DIMENSIONAL ANALYSIS

Dimensional analysis depends upon the fundamental principle that any equation or relation between variables must be *dimensionally consistent'*, that is, each term in the relationship must have the same dimensions. Thus, in the simple application of the principle, an equation may consist of a number of terms, each representing, and therefore having, the dimensions of length. It is not permissible to add, say, lengths and velocities in an algebraic equation because they are quantities of different characters. The corollary of this principle is that if the whole equation is divided through by any one of the terms, each remaining term in the equation must be dimensionless. The use of these *dimensionless groups,* or *dimensionless numbers* as they are called, is of considerable value in developing relationships in chemical engineering. The requirement of dimensional consistency places a number of constraints on the form of the functional relation between variables in a problem and forms the basis of the technique of *dimensional analysis* which enables the variables in a problem to be grouped into the form of dimensionless groups. Since the dimensions of the physical quantities may be expressed in terms of a number of fundamentals, usually mass, length, and time, and sometimes temperature and thermal energy, the requirement of dimensional consistency must be satisfied in respect of each of the fundamentals. Dimensional analysis gives no information about the form of the functions, nor does it provide any means of evaluating numerical proportionality constants.
The study of problems in fluid dynamics and in heat transfer is made difficult by the many parameters which appear to affect them. In most instances further study shows that the variables may be grouped together in dimensionless groups, thus reducing the effective number of variables. It is rarely possible, and certainly time consuming, to try to vary these many variables separately, and the method of dimensional analysis in providing a smaller number of independent groups is most helpful to the investigated.
The application of the principles of dimensional analysis may best be understood by considering an example.

Exercises:

1.

PROBLEM 1.1

98% sulphuric acid of viscosity 0.025 N s/m^2 and density 1840 kg/m^3 is pumped at 685 cm^3/s through a 25 mm line. Calculate the value of the Reynolds number.

Solution

Cross-sectional area of line = $(\pi/4)0.025^2 = 0.00049$ m^2.

Mean velocity of acid, $u = (685 \times 10^{-6})/0.00049 = 1.398$ m/s.

$\therefore$ Reynolds number, $Re = du\rho/\mu = (0.025 \times 1.398 \times 1840)/0.025 = \underline{\underline{2572}}$

2.

The power required by an agitator in a tank is a function of the following four variables:

(a) diameter of impeller,
(b) number of rotations of the impeller per unit time,
(c) viscosity of liquid,
(d) density of liquid.

From a dimensional analysis, obtain a relation between the power and the four variables.

The power consumption is found, experimentally, to be proportional to the square of the speed of rotation. By what factor would the power be expected to increase if the impeller diameter were doubled?

Solution

If the power $P = \phi(DN\rho\mu)$, then a typical form of the function is $P = kD^aN^b\rho^c\mu^d$, where k is a constant. The dimensions of each parameter in terms of **M, L,** and **T** are: power, $P = \mathbf{ML}^2/\mathbf{T}^3$, density, $\rho = \mathbf{M/L}^3$, diameter, $D = \mathbf{L}$, viscosity, $\mu = \mathbf{M/LT}$, and speed of rotation, $N = \mathbf{T}^{-1}$

Equating dimensions:

$$\begin{aligned} \mathbf{M}: &\quad 1 = c + d \\ \mathbf{L}: &\quad 2 = a - 3c - d \\ \mathbf{T}: &\quad -3 = -b - d \end{aligned}$$

Solving in terms of d : $a = (5 - 2d)$, $b = (3 - d)$, $c = (1 - d)$

$\therefore$

$$P = k\left(\frac{D^5}{D^{2d}}\frac{N^3}{N^d}\frac{\rho}{\rho^d}\mu^d\right)$$

or:

$$P/D^5N^3\rho = k(D^2N\rho/\mu)^{-d}$$

that is:

$$N_P = k\,Re^m$$

Thus the power number is a function of the Reynolds number to the power m. In fact N_P is also a function of the Froude number, DN^2/g. The previous equation may be written as:

$$P/D^5N^3\rho = k(D^2N\rho/\mu)^m$$

Experimentally: $P \propto N^2$

From the equation, $P \propto N^m N^3$, that is $m + 3 = 2$ and $m = -1$

Thus for the same fluid, that is the same viscosity and density:

$$(P_2/P_1)(D_1^5N_1^3/D_2^5N_2^3) = (D_1^2N_1/D_2^2N_2)^{-1} \text{ or: } (P_2/P_1) = (N_2^2D_2^3)/(N_1^2D_1^3)$$

In this case, $N_1 = N_2$ and $D_2 = 2D_1$.

$\therefore$ $$(P_2/P_1) = 8D_1^3/D_1^3 = \underline{\underline{8}}$$

A similar solution may be obtained using the Recurring Set method as follows:

$$P = \phi(D, N, \rho, \mu),\ f(P, D, N, \rho, \mu) = 0$$

Using **M, L** and **T** as fundamentals, there are five variables and three fundamentals and therefore by Buckingham's π theorem, there will be two dimensionless groups.

Choosing D, N and ρ as the recurring set, dimensionally:

$$\left.\begin{array}{r} D \equiv \mathbf{L} \\ N \equiv \mathbf{T}^{-1} \\ \rho \equiv \mathbf{ML}^{-3} \end{array}\right] \quad \text{Thus:} \quad \left[\begin{array}{l} \mathbf{L} \equiv D \\ \mathbf{T} \equiv N^{-1} \\ \mathbf{M} \equiv \rho\mathbf{L}^3 = \rho D^3 \end{array}\right.$$

First group, π_1, is $P(\mathbf{ML}^2\mathbf{T}^{-3})^{-1} \equiv P(\rho D^3D^2N^3)^{-1} \equiv \dfrac{P}{\rho D^5N^3}$

Second group, π_2, is $\mu(\mathbf{ML}^{-1}\mathbf{T}^{-1})^{-1} \equiv \mu(\rho D^3D^{-1}N)^{-1} \equiv \dfrac{\mu}{\rho D^2N}$

Thus: $$\mathrm{f}\left(\frac{P}{\rho D^5N^3}, \frac{\mu}{\rho D^2N}\right) = 0$$

Although there is little to be gained by using this method for simple problems, there is considerable advantage when a large number of groups is involved.

3.

It is found experimentally that the terminal settling velocity u_0 of a spherical particle in a fluid is a function of the following quantities:

particle diameter, d; buoyant weight of particle (weight of particle – weight of displaced fluid), W; fluid density, ρ, and fluid viscosity, μ.

Obtain a relationship for u_0 using dimensional analysis.

Stokes established, from theoretical considerations, that for small particles which settle at very low velocities, the settling velocity is independent of the density of the fluid except in so far as this affects the buoyancy. Show that the settling velocity *must* then be inversely proportional to the viscosity of the fluid.

Solution

If: $u_0 = kd^a W^b \rho^c \mu^d$, then working in dimensions of **M, L** and **T**:

$$(\mathbf{L}/\mathbf{T}) = k(\mathbf{L}^a(\mathbf{ML}/\mathbf{T}^2)^b(\mathbf{M}/\mathbf{L}^3)^c(\mathbf{M}/\mathbf{LT})^d)$$

Equating dimensions:

$$\begin{aligned} \mathbf{M}: &\quad 0 = b + c + d \\ \mathbf{L}: &\quad 1 = a + b - 3c - d \\ \mathbf{T}: &\quad -1 = -2b - d \end{aligned}$$

Solving in terms of b:

$$a = -1, c = (b-1), \text{ and } d = (1-2b)$$

$\therefore$ $u_0 = k(1/d)(W^b)(\rho^b/\rho)(\mu/\mu^{2b})$ where k is a constant,

or: $u_0 = k(\mu/d\rho)(W\rho/\mu^2)^b$

Rearranging:

$$(du_0\rho/\mu) = k(W\rho/\mu^2)^b$$

where $(W\rho/\mu^2)$ is a function of a form of the Reynolds number.

For u_0 to be independent of ρ, b must equal unity and $\underline{\underline{u_0 = kW/d\mu}}$

Thus, for constant diameter and hence buoyant weight, the settling velocity is inversely proportional to the fluid viscosity.