if no aliasing occurs. By applying inverse Fourier transform to $\hat{X}(j\omega)$ we can obtain $x(t)$.

does not hold for all $|\omega| < \frac{\omega_s}{2}$. Consequently, even if we apply the lowpass filter $H(j\omega)$ to $X_p(j\omega)$, the result is not $X(j\omega)$. This can be seen in Fig. 6.14.

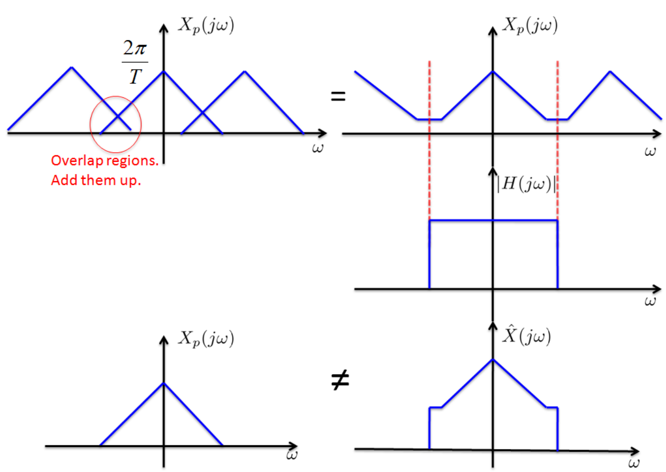
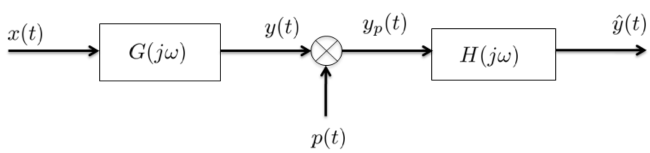
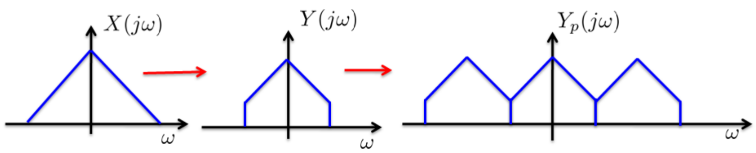


Figure 6.14: If aliasing occurs, we are unable to recover $x(t)$ from $x_p(t)$ by using an ideal lowpass filter.

6.4.3 What can I do if my sampling device does not support a very high sampling rate ?

- **Method 1:** Buy a better sampling device !
- **Method 2:** Send signals with narrower bandwidth or limit the bandwidth before sending :





- **Method 3:** Go to grad school and learn more cool methods !!

Chapter 7

The z-Transform

The z-transform is a generalization of the discrete-time Fourier transform we learned in Chapter 5. As we will see, z-transform allows us to study some system properties that DTFT cannot do.

7.1 The z-Transform

Definition 21. *The z-transform of a discrete-time signal $x[n]$ is:*

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}. \tag{7.1}$$

We denote the z-transform operation as

$$x[n] \longleftrightarrow X(z).$$

In general, the number z in (7.1) is a complex number. Therefore, we may write z as

$$z = re^{jw},$$

where $r \in \mathbb{R}$ and $w \in \mathbb{R}$. When $r = 1$, (7.1) becomes

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn},$$

which is the discrete-time Fourier transform of $x[n]$. Therefore, DTFT is a special case of the z-transform! Pictorially, we can view DTFT as the z-transform evaluated on the unit circle:

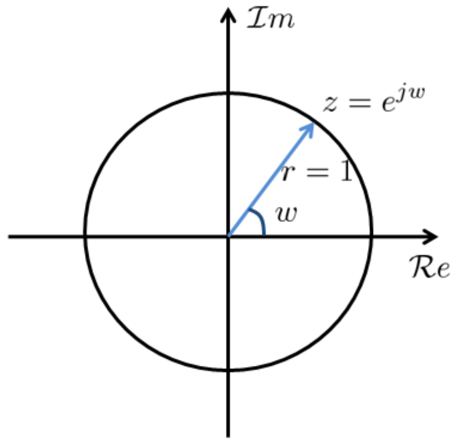


Figure 7.1: Complex z-plane. The z-transform reduces to DTFT for values of z on the unit circle.

When $r \neq 1$, the z-transform is equivalent to

$$\begin{aligned} X(re^{jw}) &= \sum_{-\infty}^{\infty} x[n] \left(re^{jw}\right)^{-n} \\ &= \sum_{-\infty}^{\infty} \left(r^{-n}x[n]\right) e^{-jwn} \\ &= \mathcal{F}\left[r^{-n}x[n]\right], \end{aligned}$$

which is the DTFT of the signal $r^{-n}x[n]$. However, from the development of DTFT we know that DTFT does not always exist. It exists only when the signal is square summable, or satisfies the Dirichlet conditions. Therefore, $X(z)$ does not always converge. It converges only for some values of r. This range of r is called the region of convergence.

Definition 22. *The Region of Convergence (ROC) of the z-transform is the set of z such that $X(z)$ converges, i.e.,*

$$\sum_{n=-\infty}^{\infty} |x[n]|r^{-n} < \infty.$$

Example 1. Consider the signal $x[n] = a^n u[n]$, with $0 < a < 1$. The z-transform of $x[n]$ is

$$\begin{aligned} X(z) &= \sum_{-\infty}^{\infty} a^n u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n. \end{aligned}$$

Therefore, $X(z)$ converges if $\sum_{n=0}^{\infty} \left(az^{-1}\right)^n < \infty$. From geometric series, we know that

$$\sum_{n=0}^{\infty} \left(rz^{-1}\right)^n = \frac{1}{1 - az^{-1}},$$

when $|az^{-1}| < 1$, or equivalently $|z| > |a|$. So,

$$X(z) = \frac{1}{1 - ax^{-1}},$$

with ROC being the set of z such that $|z| > |a|$.

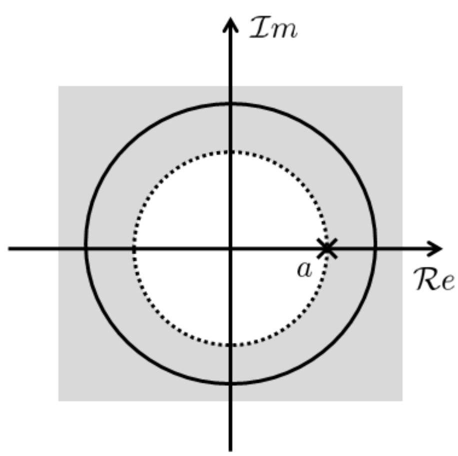


Figure 7.2: Pole-zero plot and ROC of Example 1.

Example 2. Consider the signal $x[n] = -a^n u[-n-1]$ with $0 < a < 1$. The z-transform of $x[n]$ is

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} \\ &= -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n} z^n \\ &= 1 - \sum_{n=0}^{\infty} \left(a^{-1} z\right)^n. \end{aligned}$$

Therefore, $X(z)$ converges when $|a^{-1}z| < 1$, or equivalently $|z| < |a|$. In this case,

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}},$$

with ROC being the set of z such that $|z| < |a|$. Note that the z-transform is the same as that of Example 1. The only difference is the ROC. In fact, Example 2 is just the left-sided version of Example 1!

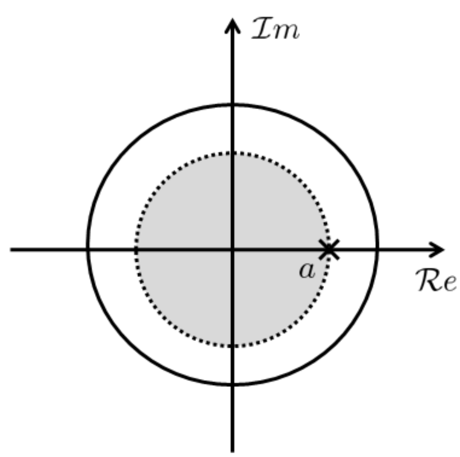


Figure 7.3: Pole-zero plot and ROC of Example 2.

Example 3. Consider the signal

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n].$$

The z-transform is

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty}\left[7\left(\frac{1}{3}\right)^n - 6\left(\frac{1}{2}\right)^n\right]u[n]z^{-n}\\
&= 7\sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^n u[n]z^{-n} - 6\sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^n u[n]z^{-n}\\
&= 7\left(\frac{1}{1-\frac{1}{3}z^{-1}}\right) - 6\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)\\
&= \frac{1-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}.
\end{aligned}$$

For $X(z)$ to converge, both sums in $X(z)$ must converge. So we need both $|z| > |\frac{1}{3}|$ and $|z| > |\frac{1}{2}|$. Thus, the ROC is the set of z such that $|z| > |\frac{1}{2}|$.

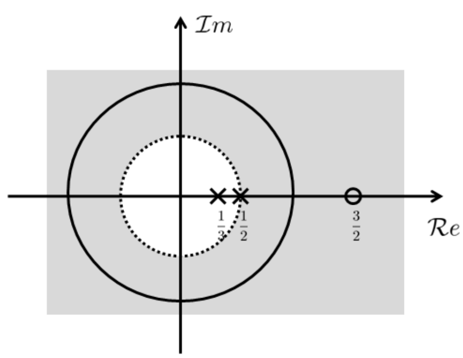


Figure 7.4: Pole-zero plot and ROC of Example 3.

7.2 z-transform Pairs

7.2.1 A. Table 10.2

1. $\delta[n] \longleftrightarrow 1$, all z
2. $\delta[n-m] \longleftrightarrow z^{-m}$, all z except 0 when $m > 0$ and ∞ when $m < 0$.
3. $u[n] \longleftrightarrow \frac{1}{1-z^{-1}}$, $|z| > 1$
4. $a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}}$, $|z| > a$
5. $-a^n u[-n-1] \longleftrightarrow \frac{1}{1-az^{-1}}$, $|z| < |a|$.

Example 4. Let us show that $\delta[n] \longleftrightarrow 1$. To see this,

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \delta[n] = 1. \end{aligned}$$

Example 5. Let's show that $\delta[n-m] \longleftrightarrow z^{-m}$:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n-m]z^{-n} \\ &= z^{-m} \sum_{n=-\infty}^{\infty} \delta[n-m] = z^{-m}. \end{aligned}$$

7.2.2 B. Table 10.1

1. $ax_1[n] + bx_2[n] \longleftrightarrow aX_1(z) + bX_2(z)$
2. $x[n-n_0] \longleftrightarrow X(z)z^{-n_0}$
3. $z_0^n x[n] \longleftrightarrow X(\frac{z}{z_0})$
4. $e^{jw_0 n} x[n] \longleftrightarrow X(e^{-jw_0} z)$
5. $x[-n] \longleftrightarrow X(\frac{1}{z})$

6. $x^*[n] \longleftrightarrow X^*(z^*)$

7. $x_1[n] * x_2[n] \longleftrightarrow X_1(z)X_2(z)$

8. If $y[n] = \begin{cases} x[n/L], & n \text{ is multiples of } L \\ 0, & \text{otherwise,} \end{cases}$
   then $Y(z) = X(z^L)$.

Example 6. Consider the signal $h[n] = \delta[n] + \delta[n-1] + 2\delta[n-2]$. The z-Transform of $h[n]$ is

$$H(z) = 1 + z^{-1} + 2z^{-2}.$$

Example 7. Let prove that $x[-n] \longleftrightarrow X(z^{-1})$. Letting $y[n] = x[-n]$, we have

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} \\ &= \sum_{m=-\infty}^{\infty} x[m]z^{m} = X(1/z). \end{aligned}$$

Example 8. Consider the signal $x[n] = \left(\frac{1}{3}\right)\sin\left(\frac{\pi}{4}n\right)u[n]$. To find the z-Transform, we first note that

$$x[n] = \frac{1}{2j}\left(\frac{1}{3}e^{j\frac{\pi}{4}}\right)^n u[n] - \frac{1}{2j}\left(\frac{1}{3}e^{-j\frac{\pi}{4}}\right)^n u[n].$$

The z-Transform is

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=0}^{\infty} \frac{1}{2j}\left(\frac{1}{3}e^{j\frac{\pi}{4}}z^{-1}\right)^n - \sum_{n=0}^{\infty} \frac{1}{2j}\left(\frac{1}{3}e^{-j\frac{\pi}{4}}z^{-1}\right)^n \\ &= \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{j\frac{\pi}{4}}z^{-1}} - \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{-j\frac{\pi}{4}}z^{-1}} \\ &= \frac{\frac{1}{3\sqrt{2}}z^{-1}}{(1-\frac{1}{3}e^{j\frac{\pi}{4}}z^{-1})(1-\frac{1}{3}e^{-j\frac{\pi}{4}}z^{-1})}. \end{aligned}$$

7.3 Properties of ROC

Property 1. *The ROC is a ring or disk in the z-plane center at origin.*

Property 2. *DTFT of $x[n]$ exists if and only if ROC includes the unit circle.*

Proof. By definition, ROC is the set of z such that $X(z)$ converges. DTFT is the z-transform evaluated on the unit circle. Therefore, if ROC includes the unit circle, then $X(z)$ converges for any value of z on the unit circle. That is, DTFT converges. □

Property 3. *The ROC contains no poles.*

Property 4. *If $x[n]$ is a finite impulse response (FIR), then the ROC is the entire z-plane.*

Property 5. *If $x[n]$ is a right-sided sequence, then ROC extends outward from the outermost pole.*

Property 6. *If $x[n]$ is a left-sided sequence, then ROC extends inward from the innermost pole.*

Proof. Let's consider the right-sided case. Note that it is sufficient to show that if a complex number z with magnitude $|z| = r_0$ is inside the ROC, then any other complex number z' with magnitude $|z'| = r_1 > r_0$ will also be in the ROC.
Now, suppose $x[n]$ is a right-sided sequence. So, $x[n]$ is zero prior to some values of n, say N_1. That is

$$x[n] = 0, \qquad n \leq N_1.$$

Consider a point z with $|z| = r_0$, and $r_0 < 1$. Then

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ &= \sum_{n=N_1}^{\infty} x[n] r_0^{-n} < \infty, \end{aligned}$$

because $r_0 < 1$ guarantees that the sum is finite.
Now, if there is another point z' with $|z'| = r_1 > r_0$, we may write $r_1 = ar_0$ for some $a > 1$. Then the series

$$\begin{aligned}\sum_{n=N_1}^{\infty} x[n] r_1^{-n} &= \sum_{n=N_1}^{\infty} x[n] a^{-n} r_0^{-n} \\ &\leq a^{N_1} \sum_{n=N_1}^{\infty} x[n] r_0^{-n} < \infty.\end{aligned}$$

So, z' is also in the ROC. □

Property 7. *If $X(z)$ is rational, i.e., $X(z) = \frac{A(z)}{B(z)}$ where $A(z)$ and $B(z)$ are polynomials, and if $x[n]$ is right-sided, then the ROC is the region outside the outermost pole.*

Proof. If $X(z)$ is rational, then by (Appendix, A.57) of the textbook

$$X(z) = \frac{A(z)}{B(z)} = \frac{\sum_{k=0}^{n-1} a_k z^k}{\prod_{k=1}^{r} (1 - p_k^{-1} z)^{\sigma_k}},$$

where p_k is the k-th pole of the system. Using partial fraction, we have

$$X(z) = \sum_{i=1}^{r} \sum_{k=1}^{\sigma_i} \frac{C_{ik}}{(1 - p_i^{-1} z)^k}.$$

Each of the term in the partial fraction has an ROC being the set of z such that $|z| > |p_i|$ (because $x[n]$ is right-sided). In order to have $X(z)$ convergent, the ROC must be the intersection of all individual ROCs. Therefore, the ROC is the region outside the outermost pole. □

For example, if

$$X(z) = \frac{1}{(1 - \frac{1}{3} z^{-1})(1 - \frac{1}{2} z^{-1})},$$

then the ROC is the region $|z| > \frac{1}{2}$.

7.4 System Properties using z-transform

7.4.1 Causality

Property 8. *A discrete-time LTI system is causal if and only if ROC is the exterior of a circle (including ∞).*

Proof. A system is causal if and only if

$$h[n] = 0, \qquad n < 0.$$

Therefore, $h[n]$ must be right-sided. Property 5 implies that ROC is outside a circle.

Also, by the definition that

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

where there is no positive powers of z, $H(z)$ converges also when $z \to \infty$ (Of course, $|z| > 1$ when $z \to \infty$!). □

7.4.2 Stablility

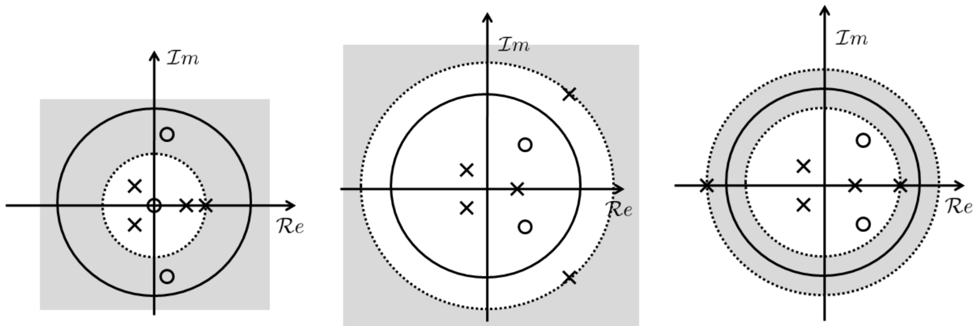
Property 9. *A discrete-time LTI system is stable if and only if ROC of $H(z)$ includes the unit circle.*

Proof. A system is stable if and only if $h[n]$ is absolutely summable, if and only if DTFT of $h[n]$ exists. Consequently by Property 2, ROC of $H(z)$ must include the unit circle. □

Property 10. *A causal discrete-time LTI system is stable if and only if all of its poles are inside the unit circle.*

Proof. The proof follows from Property 8 and 9 directly. □

Examples.



Causal, Stable　　Causal, Unstable　　Not causal, Stable