Electromagnetic Fields and Waves

And Applications

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AND APPLICATIONS

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PREFACE

Electromagnetics in theory and practice constitute the most basic principles underlying all the various aspects of electrical, electronics and computer technologies in modern times.

The foundation of electromagnetic phenomena dates back some two to three hundred years ago with the discovery of the forces of interaction between electrical charges, whether stationary or in motion, the principle of the production of magnetism from steady electrical currents, or of electricity by means of changing magnetic fields.

Three major epochs of applications based on these phenomena could be identified. First, is the commercial production of electrical motors and generators, electrical heating and lighting, voltaic cells (batteries), telegraphs and telephones. Second, is the ability to communicate instantly at long distances by wireless electromagnetic means, producing voice, images and data technologies in broadcasting and telecommunications, in the last 100 to 150 years. The ability to manipulate individual electrons and atoms to generate, amplify and detect electromagnetic signals by means of electronic devices of the vacuum tubes, diodes, transistors, integrated circuits (ICs) and optical devices and lasers are all technologies based on the principles of electromagnetics.

The third phase, which is within the last fifty years involves informatics or information communication technologies (ICT), which, essentially, is the manipulation of electrical signals, whether analogue or digital, to produce new classes of functionality in optimum signal processing, computers, mechatronics, robotics and artificial intelligence.

All of the above technologies lead, with the passage of time and advancing frontiers of knowledge, into modern state-of-the-art applications of the cellular mobile (GSM), terrestrial and satellite communications, renewable energy resources, internet facilities, radio navigation systems, space exploration and travels, guided missiles of modern warfare and myriads of biomedical applications in health-care delivery, etc. The list is inexhaustive.

The building blocks of all these edifices rely entirely on the basic principles of electromagnetics in all ramifications. It is, therefore, imperative that a competent and confident electrical, electronics or computer engineer of the future is well advised to pay serious attention to the fundamental principles of electromagnetics and applications outlined in this book.

ELECTROSTATICS

1.1 INTRODUCTION

Experimental evidences by notable scientists such as Coulomb, Gauss, Ampere and Faraday have shown that electric and magnetic interactions are intimately connected. In fact, all magnetic effects are, in the final analysis, electrical in nature.

James Clerk Maxwell (1831-1879) developed some basic equations, known as Maxwell's equations, which unify the principles of electromagnetic effects, based on the various empirical relations of previous workers. Maxwell's equations represent a synthesis of electromagnetic fields, which led to the discovery of electromagnetic waves, and that light is electromagnetic in nature, with a constant velocity in vacuum, $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$, where μ_0, ε_0 are the **permeability** and **permittivity** of vacuum or air.

Long before the discovery of electricity, human beings have observed several electrostatic and electricity phenomena within their environment. In electric eel fish for example, some 5,000 to 6,000 stacked electroplaques embedded in their nervous system are capable of producing about 860 volts and 1 ampere of current for a few milliseconds. Another naturally occurring electricity phenomenon is the lightning discharge where interaction between electrically charged regions of a cloud causes a discharge of current as high as several tens of thousands of amperes.

Early scientists therefore began to invest concerted efforts in the quest of reproducing these observed phenomena for the advancement of knowledge as well as for the improvement of the standard of living of mankind. One of such efforts, particularly in electrostatics, is that which led to the popular Coulomb's law.

The study of static and moving charged particles is critical for the understanding of several electromagnetic phenomena as the movement or storage of electric charges remains the basis on which the development of several circuit components are established. In this book and in the study of static electric field in general, it is often assumed that electric charges are in stationary positions relative to each other. This is an idealization as in reality it is not so.

1.2 COULOMB'S LAW

When two electrical charges are situated in space, there is always a force of interaction between the charges. The magnitude and direction of this force in relation to the charges is given by the expression called ***Coulomb's law***. Named after the French Physicist, Charles Augustin de Coulomb, this law gives an insight into the nature of the invisible forces of interaction between charged particles; serving as a basis on which several other electromagnetic laws are established.

Let two charges Q_1 and Q_2 be situated in space and separated by a distance r as shown in Figure 1. **Coulomb postulated that the force $\bar{F}$ between the two charges is directly proportional to the product of the charges and inversely proportional to the square of the distance *r* between them**:

$$\bar{F} \propto \frac{Q_1 Q_2}{r^2}$$

$$\bar{F} = k\frac{Q_1 Q_2}{r^2} \tag{1}$$

The constant of proportionality k itself is inversely proportional to the permittivity ε of the medium in which the charges are located. That is,

$$k = \frac{1}{4\pi\varepsilon}$$

Therefore, in terms of the charges, the separating distance and the permittivity ε of the medium, the force is expressed as:

$$\overline{F} = \hat{r}\,\frac{1}{4\pi\varepsilon}\,\frac{Q_1 Q_2}{r^2} \tag{2}$$

where $\hat{r}$ is a unit vector in the direction of the line joining the two charges. For free space, the permittivity ε_0 is taken as $8.85 \times 10^{-12} Fm^{-1}$. Distance r is of course, larger than the size of the charges concerned for the relationship to hold.

The equality of the right hand side and the left hand side of Equation 1 can be verified using dimensional analysis by employing the fundamental dimensions of the quantities involved.

The following statements which describe the electrostatic phenomenon can be deduced from equation 2. Figure 1.1 (a),(b),(c) illustrate the force between two charges with different and same polarity as well as the inverse-square relationship of the force with the distance of separation of the charges, respectively.

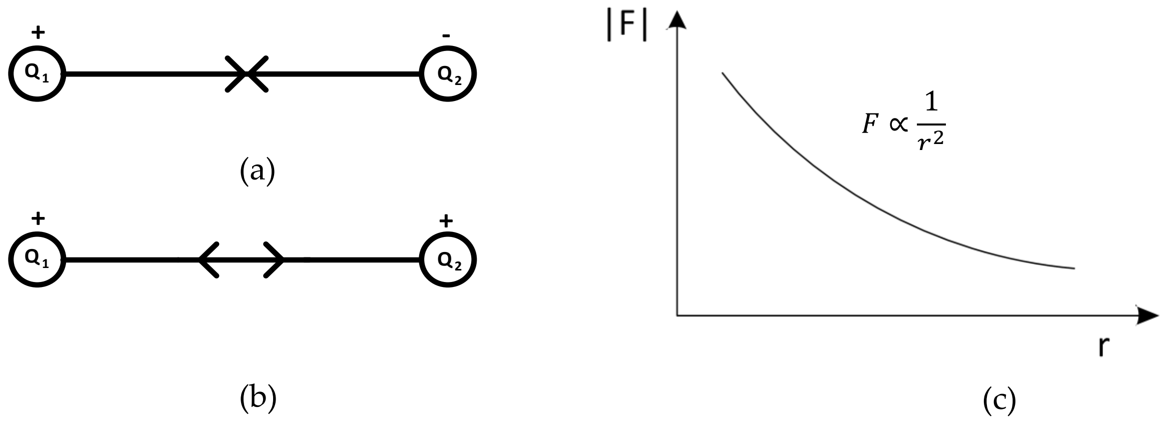


Figure 1.1: The force between two charges of (a) different signs (b) same signs. (c) The inverse square relationship between force F and distance r.

If the separating distance r is large enough to be considered infinite, the force $\bar{F}$ between the two charges becomes zero, this means neither of the charges exerts any measureable influence on the other.

Also, owing to the inverse square relationship between the force and the separating distance, the magnitude of the force between two charges reduces quite fast with increase in distance r. (See Figure 1.1c).

It should be noted as illustrated in Figure 1.1 that, if the two charged particles are of **same sign**, the force of interaction between them is **repulsive**, but if the two charges are of **different polarities**, the force between them is **attractive**.

The force between two charges of $1C$ each separated by $1m$ is about a million tons $(9 \times 10^9 N)$; $[1\ ton = 9 \times 10^3 = 9kN]$.

The pattern of the electric lines of forces for two similar charges and two dissimilar charges are shown in Figure 1.2 and Figure 1.3, respectively.

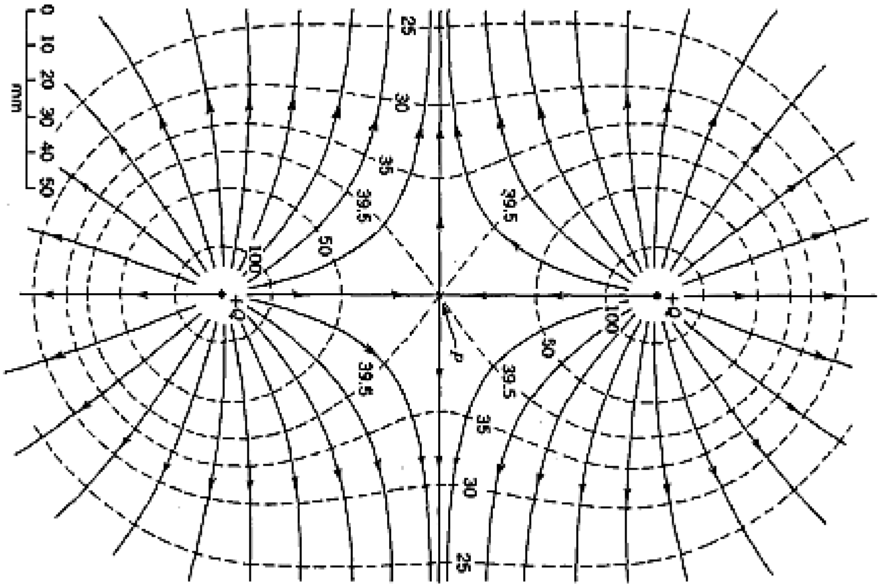


Figure 1.2. Lines of force around two positive charges, shown as continuous lines.

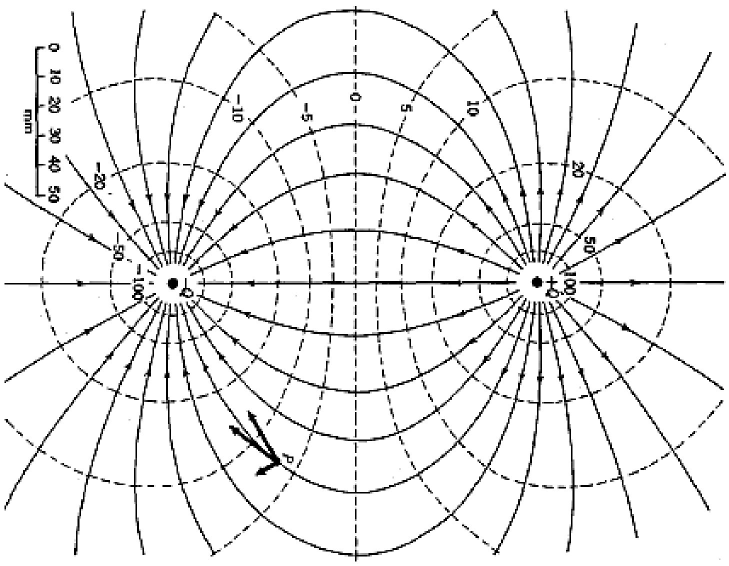


Figure 1.3. Lines of force around two charges of opposite polarities

Example 1.1

A positive point charge of 2 μC is located 650mm in space away from another positive charge of 1 μC . Determine the magnitude of the force between the charges.

Solution

From Coulomb's law (Equation 1), the force $\bar{F}$ between the charges is given as

$$F = k\frac{Q_1 Q_2}{r^2}$$

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2}$$

$$= \frac{1}{4\pi \times 8.85 \times 10^{-12}} \times \frac{2 \times 10^{-6} \times 1 \times 10^{-6}}{(650 \times 10^{-3})^2}$$

$$F = 0.043N$$

In many physical scenarios, however, there are usually more charges present in the space of interest than mere two charges separated by a distance r. These complex configurations can be aggregated by applying the principle of resolution and superposition of forces.

1.3 Electric Field Intensity, *E*

When there are numerous different charges located in a particular space of interest as is often the case in real world applications, it may be of interest to determine, in terms of force, the overall effect of all other charges at a particular point in the domain. This brings us to the concept of electric field.

From an illustrative point of view, if the charge Q_2 in Equation 1.2.1 is a positive test charge, then the force per unit charge experienced by this test charge would be

$$\frac{\bar{F}}{Q_2} = \hat{r}\frac{1}{4\pi\varepsilon}\frac{Q_1}{r^2} \tag{1}$$

The quantity $\frac{\bar{F}}{Q_2}$ is called the **electric field intensity** due to charge Q_1, at a distance r from it. Denoted by E, ***E*****lectric field intensity due to a static charge is defined as the electric force per unit charge experienced by a test charge placed at a given distance from the charge.** This quantity helps us to know the magnitude and direction of the resultant force experienced by a test charge located at any point within the field of influence of other charges.

Example 1.2

Three charges are arranged at three corners of a square of length $2.5cm$. The charge at the top left corner is $-3pC$ while a charge of $+3pC$ is located at the bottom right and bottom left corners. Determine the total electric field intensity at the fourth corner. ($\varepsilon = 8.85 \times 10^{-12} Fm^{-1}$).

Solution

The arrangement of these charges and the direction of the interacting forces are as shown in Figure 1.4. From definition, the electric field intensity is the net force exerted on a unit positive charge situated at the location of interest as a result of the presence of other charges. Thus, in our example, a charge Q_4 of 1C is assumed to be placed at the top right corner of the square.

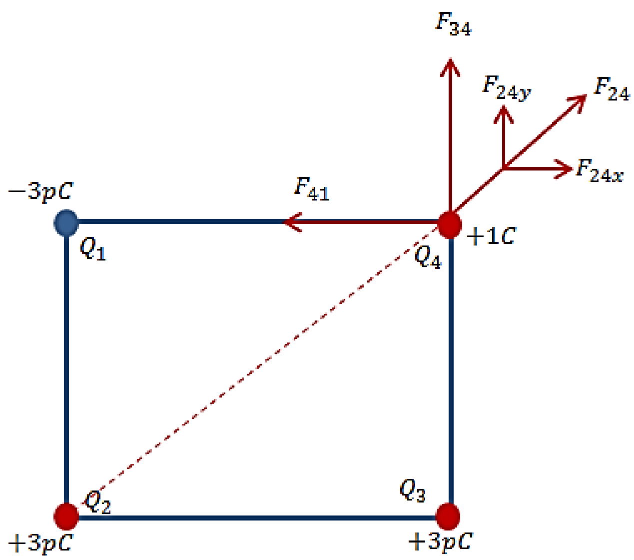


Figure 1.4. Arrangement of the charges for example 2 showing the direction of the forces.

By applying equation (2), the magnitude of the force F_{41} exerted on the unit positive charge due to charge Q_1 is:

$$F_{41} = \frac{1}{4\pi\varepsilon}\frac{Q_1Q_4}{r^2} = k\frac{Q_1Q_4}{r^2} = \frac{(8.99 \times 10^9)(-3.0 \times 10^{-12})(1)}{(2.5 \times 10^{-2})^2} = -43.15N$$

Similarly, the force $\bar{F}_{34}$exerted on the unit positive charge due to charge Q_3 is:

$$F_{34} = \frac{1}{4\pi\varepsilon}\frac{Q_3Q_4}{r^2} = k\frac{Q_3Q_4}{r^2} = \frac{(8.99 \times 10^9)(3.0 \times 10^{-12})(1)}{(2.5 \times 10^{-2})^2} = 43.15N$$

In the same vein, we can compute the remaining force $\bar{F}_{24}$ taking note of the separating distance r_k.

$$F_{24} = \frac{1}{4\pi\varepsilon}\frac{Q_2Q_4}{r_k^2} = k\frac{Q_2Q_4}{r_k^2} = \frac{(8.99 \times 10^9)(3.0 \times 10^{-12})(1)}{(3.536 \times 10^{-2})^2} = 21.57N$$

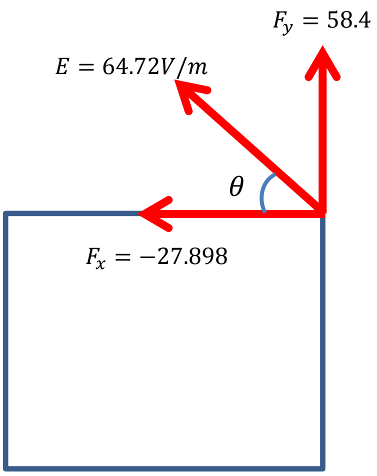
The effect of F_{24} on the unit positive charge must be resolved into vertical and horizontal components as shown in Figure 1.4. We can take the advantage of the structural symmetry of a square to know that the horizontal component of F_{24} is $F_{24}\,cos\,45^0$. While the vertical component is $F_{24}\,sin\,45^0$.

The net force exerted on the unit positive charge must be resolved vectorally because information on both magnitude and direction are crucial for complete description of vector fields. We can therefore resolve into horizontal (x-axis) and vertically (y-axis) component.

Force	x-components	y-components
F_{41}	-43.15	0
F_{34}	0	43.15
F_{24}	$21.57\cos 45^0$	$21.57\sin 45^0$
	$\mathbf{-27.8977}$	$\mathbf{58.4023}$

Therefore,

$$E = \sqrt{F_x^2 + F_y^2}$$



$$E = \sqrt{(-27.8977)^2 + (58.4023)^2} = 64.72N$$

At an angle of $\tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{58.4023}{27.8977}\right) = 64.47^0$ to the negative x-axis.

1.4 Electric Field of a Finite Line Charge

If a charge Q is uniformly distributed along the length of a thin wire of length L, the wire is therefore considered as a uniformly charged line. To obtain the electric field at a point P distance d away from the wire (see Figure 1.5),

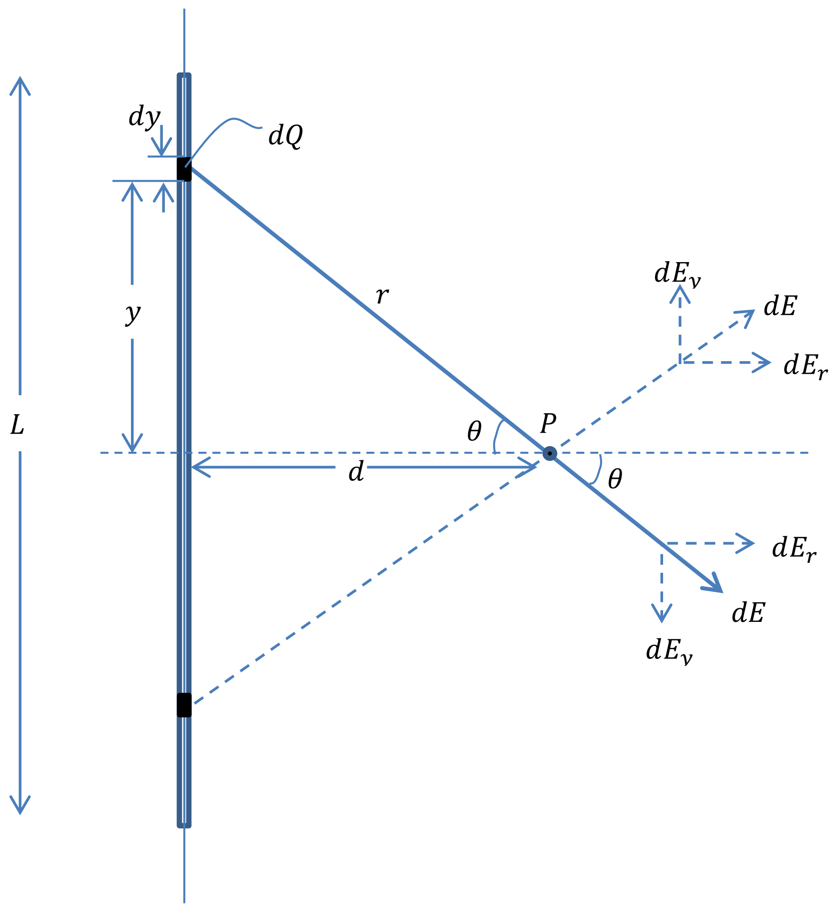


Figure 1.5: Illustration of the electric field due to a finite line charge

we take an infinitesimal portion of the wire dy with corresponding infinitesimal charge dQ and evaluate the electric field at point P distance r due to this portion. From Coulomb's law,

$$dE = k\frac{dQ}{(y^2 + d^2)} \tag{1}$$

where $= \frac{1}{4\pi\varepsilon}$. Each infinitesimal electric field vector dE can be resolved into vertical and horizontal components dE_y and dE_r respectively. However, because of symmetry, the vertical components cancel out one another such that only the horizontal component dE_r eventually survives. So that we can write:

$$dE_r = k\frac{dQ}{(y^2 + d^2)}\cos\theta \tag{2}$$

Expressing dQ and $\cos\theta$ in terms of dy using ratio of proportion, we have:

$$\frac{Q}{L} = \frac{dQ}{dy} \tag{3}$$

such that

$$dQ = \frac{Q}{L}dy \tag{4}$$

Also, in terms of the distance *d*, the quantity $\cos\theta$ can be written as:

$$\cos\theta = \frac{d}{\sqrt{(y^2 + d^2)}} \tag{5}$$

Now we can write Equation (2) as

$$dE_r = k\frac{\left(\frac{Q}{L}dy\right)}{(y^2+d^2)}\frac{d}{(y^2+d^2)^{1/2}} \tag{6}$$

To determine the net electric field at the distance d away from the line charge, we integrate over the entire length of the line.

$$E_r = \frac{kQd}{L}\int_{-L/2}^{L/2}\frac{dy}{(y^2+d^2)^{3/2}} \tag{7}$$

Evaluating Equation (7) using table of integrals, we obtain:

$$E_r = 2\left(\frac{kQd}{L}\frac{1}{d^2}\frac{y}{\sqrt{y^2+d^2}}\right)\Bigg]_0^{L/2} = \frac{2kQd}{L}\frac{1}{d^2}\frac{\frac{L}{2}}{\sqrt{\left(\frac{L}{2}\right)^2+d^2}} \tag{8}$$

$$E_r = \frac{kQ}{d\sqrt{\left(\frac{L}{2}\right)^2+d^2}} \tag{9}$$

To verify this, let's assume that the length L of the line charge tends to zero such that the line charge essentially becomes a point charge, the electric field from Equation 9 approximately becomes

$$E_r = \frac{kQ}{d^2} = \frac{Q}{4\pi\varepsilon d^2} \tag{10}$$

which is clearly the expression for the electric field at a distance d due to a point charge, Q.

1.4.1 Electric Field of an Infinite Line Charge

If the length of the line charge in Figure 1.5 is extended to infinity, it becomes an infinitely long positive line charge. Equation 1.4. (9) can be re-written as:

$$E_r = \frac{kQ}{\frac{d}{2}\sqrt{L^2 + 4d^2}} = \frac{kQ}{\frac{dL}{2}\sqrt{1 + \frac{4d^2}{L^2}}} \tag{1}$$

By factoring-in the infinite length of the line into equation 1, such that L tends to infinity ($L \rightarrow \infty$), the term $\frac{4d^2}{L^2}$ tends to zero. Hence Equation 1 becomes

$$E_r = k\frac{2Q}{dL} = \frac{1}{4\pi\varepsilon}\frac{2Q}{dL} \tag{2}$$

we obtain:

$$E_r = \frac{1}{4\pi\varepsilon}\frac{2Q}{dL} = \frac{Q}{2\pi\varepsilon dL} \tag{3}$$

$$E_r = \frac{Q}{L}\frac{1}{2\pi\varepsilon d} \tag{4}$$

The term $\frac{Q}{L}$ represents the charge per unit length (the line charge density ρ_L) of the line. Hence,

$$E_r = \frac{\rho_L}{2\pi\varepsilon d} \tag{5}$$

Equation (5), therefore, represents the electric field inensity at a distance d away from an infinite line charge, in terms of the charge per unit length ρ_L and the permittivity ε of the medium.

1.4.2 Electric Field of an Infinite Surface Charge

The operation of some electronic components are based on the principle of charged surfaces, these include strip transmission lines and parallel-plate capacitors. The understanding of the distribution of charges on surfaces (surface charge density) as a measure of the quantity of charged particles per unit surface area, is, therefore, important.

Consider an infinite yz-plane sheet of charge with the aim of obtaining the electric field intensity at a point P on the axis due to the sheet, Figure 1.6.

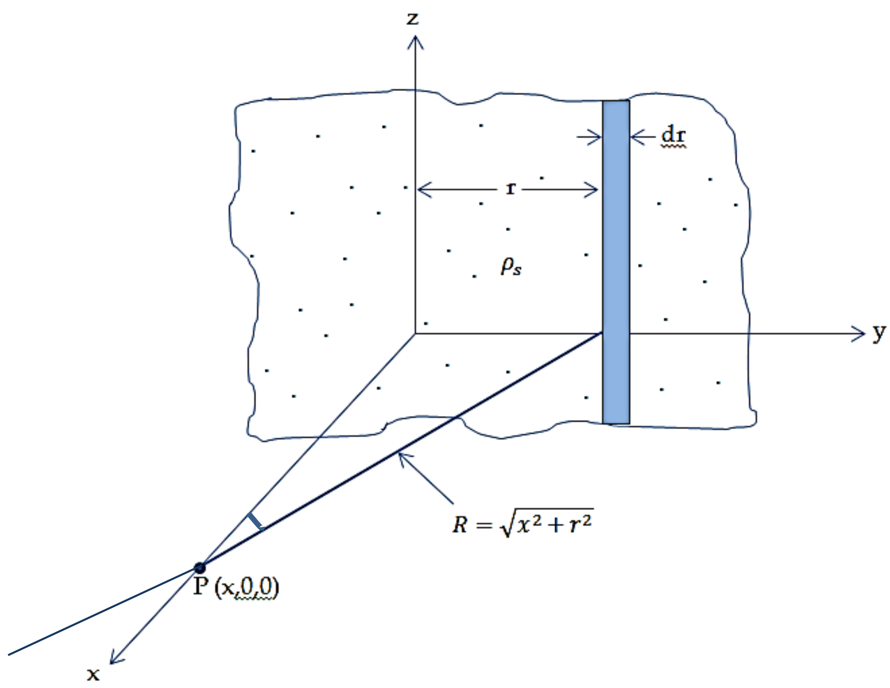


Figure 1.6: Electric field of an infinite sheet of charge

The electric field does not vary in z-direction or y-direction, thus, only the x-component (E_x) of the electric field due to the infinite sheet exists.

Consider a strip of width dr on the plane at distance r from the z-axis.

Let the charge per unit area of the sheet be ρ_s. Then the charge per unit length ρ_L is expressed as

$$\rho_L = \rho_s dr \qquad (1)$$

The distance between the differential width dr and the point P is R = $\sqrt{x^2 + r^2}$. Applying Equation 1.4.1 (5), the field E_x at P can be expressed as:

$$dE_x = \frac{\rho_s dr}{2\pi\varepsilon_0\sqrt{x^2 + r^2}} \cos\theta \qquad (2)$$

Now,

$$\cos\theta = \frac{x}{R} = \frac{x}{\sqrt{x^2 + r^2}}$$

Therefore,

$$dE_x = \frac{\rho_s}{2\pi\varepsilon_0} \left(\frac{xdr}{x^2 + r^2}\right) \qquad (3)$$

In order to put the overall width of the sheet into consideration, we can write

$$E_x = \frac{\rho_s}{2\pi\varepsilon_0} \cdot \int_{-\infty}^{+\infty} \frac{xdr}{x^2 + r^2} = \frac{\rho_s}{2\pi\varepsilon_0} \tan^{-1} \frac{r}{x} \bigg|_{-\infty}^{+\infty} = \frac{\rho_s}{2\varepsilon_0} \tag{4}$$

If the point P is located on the negative x-axis, we can show that

$$E_x = -\frac{\rho_s}{2\varepsilon_0} \tag{5}$$

Therefore, the electric field is generally expressed as

$$E_x = \frac{\rho_s}{2\varepsilon_0} \hat{\boldsymbol{x}} \tag{6}$$

where $\hat{\boldsymbol{x}}$ is a unit vector outward normal to the surface of the sheet.

The significance of this equation is that, provided the charged sheet is considered infinite, the electric is independent of the distance from the plane surface. The field several kilometers away from the surface of the sheet is as strong as the field near the surface of the sheet. The electric field is, therefore, constant in magnitude and direction.

If a second infinite sheet of charge density $-\rho_s$ is situated at $x = a$ in a similar plane and beside the first infinite charged sheet, the electric field in the region within the two sheets is obtained as the addition of the individual fields due to each charge, as illustrated in Figure 1.7.

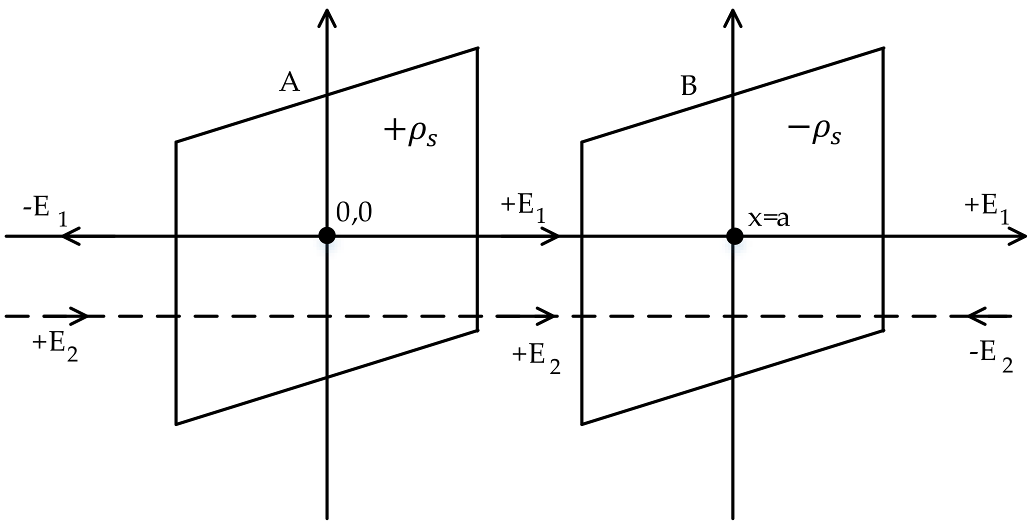


Figure 1.7. Electric field between two oppositely-charged infinite sheets of charge

Let A be an infinite sheet of charge of surface density $+\rho_s\ Cm^{-2}$ located at $x = 0$ on the yz plane. Denote the electric field in the positive x-direction due to $+\rho_s$ as E_1; the field is $-E_1$ in the negative x-direction as shown (full line).

Let B be a second infinite sheet of charge, with charge density $-\rho_s$, located at a distance $x = a$, parallel to sheet A, as shown. The electric field of this negatively charged sheet is directed inwards, as shown with broken lines. The field due to this is $-E_2$ in the positive x-direction.

The total field to the left of sheet A is $-E_1 + E_2 = 0$ when $|E_1| = |E_2|$. Similarly, the total field to the right of sheet B is $+E_1 - E_2 = 0$.

However, the total field between the sheets A and B is E = $+E_1 + E_2 = 2E_1$ or $2E_2$.

Since $E_1 = E_2 = \frac{\rho_s}{2\varepsilon_0}$, $E = \frac{\rho_s}{\varepsilon_0}$ in the positive x-direction.

That is,

$$E = \frac{\rho_s}{2\varepsilon_0}\hat{\boldsymbol{x}} + \frac{\rho_s}{2\varepsilon_0}\hat{\boldsymbol{x}} = \frac{\rho_s}{\varepsilon_0}\hat{\boldsymbol{x}} \quad (7)$$

At every other region apart from in-between the two charged sheets, the net field is zero. It should be noted that the assumption is that the sheets are infinitely wide, or much wider than the separation distance between the two sheets, and that the surface charge density of the two sheets are of opposite polarity. Additionally, it is assumed that the distance of separation a, is small.

Assignment:

Q. 1: Determine the electric field intensity at a distance d from the axis of a ring-shaped conductor with a radius R carrying total charge Q uniformly distributed around the ring.

Q. 2: Determine the electric field intensity caused by a disc of radius R placed in the yz plane with a uniform positive surface charge density ρ_s (charge per unit area) at a distance d from the centre of the disc along the positive x-axis.

1.5 Total Charge of a Volume Charge Distribution

To understand the concept of volume charge distribution, consider a tremendous number of charged particles spread over a definite volume of space. With the safe assumption that the distances separating the individual particles in the volume are negligible, the charge per unit volume can be computed in order to estimate the volume charge density ρ_v.

Consider a small amount of charge ΔQ in a small volume ΔV, then

$$\Delta Q = \rho_v \Delta V \tag{1}$$

If we take the limit of the charge-volume ratio = $\frac{\Delta Q}{\Delta V}$ as ΔV tends to zero, we obtain:

$$\rho_v = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V} \tag{2}$$

such that the total charge enclosed within a specified volume could be obtained by a volume integral with limits covering the entire volume.

$$Q = \iiint \rho_v \Delta V \tag{3}$$

Consider a cylindrical volume (shown in Figure 1.8), the cylinder is centered along the z-axis. If the interest is to determine the total charge enclosed by the volume given a volume charge density ρ_v.

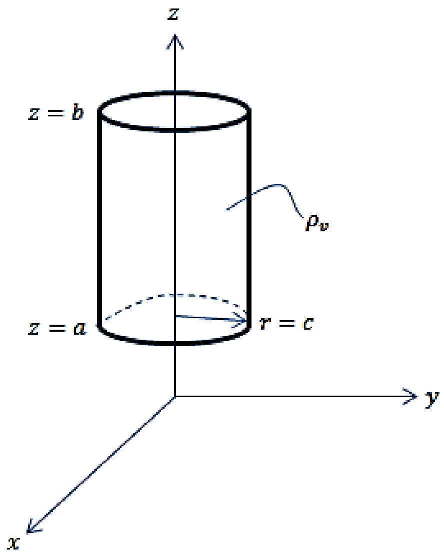


Figure 1.8: A charged volume

It follows that the total volume is the triple integral, using cylindrical coordinate system, expressed as:

$$Q = \int_a^b \int_0^{2\pi} \int_0^c \rho_v r dr d\varphi dz \tag{4}$$

$$Q = 2\pi\rho_v \int_a^b dz \int_0^c r dr \tag{5}$$

ELECTRIC POTENTIAL

2.1 The Concept of Electric Potential

Consider the uniform electric field depicted in Figure 2.1. When charges are moved from one point to another against the direction of the electric field ***E***, expectedly, work is done against the field. Therefore, a measure of the work done in transporting a test charge from one point to another in a direction parallel to a uniform electric field is termed **electric potential** difference between the two points**.** In other words, the electric potential between two points A and B is defined as the work done per unit charge in moving a test charge from A to B.

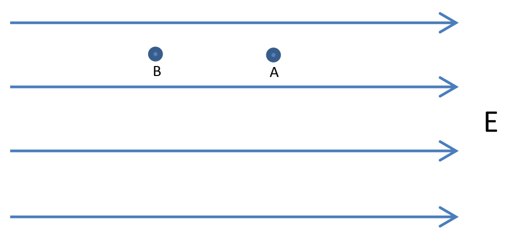


Figure 2.1 Two points A and B in a uniform electric field

However, if the initial location of the test charge is so significantly far away as to be considered infinity, the potential at point B is considered absolute. Therefore, **absolute potential at a point** is defined as the work done against the electric field in carrying a unit positive test-charge from infinity to that **point**, where the potential at infinity is regarded as zero.

When moving the test charge against the direction of the electric field $\boldsymbol{E}$, more energy is expended, thus there is an increase in electric potential. Conversely, when moving the test charge along the direction of the field $\boldsymbol{E}$, there is a reduction in potential because the field does the work. It should be noted that potential is a scalar quantity with a unit of joules per coulomb (JC^{-1}).

Depending on the distribution of electric charges in a uniform electric field, there could be surfaces or lines that are of same potential. These are called **equipotential surfaces** or **equipotential lines**. When a test charge is traversed along these lines, there is no change in potential.

Now consider a point charge Q, located at O, the origin of the Cartesian coordinate, as shown.

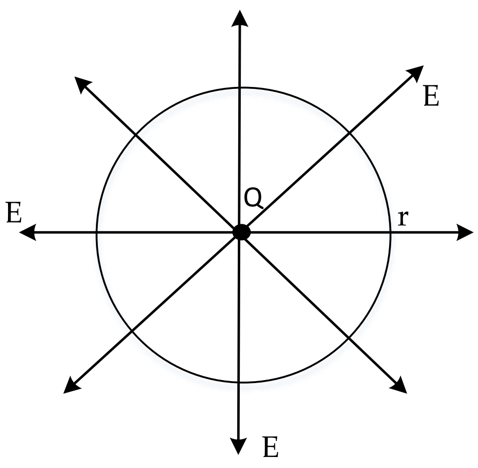


Figure 2. 2: The potential at a point *r* away from a charge *Q*

Any distance *r* from *O* will describe a spherical surface centered on *O*, and the absolute potential at any point on this surface will be equal. That is, the spherical surface is described as an equipotential surface. In fact, it does not matter the path taken by the test charge in moving from infinity to that point, the potential on the surface is still the same.

It follows therefore that no work is done in moving a test charge from one point to another on an equipotential line or surface.

Note also that the electric field from charge *Q* is everywhere in the radial direction, and cuts the equipotential surface at right angles. **In general, it is an important property of electric fields that equipotential line or surface is at right angles or orthogonal to the direction of the electric field.**

Since the electric field around a positive charge Q is not uniform, the potential difference in moving a test charge from a position r_b to r_a as illustrated in Figure 2.3, can be regarded as a summation of the incremental potential differences, $E_r dr$, within the two end points.

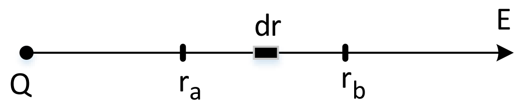


Figure 2.3: Potential between two points

$$V_{ab} = -\int_{r_b}^{r_a} E_r dr \qquad (1)$$

where dr is an element of distance between r_b and r_a and the summation becomes an integral when $dr \rightarrow 0$;

The negative sign in Equation 1 is an indication that work is done against the electric field in moving from r_b to r_a.

$$\text{Now,} \quad E_r = \frac{Q}{4\pi\varepsilon_0 r^2} \qquad (2)$$

Substituting Equation (2) into Equation (1), we have

$$V_{ab} = -\int_{r_b}^{r_a} \frac{Q}{4\pi\varepsilon_0 r^2} dr \qquad (3)$$

$$V_{ab} = -\int_{r_b}^{r_a} \frac{1}{4\pi\varepsilon}\frac{Q}{r^2}dr = -\frac{Q}{4\pi\varepsilon}\int_{r_b}^{r_a}\frac{1}{r^2}dr \tag{4}$$

$$V_{ab} = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \tag{5}$$

Thus, the absolute potential at a point r from a charge Q is given by

$$V_r = \frac{Q}{4\pi\varepsilon_0 r} \tag{6}$$

when $r_b \rightarrow \infty$

It is immaterial what path is taken by a test charge to arrive at point r. In other words, the work done to move a test charge around a closed path in a static field is zero, since the path starts and ends at the same point as illustrated in Figure 2. 4.

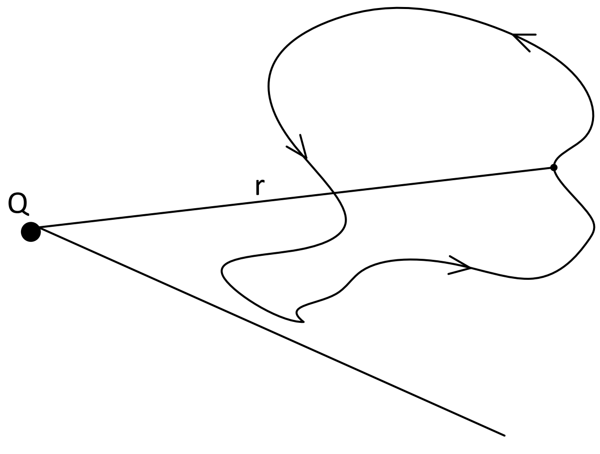


Figure 2.4: A test charge moved along a closed path in an electric field

Or, considering Equation (5), $V_{ab} = 0$ when $r_a = r_b$.

A property of the static electric fields is that the line integral of the field around a closed path is zero. i.e. expressing in vector form,

$$\oint \boldsymbol{E} . d\boldsymbol{l} = 0 \qquad (7)$$

A field for which Equation (7) holds is called a **conservative** or **lamellar** field. The potential difference between any two points of a conservative field is independent of the path.

Potential is a scalar quantity. This means when several point charges produce different potentials at a given point, the resultant potential is a simple, scalar, addition of the individual potentials of the charges.

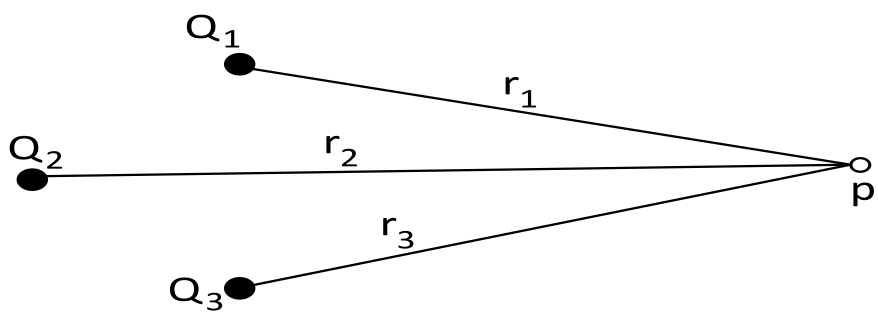


Fig. 2.5 Potential of a number of point charges

$$V_p = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right] \qquad (8)$$

The concept of line and surface charges have been discussed earlier. Let us now assume that a line charge of density ρ_l, a surface charge of density ρ_s and four point charges (Q_1, Q_2, Q_3, and Q_4) are arranged as shown in Figure 2.6. The electric potential at point P is the scalar addition of the individual potentials due to each of the six charge elements. The principle behind this approach is called the superposition principle as applied to electric potential.

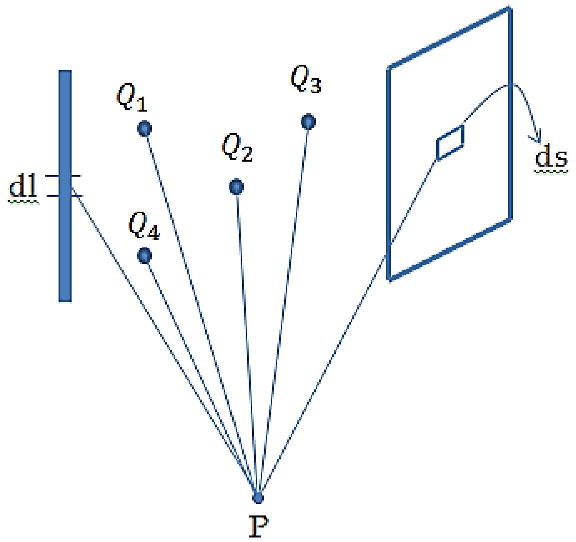


Figure 2.6: Superposition principle illustrated with a line charge, point charges and surface charge

This principle states that the total electric potential at a point is the algebraic sum of the individual component potentials at the point. In other word, the total electric potential at point P is a sum of the potentials at point P as a result of the line charge, the point charges and the surface charge.

$$V_{total} = V_{line} + V_{points} + V_{surface} \tag{9}$$

The potential at point P due to the line charge situated r_{PL} meter away from P is written as:

$$V_{line} = \int \frac{\rho_l}{4\pi\varepsilon r_{PL}} dl = \frac{1}{4\pi\varepsilon} \int \frac{\rho_l}{r_{PL}} dl \tag{10}$$

Similarly, the potential at P due to the surface charge is:

$$V_{surface} = \int \frac{\rho_s}{4\pi\varepsilon r_{Ps}} ds = \frac{1}{4\pi\varepsilon} \iint \frac{\rho_s}{r_{Ps}} ds \tag{11}$$

While that at point P due to the four point charges is:

$$V_{points} = \frac{Q_1}{4\pi\varepsilon r_{P1}} + \frac{Q_2}{4\pi\varepsilon r_{P2}} + \frac{Q_3}{4\pi\varepsilon r_{P3}} + \frac{Q_4}{4\pi\varepsilon r_{P4}} = \frac{1}{4\pi\varepsilon} \sum_{1}^{N} \frac{Q_n}{r_{Pn}} \tag{12}$$

Therefore the total potential (V_{total}) is:

$$V_{total} = \frac{1}{4\pi\varepsilon} \int \frac{\rho_l}{r_{PL}} dl + \frac{1}{4\pi\varepsilon} \sum_{1}^{N} \frac{Q_n}{r_{Pn}} + \frac{1}{4\pi\varepsilon} \iint \frac{\rho_s}{r_{Ps}} ds \tag{13}$$

2.1.1: Application of the electric dipole principle: The Lightning Flash

Lightning is a natural phenomenon. It is one of the natural events and forces of nature that have been observed by man from time immemorial. The fearful and devastating nature of lightning and thunder had aroused man's curiosity and contemplation over the years. Many gods and goddesses have been ascribed the sources of the forces and powers, and the only way man knew how to combat the supernatural phenomenon was through divinations and prayers.

Scientific study of this phenomenon began in the second half of the 18th century following the discovery by laboratory experimentation of electrostatics in Europe. It was concluded that the phenomenon of electrical discharges observed in the sky was of the same nature as the laboratory electrical discharges, though of much greater magnitude.

The thundercloud that produces lightning is now regarded as a huge electrostatic generator which produces electrical charges, both positive and negative. The positive charge is concentrated in one region of the cloud and the negative in another region, a kind of a **giant electrical dipole.** The charge separation occurs due to aerodynamic motions of atmospheric particles and the wind. As the separation between the charges proceeds, the electrical field between them, or between one of them and the earth, grows until an electrical breakdown of the air occurs, resulting in lightning flashes, intra-cloud, cloud-to-cloud or cloud-to ground as illustrated in Fig. 2.11.

When the discharge occurs, an intense electrical current flows through the channel producing high pressure shock wave explosion and loud audible sound, which is the thunder.

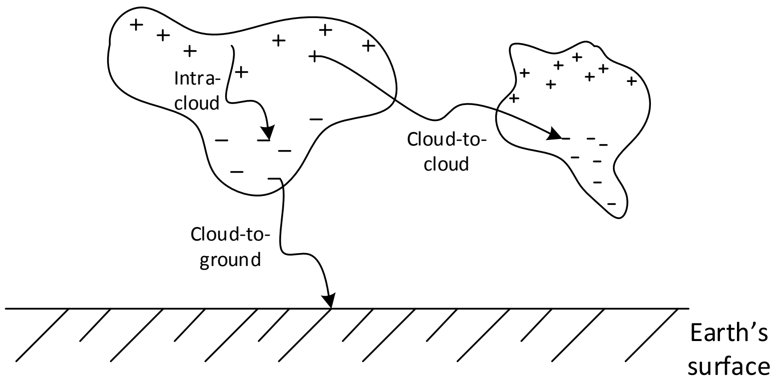


Fig. 2.11: Model of thunder clouds and lightning discharges.

The best protection of a building or structure against lightning is by the installation of lightning conductors on top of the building. The design and installation must be done carefully taking into consideration the size, shape and location of the building. Copper is almost universally chosen as material for a lightning conductor on account of its good conductivity and resistance to corrosion. The tip of the rod is made sharp so as to enhance the concentration of the electric field on it, thereby producing an easy path for the electric current to ground, bypassing the structure being protected. Actually, the lightning conductor does not prevent a lightning discharge occurring, contrary to popular belief, it merely intercepts the path of the ground flashes and harmlessly diverts the current to earth, the more reason why the resistance of the ground at the base of the conductor must be very low.

2 .1 The Electric Dipole

An electric dipole consists of two point charges of equal magnitude $\boldsymbol{Q}$ and different signs separated by a short distance $\boldsymbol{d}$. This separating distance is considered to be small compared to the region away from the dipole in which the electric field and potential is to be evaluated.

A good understanding of the concept of electric dipole is crucial in electromagnetics as it serves as one of the basis upon which several phenomena are based. These include mirroring, conducting planes, dipole antennas and infinite conductors carrying current in opposite directions.

The electric dipole described so far is illustrated if Fig. 2.8, where r_1 is the distance between charge $+Q$ and point P, at which the electric field and/or potential is to be determined, and r_2 is the distance between charge $-Q$ and point P, while r is the distance between the midpoint of the dipole length and point P.

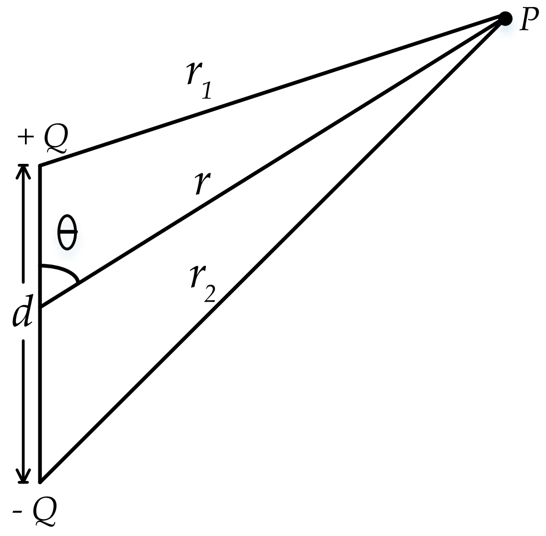


Figure 2.8: The electric dipole illustrating the fields at point P

In order to obtain the fields at a point P, we can minimize the complexity of the problem by first determining the electric potential at that point first.

Now, we proceed by taking the first assumption that r is long enough such that the distances r_1, r and r_2 can be considered parallel lines as shown in Figure 2.9.

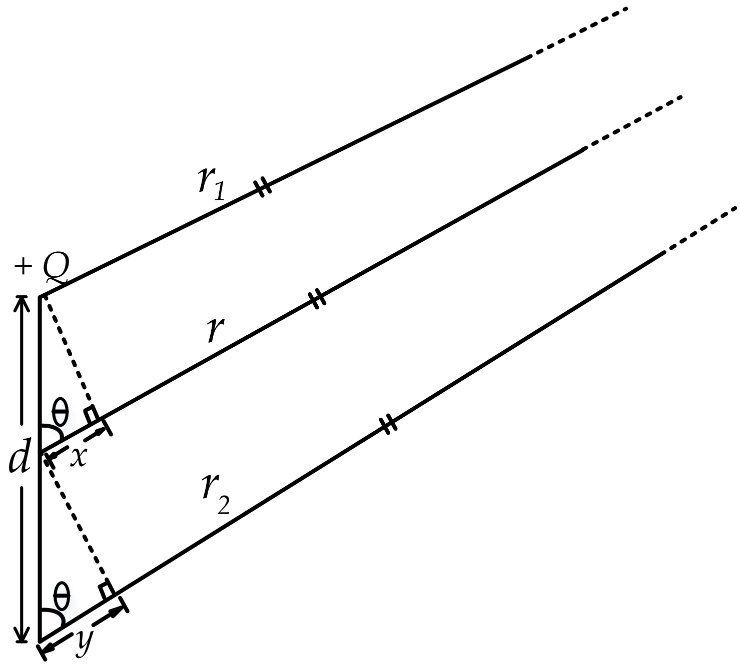


Figure 2.9: Extension of the location of point P such that the connecting distances are almost parallel.

The difference between distances r and r_1 is x, and The difference between distances r_2 and r is y such that we can write:

$$x = \frac{d}{2}\cos\theta \qquad and \qquad y = \frac{d}{2}\cos\theta$$

And

$$r_1 = r - \frac{d}{2}\cos\theta \qquad and \qquad r_2 = r + \frac{d}{2}\cos\theta$$

Therefore, the total electric potential V at far away point P due to the two charges is:

$$V = V_{+Q} + V_{-Q} = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{1}$$

Substituting for the values of r_1 and r_2 in Equation (1), we obtain:

$$V = V_{+Q} + V_{-Q} = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r - \frac{d}{2}\cos\theta} - \frac{1}{r + \frac{d}{2}\cos\theta}\right) \tag{2}$$

Simplifying Equation (2) and ignoring the terms with multiple powers of d because r is very large compared to d, we find that:

$$V = \frac{Qd\cos\theta}{4\pi\varepsilon r^2} \tag{3}$$

The term Qd is referred to as the **dipole moment**.

It is therefore evident that the potential at point P due to the dipole is inversely proportional to the square of the distance r between the centre of the dipole and P. This is in slight contrast with the expected relationship between the electric potential at point P due to only one of the charges, wherein the electric potential maintains an inverse relation with the separating distance between the charge and point P.

We remember that the electric potential between two points is generally denoted by:

$$\mathrm{V} = -\int \overline{E}.\ \overline{dr} \tag{4}$$

Note that Equation (4) is the integral of the product of two vectors, namely, the electric field intensity $\bar{E}$ and the displacement $\overline{dr}$ along which the test charge is moved. This implies that when the two vectors are not in the same direction, the magnitude of the potential is Edr (Cosθ) where θ is the angle between them; moreover, when θ = 90^0, i.e., the two vectors are at right angles to each other, the potential difference along the displacement is zero, *an equi-potential line.*

In the reverse operation, we can write:

$$E = -\frac{dV}{dr}\,\hat{\boldsymbol{r}} \tag{5}$$

The right hand side of equation (5) is analogous to finding the ***gradient*** of the electric potential V. Thus,

$$\mathrm{E} = -\mathrm{grad\ V} \tag{6}$$

And in spherical coordinates, we can present the electric field $\boldsymbol{E}$ at point $P(r,\theta)$ with symmetry or no variation in the azimuthal (ϕ) direction, as:

$$\boldsymbol{E} = -\hat{\boldsymbol{r}}\frac{\partial V}{\partial r} - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial V}{\partial \theta} \tag{7}$$

Substituting V from equation (3) into Equation (7) and differentiating appropriately with respect to r and θ, we have:

$$\boldsymbol{E} = \hat{\boldsymbol{r}} \frac{Qd\cos\theta}{2\pi\varepsilon r^3} + \hat{\boldsymbol{\theta}} \frac{Qd\sin\theta}{4\pi\varepsilon r^3} \tag{8}$$

Equations (3) and (8) are the relations for the electric potential and electric field intensity at point $P(r, \theta)$ from the electric dipole. Once again, it should be noted that these relations are obtained with the assumption that the length d of the electric dipole is very small compared to the distance between point P and the centre of the dipole.

The plot of the potential and the electric field of a dipole is shown in Figure 2.10, which is the same as was presented in Figure 1.3 at the earlier portion of this course.

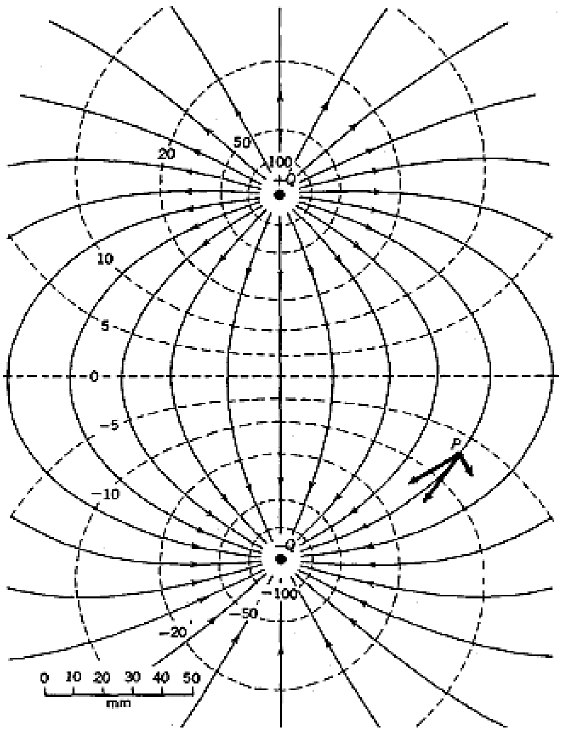


Figure 2.10: The plot of the potential and electric field of an electric dipole

The potentials are in dotted lines while the electric fields are in solid lines. It would be observed that at any given point, the equipotential line (or surface) is at right angles (or orthogonal) to the electric field lines.

GAUSS' LAW AND THE ELECTRIC FLUX

When charged particles are close enough as to cause significant interaction, lines of force are generated between them. These lines of force (otherwise called "flux") generally originate from positive charges and terminate on negative charges. If a section of the surface is cut by a plane (see Figure 3.1), an estimate of the total number of flux lines passing through this surface can be obtained.

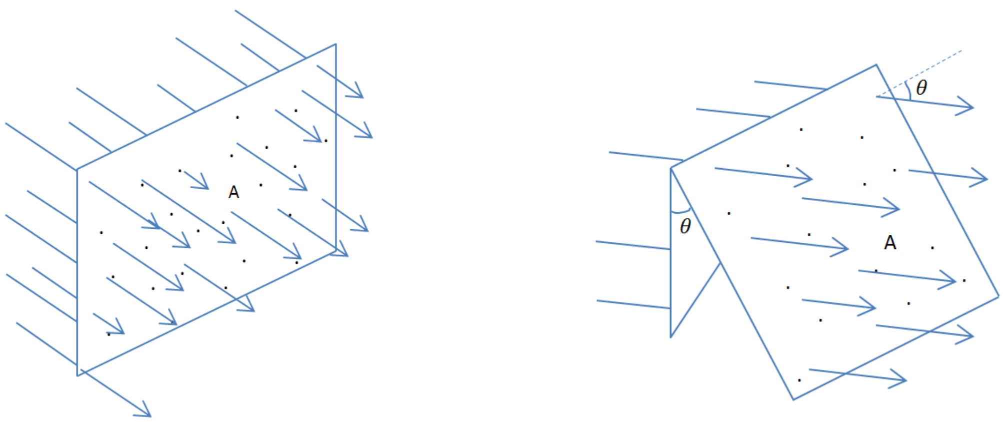


Figure 3.1: The electric flux through a normal and inclined surface.

The electric flux ψ, through any section of this surface can be obtained as a product of the average flux density, $\bar{D}$, a vector, in Coulomb per square meter, (Cm^{-2}) and the area $\bar{A}$, also a vector, of the section (in square meter). The direction of the vector $\bar{A}$ is taken to be the outward normal to the surface. It, therefore, follows that the electric flux through any section of this surface is the integral of the flux density over the area of the section of surface.

In general, for any surface described by, $\overline{ds}$, and located at a distance r away from a charge Q, the electric flux ψ through it is related to the flux density $\bar{D}$ by:

$$\psi = \iint \bar{D} \cdot \overline{ds} \qquad (1)$$

In spherical coordinate system depicted by Figure 3.2,

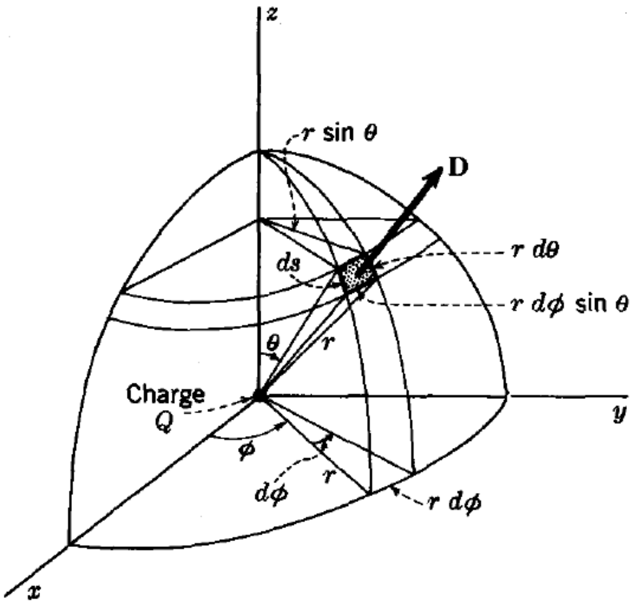


Figure 3.2: The spherical coordinate system showing location of an infinitesimal surface ds

$$ds = r^2 \sin\theta \, d\theta d\varphi \qquad (2)$$

As such, we can re-write the equation of the electric flux emanating from a charge Q located at the centre of a sphere as:

$$\psi = \int_0^r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} D \, ds$$

$$= \int_0^r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} D \, r^2 \sin\theta \, d\theta d\varphi$$

$$= \mathrm{D} \, (4\pi r^2) \qquad (3)$$

Equation (3) describes the total flux from the surface of the sphere of radius r, emanating from the charge at the centre of the sphere. Since $4\pi r^2$ is the surface area of a sphere of radius r, we can equate the total flux emanating from the spherical surface to the charge Q producing the flux, and write,

$$\psi = \iint \overline{D} \, . \, \overline{ds} = \mathrm{D}(4\pi r^2) = \mathrm{Q} \qquad (4)$$

$$\text{Or} \quad \mathrm{D} = \frac{Q}{4\pi r^2} \qquad (5)$$

Noting, also, that the magnitude of the electric field intensity, $\bar{E}$, around a single isolated charge, Q, at a distance r from the charge, is given by,

$$\mathrm{E} = \frac{Q}{4\pi \varepsilon r^2}, \qquad (6)$$

we can establish the relation between the flux density D emanating from charge Q to the electric field intensity E as,

$$\overline{D} = \varepsilon \, \overline{E} = \frac{Q}{4\pi r^2} \, \hat{r} \qquad (7)$$

Equation (5) implies that the magnitude of $\overline{D}$ at radius r is identical to the surface charge density ρ_s in C m^{-2} were charge Q to be distributed uniformly over the spherical surface instead of being centralized at the centre of the sphere.

The expression, Equation (4), above is, therefore, the basis upon which Karl Freidrich Gauss (1813) formulated the law, popularly known as Gauss' law, stated as follows:

3.1 Gauss' Law

the surface integral of the outward normal component of the electric flux density over any closed surface is equal to the charge enclosed.

The closed surface need not be spherical in shape; it is a law applicable to any closed surface within which the charge Q resides; moreover, Q need not be a single charge. If there are multiple charges, positive and negative, Q will be the scalar addition of the charges or the net charge.

A simple illustration which makes Gauss' law easier to visualize is that of a uniformly permeable spherical perfume container. It would be observed that the total perfume fragrance oozing out of the sphere will be same as the initial concentration of perfume within the sphere.

Assuming the charge Q were to be uniformly distributed within the volume of the sphere instead of being concentrated at the centre with the volume charge density ρ_v, in C m^{-3}, and the incremental volume dv, Gauss' law can be expressed as:

$$\oint_s \bar{D}\,.\,\overline{ds} \;=\; \oint_v \rho_v dv = Q \tag{8}$$

The laws propounded by Coulomb and Gauss are important laws in the study of charges of different configurations; they represent handy tools in understanding the field distributions at different regions around charged structures of different shapes. The application of Gauss' law greatly simplifies the calculation of the electric field intensity and/or the potential of charge distribution in the vicinity of conductors of simple geometries, instead of solving the integral equations involved.

3.2 Electric Field and Potential of a Charged Spherical Shell

Consider a uniformly charged spherical shell of radius a, presumably of zero thickness with a charge Q uniformly distributed on its surface. It is required to calculate the electric field intensity and the potential distribution within and outside of the sphere. Assume the sphere is in air.

Because there is no charge enclosed in the region $r \le a$ (inside the sphere), the integral of $\mathbf{D}$ around any closed surface within the sphere is zero. It becomes understandable, therefore, that $\mathbf{E} = 0$ inside the sphere. However, for regions described by $r \geq a$, the electric flux density $\overline{D} = \hat{\boldsymbol{r}}\frac{Q}{4\pi r^2}$, applying Gauss' law and noting that the total charge enclosed is Q. The electric field intensity $\bar{E} = \hat{\boldsymbol{r}}\frac{Q}{4\pi\varepsilon_0 r^2}$. That is, the electric field at any point outside the charged sphere is same as the field due to a point charge Q located at the centre of the sphere.

For the electric potential at different regions due to the charged sphere, we can write:

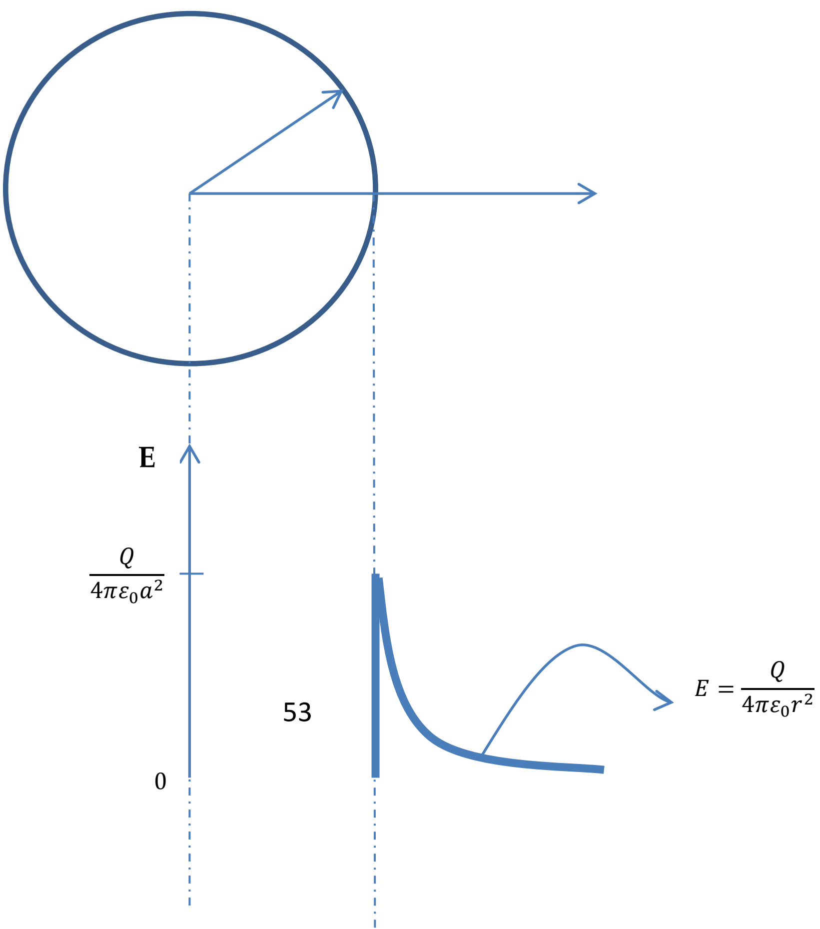
$$V = -\int_{\infty}^{r} \boldsymbol{E}.d\boldsymbol{r} = -\int_{\infty}^{r} \frac{Q}{4\pi\varepsilon_0 r^2}.dr = \frac{Q}{4\pi\varepsilon_0 r} \qquad (V/m) \qquad (1)$$

The potential inside the sphere is constant because it requires no additional work to bring a test charge from infinity to any region within the sphere than the surface of the sphere. The potential within and outside the sphere is expressed as,

$$V_r = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 a} & r \leq a \\ \\ \dfrac{Q}{4\pi\varepsilon_0 r} & r \geq a \end{cases} \qquad (2)$$

Figure 3.3 a-c portray these information including the expected discontinuity observed in the electric field distribution and the continuity of the potential distribution at the sphere-air interface.

(*a*)



(*b*)

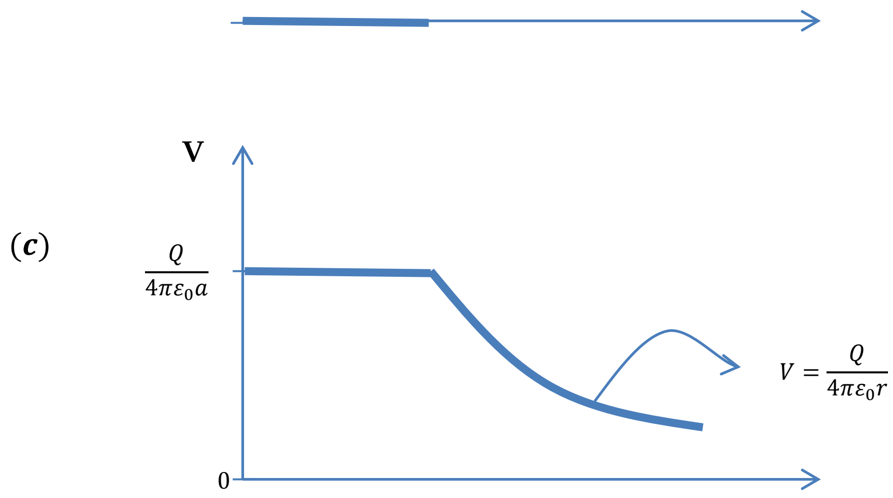


Figure 3.3 Diagram showing the distribution of the electric field and potential of a charged spherical shell.

3.3. The Coaxial Cable

A coaxial cable is made up of an inner conductor surrounded by an outer conductor wherein both conductors are separated by a dielectric material. The term "coaxial" comes from the understanding of the fact that the inner and outer conductors share a common geometric axis.

A good knowledge of the behaviour of the electric field and potential everywhere around a coaxial cable is very important. This is because the coaxial cable is a very common type of transmission line with a wide range of applications. These include their use for distributing TV signals, computer network connections, as well as feed lines for radio transmitters and receivers.

Consider the coaxial cable shown in Figure 3.4, where the inner conductor is of radius, a, and the outer conductor is of radius, b.

The thin inner conductor has a linear positive charge per unit length ρ_L. The two conductors are separated by a dielectric material of permittivity ε. To obtain the electric field at a region described by, $r < b$, i.e. any region in-between the two conductors, we need to draw a Gaussian cylindrical surface of radius, r, surrounding the inner conductor, and apply Gauss' law to this surface, to obtain,

$$\frac{Q}{\varepsilon} = \oint \boldsymbol{E} \cdot d\boldsymbol{A} \qquad (1)$$

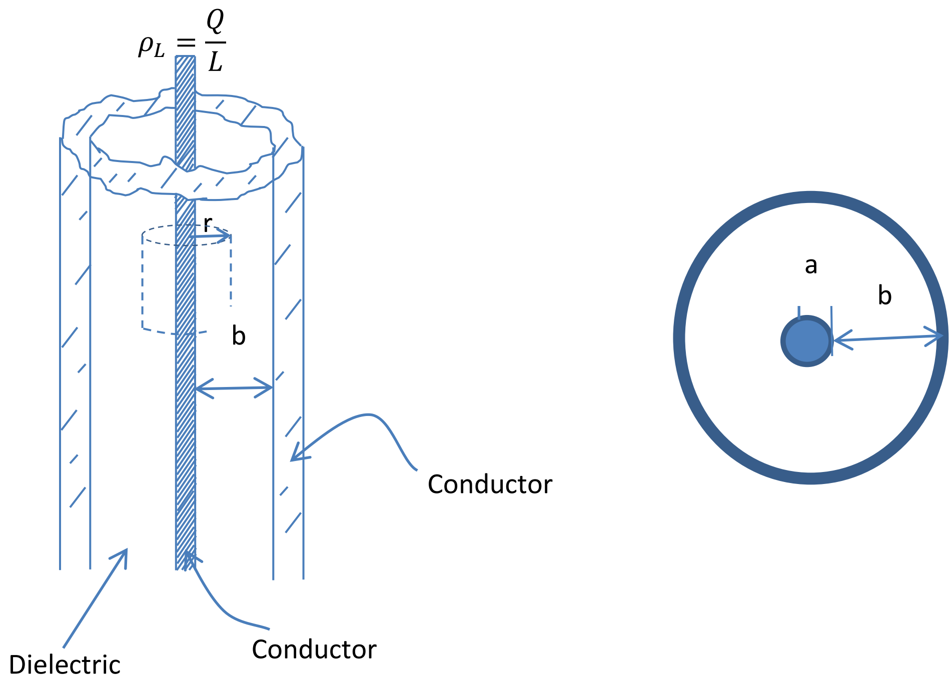


Figure 3.4: A coaxial cable showing the inner and outer conductors

Replacing the charge Q in Equation (1) above with the charge per unit length, and a length, h, of the conductor, we have,

$$\frac{\rho_L h}{\varepsilon} = \oint \boldsymbol{E} \cdot d\boldsymbol{A} \qquad (2)$$

We can eliminate the dot product in Equation (2) because $\boldsymbol{E}$ and $d\boldsymbol{A}$ are in the same radial direction. Furthermore, the $\boldsymbol{E}$ can be taken outside the integrand because the electric field is constant along the cylindrical length owing to the symmetry of the figure; and there is no flux through the bottom and the top of the cylinder. Therefore, Equation (2) becomes:

$$\frac{\rho_L h}{\varepsilon} = E \oint dA \tag{3}$$

For a cylinder, the surface area is circumference times the height, neglecting the top and bottom faces through which no flux emanates,

i.e.,

$$\frac{\rho_L h}{\varepsilon} = E \oint dA = E(2\pi r h) \tag{4}$$

$$E = \frac{\rho_L}{2\pi r \varepsilon} \tag{5}$$

The potential difference between the inner and outer conductors of the coaxial cable is obtained from,

$$V_{ab} = -\int_b^a \frac{\rho_L dr}{2\pi\varepsilon r} = \frac{\rho_L}{2\pi\varepsilon}\int_a^b \frac{dr}{r} \tag{6}$$

$$= \frac{\rho_L}{2\pi\varepsilon} \log_e \frac{b}{a} \tag{7}$$

QUIZ:

Two concentric hollow spherical conductors in free space, centred at the origin of the Cartesian coordinates, have radii, a, and, 2a, respectively. The inner sphere carries a charge of, +3Q, and the outer one, -2Q. Determine

(i) the potential difference between the conductors, and

(ii) the electric field intensity and the potential at a point distance, 3a, from the centre.

3.4 The Divergence Theorem from Gauss' Law

In vector calculus, the divergence theorem, also known as Gauss' theorem, is a result that relates the flow (or flux) of a vector field through a surface to the behavior of the vector field inside the surface.

Applied to electrostatics, the vector field is that of the electric flux density $\overline{D}$; and the theorem is expressed as,

$$\iint \overline{D}.\overline{ds} \;=\; \iiint \nabla \,.\, \overline{D} \text{ dv}$$

(1)

where the LHS of Eqn. (1) is the surface integral of the flux density over the closed surface surrounding the volume from which the flux emanates, as illustrated in Fig. 3.6.

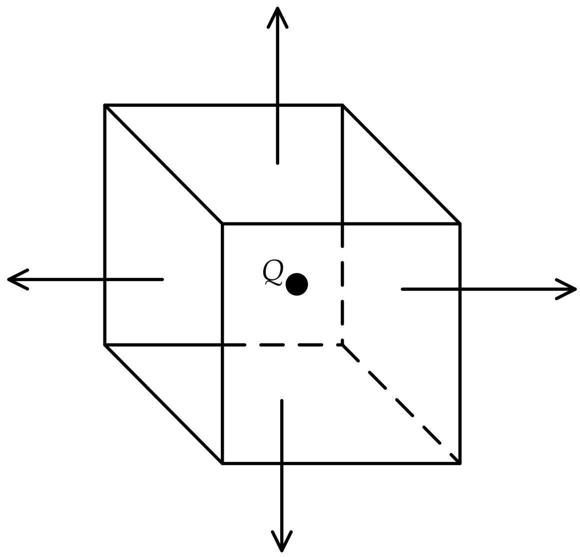


Figure 3.6: Electric flux from a volume enclosing a charge Q

Thus, the divergence theorem may be stated as follows:

The integral of the normal component of the electric flux density over a closed surface is equal to the integral of the divergence of the flux density throughout the volume enclosed by the surface.

By definition, the divergence of the vector function $\overline{D}$ is expressed as,

$$\nabla . \ \overline{D} \ = \ \lim \mathrm{dv} \rightarrow 0 \ \frac{\iint \overline{D}.\overline{ds}}{dv} \tag{2}$$

If we assume a small element of volume, dv, containing charge , Q, which is uniformly distributed within the volume with volume charge density, ρ_v, (C m^{-3}), then, by applying Gauss' law, we have,

$$\iint \overline{D}.\overline{ds} = \iiint \rho_v \, \mathrm{dv} = \mathrm{Q} \quad (3)$$

Combining Eqns (1) and (3) we have,

$$\iiint \nabla.\overline{D} \, \mathrm{dv} = \iiint \rho_v \, \mathrm{dv}$$

(4)

Or, $$\boldsymbol{\nabla} \,.\, \overline{\boldsymbol{D}} = \boldsymbol{\rho_v} \quad (5)$$

In the Cartesian coordinate system,

$$\nabla \,.\, \overline{D} = \hat{x}\frac{\partial D_x}{\partial x} + \hat{y}\frac{\partial D_y}{\partial y} + \hat{z}\frac{\partial D_z}{\partial z} \quad (6)$$

where the symbol, (del or lamda), ∇, treated as a vector, is expressed as,

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}, \text{ and}$$

(7)

$$\overline{D} = \hat{x}\, D_x + \hat{y}\, D_y + \hat{z}\, D_z \quad (8)$$

Eqn (5) is one of the most important equations of electrostatics, relating the divergence of the electric flux density to the volume charge density.

Eqns (3) and (5) are the same in concept, expressing Gauss' law in two different forms, first, in **integral or macroscopic** form, Eqn (1), and second, in **differential or_microscopic** form, Eqn (5).

3.4.1 Poisson's and Laplace's equations

Eqn (5) may be further expanded, given the relatioships between $\bar{D}$, $\bar{E}$ and V, already established, as follows,

$$\nabla . \bar{D} = \nabla . \varepsilon \bar{E} = \varepsilon(\nabla . \overline{E)} = \varepsilon \nabla. (- \nabla V) = - \varepsilon \nabla^2 V = \rho_v ,$$

(9)

Or, $$\nabla^2 V = - \frac{\rho_v}{\varepsilon} \quad (10)$$

known as Poisson's Equation.

In a volume of space where there $\rho_v = 0$, no charge density, we have,

$$\nabla^2 V = 0 \quad (11)$$

known as Laplace's Equation.

Eqns (10)and (11) are very handy in the solution of a number of electrostatic problems of various configurations subject to certain boundary conditions.

3.5 Boundary Conditions at the Interface of Two Dielectric Materials

Electric field behaviour in different materials differs as their permittivities differ. Therefore, when two different materials form a composite, a dielectric boundary with interesting properties is created. As a result, the properties of these boundaries need to be understood in order to understand the behaviour of the field around the boundaries as electromagnetic waves traverse through them. Electrostatic boundary condition is very important in many ways, in optics, for example, boundary condition allows us to derive the various equations bordering on reflections and refractions at the interface between two different media and their refractive indices, and so on. Remember, light is a member of the family of electromagnetic waves!

The electric field entering the interface from medium 2, from any arbitrary angle, as shown in Fig. 3.7, could be resolved into two components, namely, a component parallel to the surface (i.e the tangential component) and the component normal to the surface (the normal component). Similarly, for the electric field emanating from medium 1.

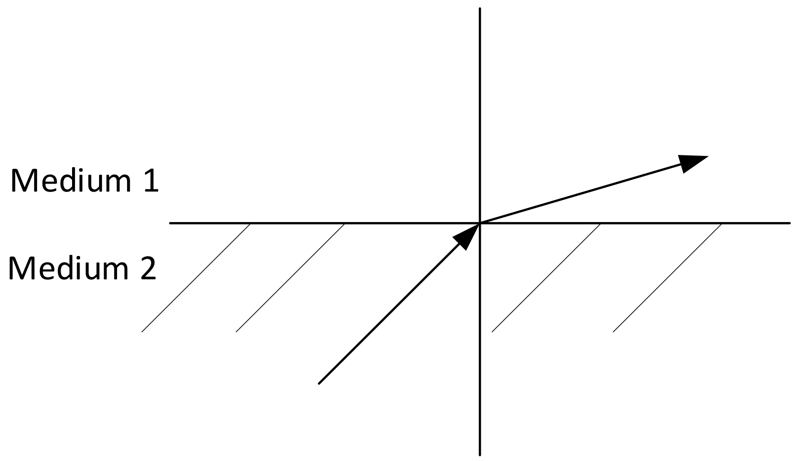


Fig. 3.7: Electric field entering and emanating the interface between two media

First, consider the tangential component of the electric field, illustrated in Fig. 3.8. Let the media have permittivities, ε_1, ε_2, and conductivities, σ_1, σ_2, respectively, as shown. If both media are perfect dielectrics, the conductivities are, of course, zero. Consider a rectangular path, half in each medium, of length Δx parallel to the interface and length Δy normal to the interface, as illustrated. Let the electric field intensity tangent to the boundary in medium 1 be E_{t1}, similarly E_{t2} for medium 2. The work per unit charge required to move a positive test charge round the rectangular closed path is the line integral $\oint \bar{E}.\bar{dl}$. By making Δy tend to zero, since we are considering the interface between the two media,, the work along this perpendicular section is zero.

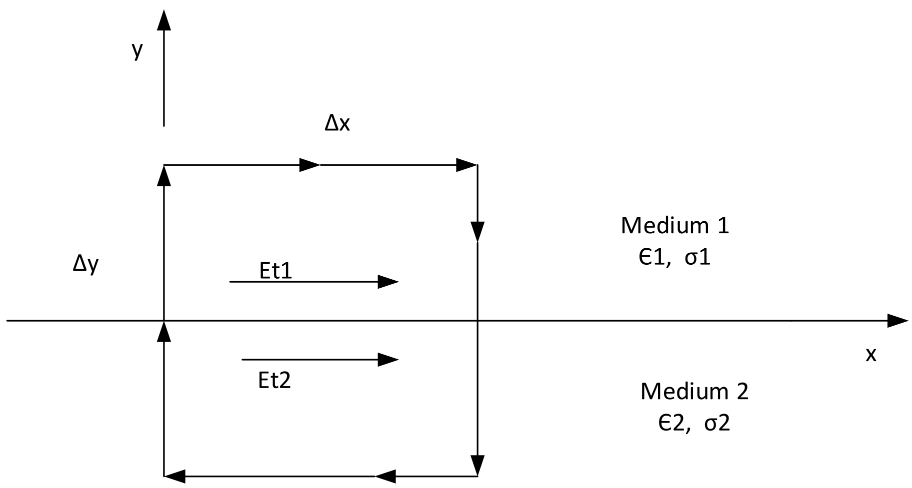


Fig. 3. 8: Tangential components of the electric field at interface of two media

The line integral of the electric field intensity around the rectangle, in the direction of the arrows,will then be,

$$E_{t1}\,\Delta x \ - \ E_{t2}\,\Delta x \ = 0 \qquad (1)$$

or $$\boldsymbol{E_{t1}} \ = \ \boldsymbol{E_{t2}} \qquad (2)$$

According to Eqn (2), we conclude that **the tangential components of the electric field intensity are the same on both sides of the boundary between two dielectrics, or, that the tangential electric field intensity is continuous across such a boundary.**

It is significant to note that if medium 2 happens to be a metal or any perfect conductor, the static electric field intensity within such a medium is zero. Hence, that in medium 1 will also be zero by implication of Eqn (2). That is, *the tangential component of the electric field intensity at a conductor/dielectric (or air) interface is zero; only the normal component exists.*

3.6. Electric Field Normal to the Interface

To determine the electric field intensity normal to the interface, we could take the Gauss' law approach. This approach requires us to employ some form of

Gaussian surface. For this thin interface, we can pick a *Gaussian pillbox* with an area S parallel to the surface and thickness h across the interface.

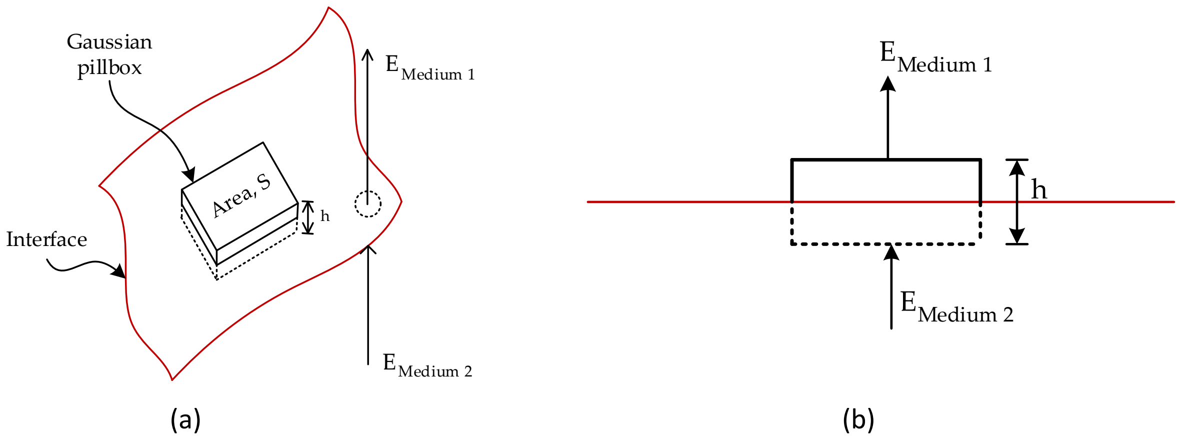


Figure 3.9: (a) The normal component of electric flux at an interface (b) A section of the composite showing the pillbox centered on the interface

This pillbox is centered on the interface such that the part with continuous line falls into medium 1 while the part with the dotted line falls into medium 2. From Gauss' law, we have:

$$\oint_s \boldsymbol{E} \cdot d\boldsymbol{s} = \frac{1}{\varepsilon} \int \boldsymbol{\rho_v} \mathrm{dv} \qquad (1)$$

The left hand side of Equation (1) represents the total outward electric flux from the pillbox, which, clearly, is the difference between the flux emanating from the top of the pillbox in medium 1 minus the flux entering the pillbox in medium 2. Therefore, the net normal component of the flux is:

$$\varphi = \left(\boldsymbol{E}^{normal}_{medium\ 1} - \boldsymbol{E}^{normal}_{medium\ 2}\right)S \tag{2}$$

Now, if we shrink the height of the Gaussian pillbox to zero leaving only the top and bottom of the pillbox (i.e. $h \rightarrow 0$) such that the sides no longer contribute to the electric flux, we would no longer be talking about a volume, rather the pillbox becomes essentially a surface, such that the right hand side of Equation (1) becomes the integral of a surface charge, with a surface charge density ρ_s (C m^{-2}) , i.e,

$$\frac{1}{\varepsilon}\int \rho_v dv = \frac{1}{\varepsilon}\int \rho_s ds = \frac{\rho_s S}{\varepsilon} \tag{3}$$

$$\left(\boldsymbol{E}^{normal}_{medium\ 1} - \boldsymbol{E}^{normal}_{medium\ 2}\right) = \frac{\rho_s}{\varepsilon} \tag{4}$$

The implication of Equation (4) is that **the normal component of the electric field intensity is discontinuous across the boundary by an amount $\frac{\rho_s}{\varepsilon}$.**

3.7 Capacitors and Capacitances

Two conductors separated by an insulating or dielectric material constitute a capacitor; the popular shapes of which are parallel- plate, cylindrical or spherical. By definition, the capacitance, C, of a capacitor is the ratio of the electrical charge, Q, on one of the conductors to the potential difference or voltage, V, between the conductors. That is,

$$C = \frac{Q}{V} \qquad (1)$$

3.7.1 Parallel-plate capacitor

Consider a parallel-plate capacitor of area A,carrying charge +Q on the upper plate, charge -Q on the lower plate and voltage, V, between the plates. Let the plate separation be, d. The electric field intensity, E, the flux density, D and the permittivity, ε, of the medium between the plates, are all indicated in the diagram of Fig. 3.10.

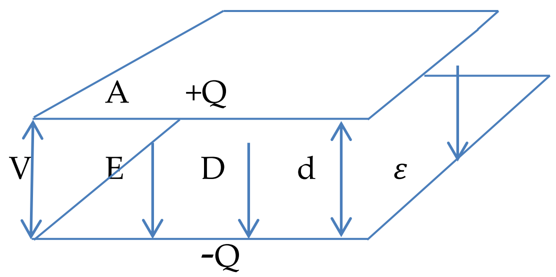


Fig. 3.10: Capacitance of a parallel-plate capacitor

From the analysis made in section 1.4.2, (see p. 26), we find that the electric field intensity between two oppositely-charged infinite, or large, sheets of charge is given by,

$$E = \frac{\rho_s}{\varepsilon} \qquad (1)$$

where, ρ_s, is the surface charge density (C m^{-2}). We can apply this relation to the parallel-plate capacitor situation, assuming area, A, is much greater than the plate separation, d.

We have the following relations:

$$\rho_s = \frac{Q}{A} \qquad (2)$$

$$E = \frac{V}{d} \qquad (3)$$

Combining Eqns (1), (2) and (3), we have,

$$E = \frac{Q}{\varepsilon A} = \frac{V}{d} \qquad (4)$$

Hence, the capacitance, $$\mathbf{C} = \boldsymbol{\frac{Q}{V}} = \boldsymbol{\frac{\varepsilon A}{d}} \qquad (5)$$

Example 1:

Calculate the capacitance of a parallel-plate capacitor in air of plate area of 1 m^2 and plate separation of 1 m.

Ans:

$$C = \frac{\varepsilon_0 A}{d} = \frac{8.854\, x 10^{-12}\; x\; 1}{1} \quad = 8.854 \text{ pF}$$

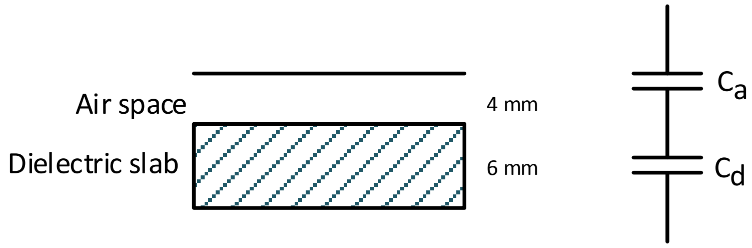
Note: The capacitance remains the same value if the dimensions are reduced proportionately, e.g., if A = 0.01 m^2 (or 10 $mm^{2)}$, and d =0.01m (or 10 mm). This is a way of devising a very small capacitance for use in a laboratory experiment if manufactured capacitor of this small value is not available.

Example 2:

A parallel-plate capacitor is made of two square plates each of dimension 600 mm by 600 mm and plate separation of 10 mm. A dielectric slab of relative permittivity of 4 and thickness of 6 mm is placed on the lower plate, leaving an air space of 4 mm between the upper plate and the slab. Calculate the capacitance of the capacitor.

Answer:

The arrangement is as shown in the following diagram.



It consists of two parallel-plate capacitors in series, an air capacitor of capacitance C_a and the other, filled with the dielectric, of capacitance, C_d.

$$C_a = \frac{\varepsilon_0 A}{a} = \frac{8.854 \times 0.6 \times 0.6}{0.004} = 796.86 \text{ pF}$$

$$C_d = \frac{\varepsilon_r \varepsilon_0 A}{d} = \frac{4 \times 8.854 \times 0.6\ x{\times}0.6}{0.006} = 2{,}125 \text{ pF}$$

The effective capacitance, C, is given by,

$$\frac{1}{C} = \frac{1}{C_a} + \frac{1}{C_d} \quad \text{or} \quad C = \frac{C_a C_d}{C_a + C_d} = 579.54 \text{ pF}$$

3.7.2: Capacitance of a cylindrical or coaxial cable capacitor

Refer to Figure 3.4 of Section 3.2, where the inner conductor of a coaxial cable carries a charge per unit length, $\rho_L = \frac{Q}{L}$

The potential difference between the inner and outer conductors is calculated to be,

$$V_{ab} = \frac{\rho_L}{2\pi\varepsilon} \log_e \frac{b}{a} \quad \text{(Eqn 3.3.7, p.57)} \qquad (1)$$

It follows, therefore, that *the capacitance per unit length* of the coaxial cable is,

$$\mathbf{C} = \frac{\rho_L}{V_{ab}} = \frac{\mathbf{2\pi\varepsilon}}{\mathbf{\log_e \frac{b}{a}}} \quad F\, m^{-1} \qquad (2)$$

Example 3:

Calculate the capacitance per unit length of a coaxial cable for which the radius of the outer conductor is twice the radius of the inner conductor and the space between them is filled with a dielectric material of relative permittivity $\varepsilon_r = 4$.

Ans:

$$\text{C} = \frac{\mathbf{2\pi\varepsilon}}{\mathbf{\log_e \frac{b}{a}}} = \frac{2\pi\,(4)\times 8.854\times 10^{-12}}{\ln 2} = 321 \text{ pF } m^{-1}$$

3.7.3: Energy stored in a capacitor

It requires energy, or work, to charge up a capacitor. Suppose at an instant of time the plate of a parallel-plate capacitor is charged to a potential, V, between the plates while the charge on the plate is, q.

Let, dW, be the amount of work done to increase the charge by, dq, then we have,

$$dW = Vdq \qquad (1)$$

Using $q = CV$ we have, (2)

$$dW = \frac{q}{C}\, dq \qquad (3)$$

If the process of charging starts from q = 0, till a final charge, Q, is delivered on the plate, the amount of work done is,

$$\mathbf{W} = \frac{1}{C}\int_0^Q q\,dq = \frac{\mathbf{1}}{\mathbf{2}}\frac{\mathbf{Q^2}}{\mathbf{C}} \tag{4}$$

This is the energy stored in the capacitor.

Eqn (4) is written in various forms, using the relation, Q = CV, as follows:

$$\mathbf{W} = \frac{\mathbf{1}}{\mathbf{2}}\frac{\mathbf{Q^2}}{\mathbf{C}} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{C}\,\boldsymbol{V^2} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{Q\,V} \tag{5}$$

3.7.4 Energy density within a capacitor

The energy referred to above is stored in the electric field between the plates of the capacitor.

Suppose we take a small cube of space of length, Δl, within the space between the plates, as shown in Fig. 3.11, such that the top and bottom faces of area (Δl^2) are parallel to the capacitor plates. If thin sheets of a metal foil are placed coincident with the top and bottom faces of the volume, the electric field, $\bar{E}$, will be undisturbed, and the arrangement constitutes a small parallel-plate capacitor.

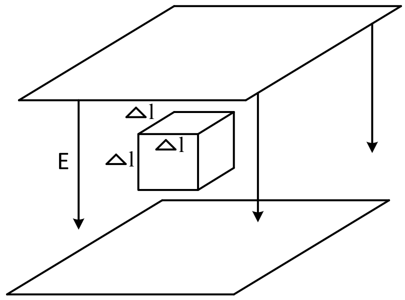


Fig. 3.11 Imaginary cubic space between parallel-plate capacitor

The potential difference between the top and bottom of the cube is,

$$\Delta V = E\,\Delta l \qquad (1)$$

and the capacitance, $\Delta C = \frac{\varepsilon(\Delta l)^2}{\Delta l} = \varepsilon\,(\Delta l\,)$, using Eqn 3.7.1 (5) (2)

The energy stored within the cube of space is,

$$\Delta W = \frac{1}{2}\,(\Delta C\,)\,(\Delta V\,)^2 = \frac{1}{2}\,\varepsilon\,(\Delta l\,)\,(\,E\,\Delta l)^2$$

$$= \frac{1}{2}\,\varepsilon\;E^2\,(\,\Delta l)^3 \qquad (3)$$

Hence, the **energy per unit volume, or the energy density** is defined as,

$$\boldsymbol{\omega} = \lim \Delta l \to 0\;\frac{\Delta W}{(\Delta l)^3} = \frac{\mathbf{1}}{\mathbf{2}}\,\boldsymbol{\varepsilon}\;\mathbf{E^2} \qquad (\mathbf{J}\;\boldsymbol{m}^{-3}\,) \qquad (4)$$

The above expression holds for an isotropic and homogeneous medium where $\bar{D}$ and $\bar{E}$ are in the same direction, and D = ε E. In general, for a nonisotropic medium where $\bar{D}$ and $\bar{E}$ may not be in the same direction, the energy density is expressed vectorially as,

$$\omega = \frac{1}{2}\bar{D}.\bar{E} \qquad (5).$$

4.0 ELECTRODYNAMICS

4.1 Introduction

Thus far, under electrostatics, we have considered electrical charges that are static or stationary and the corresponding electric fields and potentials resulting therefrom. We now wish to consider moving charges and their effects. A moving charge constitutes an electric current. In metallic conductors the charge is carried by electrons. In liquid conductors as in electrolytes in batteries the charge is carried by ions, both positive and negative. In semiconductors the charge is carried by electrons and holes; the holes behaving like positive charges. Here, we start by considering steady direct electric currents flowing in conductors and fields associated with them.

Historically, Hans Christian Oersted (1819) first discovered that a wire carrying a current, I, is surrounded by a region of magnetic field, the presence of which could be detected by a compass needle or iron filings. Tracing the direction of the field by the compass, it is observed that the needle always turns in a direction that is perpendicular to the wire and to the radial line extending out from the wire as the needle moves round in a closed circle round the wire, as illustrated in Fig. 4.1. The magnetic field strength is characterized by a magnetic flux density, $\overline{B}$, analogous to the electric flux density, $\overline{D}$.

Consider a wire passing through the page such that the cross-sectional area is represented by the small circle at the centre of Fig. 4.1. The letter X indicates that the direction of the current is into the page while the flux density B goes round the wire in a clockwise direction. If the current changes direction, B also changes to the anticlockwise direction, *the familiar right-hand rule*; *the thumb pointing in the direction of the current while the fingers of the right hand encircle the wire in the direction of the lines of magnetic flux*!

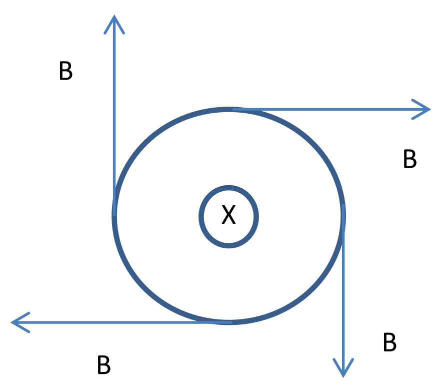


Fig. 4.1 Magnetic field round a current-carrying wire

4.2 The magnetic field of a current-carrying wire: the Biot-Savart law

The magnitude of the magnetic flux density, B, is found to depend on the distance from the current-carrying wire, the value of the current and the length of the wire. Considering a small element of length, Δl, of wire carrying a current, I. The incremental value of B at a point, P(r,θ), for r ≫ Δl, as shown in Fig. 4.2, is given by the relation,

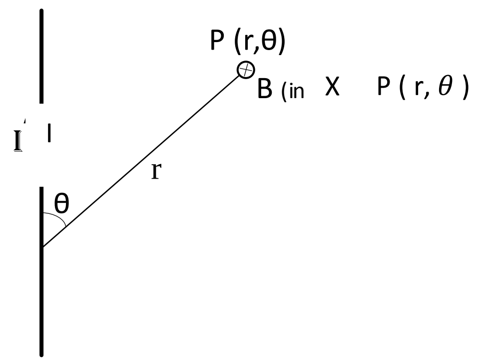


Fig.4.2 Determination of B at point P(r,θ)

$$\Delta B = \mathrm{k}\,\frac{I\,\Delta l \sin\theta}{r^2} \qquad (1)$$

where, k, is a constant of proportionality given by,

$$\mathrm{k} = \frac{\mu}{4\pi} \qquad (2)$$

and, μ, is the permeability of the medium. The permeability of vacuum, or air, is,

$$\mu_0 = 4\pi \times 10^{-7} \text{ H } m^{-1}$$

Combining Eqns (1) and (2) we have the fundamental relation,

$$\boldsymbol{dB} = \frac{\boldsymbol{\mu}}{\boldsymbol{4\pi}}\,\frac{\boldsymbol{I\,dl\sin\theta}}{\boldsymbol{r^2}} \qquad (3)$$

Eqn (3) is written in infinitesimals rather than incrementals. The direction of dB is normal to the page, inward at point P.

In order to determine the value of B due to a current I in a long, straight or curved conductor placed on the plane of the page, Eqn (3) will be integrated with respect to the infinitesimal length dl to obtain,

$$B = \frac{\mu I}{4\pi} \int \frac{\sin\theta}{r^2}\, dl \qquad (4)$$

Eqns (3) and (4) are the expressions of the Biot-Savart law.

4.2.1 The magnetic field of an infinite linear conductor

The magnetic field or flux density $\bar{B}$ at a distance R from a thin linear conductor of infinite length carrying a current I can be obtained from eqn. 4.2.4 above, and the diagram is illustrated in fig. 4.3.

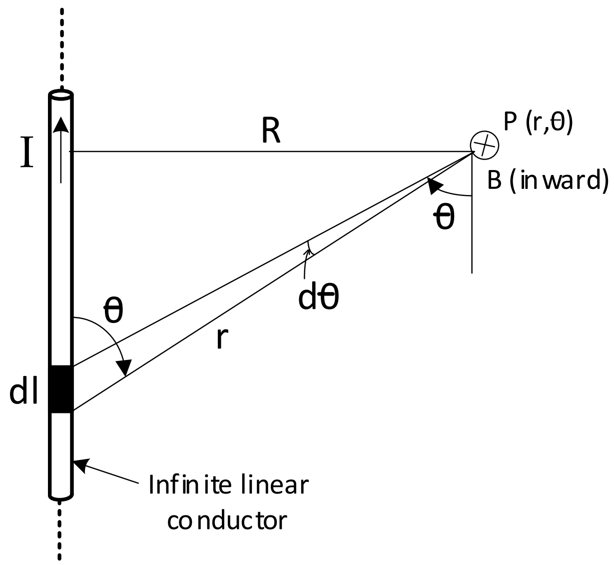


Fig. 4.3: Flux density from an infinite linear conductor

From the geometry of the figure, the following substitutions are made:

dl sin θ = r dθ, R = r sin θ and,

$$B = \frac{\mu I}{4\pi} \int_0^{\pi} \frac{1}{r} d\theta = \frac{\mu I}{4\pi R} \int_0^{\pi} \sin\theta d\theta \qquad (1)$$

$$= \frac{\mu I}{4\pi R}[-\cos\theta]_0^{\pi} = \frac{\mu I}{4\pi R} \times 2 = \frac{\mu I}{2\pi R} \qquad (2)$$

4.2.2 The magnetic field of a current-carrying loop

Let the loop of radius R be placed on the x – y plane and centred on the origin of the Cartesian coordinates, while the axis of the loop coincides with the z-axis as shown in Fig. 4.4

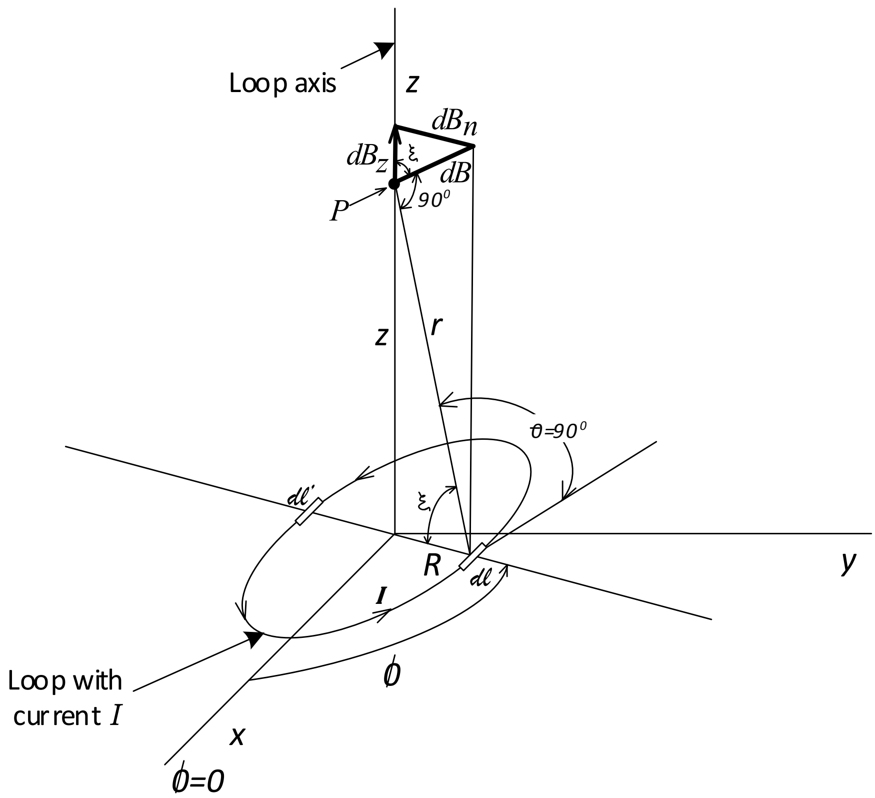


Fig. 4.4 Flux density from a current-carrying loop

At a point P on the loop axis, the contribution dB produced by an element of length dl of the loop is,

$$dB = \frac{\mu\, I\, dl \sin\theta}{4\, \pi\, r^2} \tag{1}$$

where θ is the angle between dl and the radius vector of length r.

The direction of $\overline{dB}$ is at right angle to the radius vector of length r.

From the geometry of the figure, it can be shown that the component dB_z of dB in the direction of the z-axis is given by,

dB_z = dB cos ξ = dB $\frac{R}{r}$;

With $\theta = 90^0$; dl = R dφ and r = $(R^2 + z^2)^{\frac{1}{2}}$ we have,

$$dB_z = \frac{\mu\, I\, R^2}{4\pi\, (R^2 + z^2)^{3/2}}\, \mathrm{d}\varphi \qquad (2)$$

By the symmetry of the figure, the total flux density B = B_z , hence

$$\mathrm{B} = B_z = \frac{\mu\, I\, R^2}{4\pi\, (R^2 + z^2)^{3/2}} \int_0^{2\pi} \mathrm{d}\varphi = \frac{\mu\, I\, R^2}{2\, (R^2 + z^2)^{3/2}} \qquad (3)$$

At the centre of the loop, i.e, at z = 0, we have

$$\mathbf{B} = \frac{\boldsymbol{\mu\, I}}{\mathbf{2}\, \boldsymbol{R}} \qquad (4)$$

4.3. Effect of a magnetic field on a current-carrying wire

If a current-carrying wire is placed in a uniform magnetic field of flux density B, it experiences a force. The relationship between the force $\bar{F}$, current $\bar{I}$ and flux density $\bar{B}$ is expressed vectorially as,

$$\bar{F} = (\bar{I} \times \bar{B})\, \mathrm{L} \qquad (1)$$

where L is the length of the wire lying within the magnetic field.

For an element of length dl of the conductor, giving rise to the element of force $\overline{dF}$, we may write,

$$\overline{dF} = (\bar{I} \times \bar{B})\, \mathrm{dl} \qquad (2)$$

In magnitude, we have,

$$\mathrm{dF} = \mathrm{I\, B\, dl} \sin\theta \qquad (3)$$

where θ is the angle between the direction of I and that of B, and the direction of dF is at right angles to the plane containing I and B such that the three vectors $\bar{F}$, $\bar{I}$ and $\bar{B}$ form a right-handed set, or turning from $\bar{I}$ to $\bar{B}$ through angle θ, $\bar{F}$ is in the direction of motion of a cock-screw, as in Fig. 4.5.

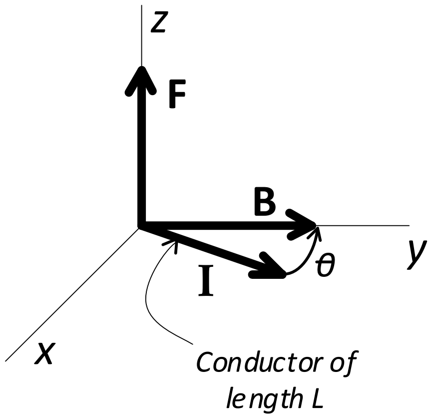


Fig. 4.5 Vectorial diagram of directions of I, B and F.

NOTE: If a point charge Q moves with a velocity u, covering a distance dl in time dt, we may write Qu = Q($\frac{dl}{dt}$) = $\frac{Q}{dt}$ dl = I dl. Hence Eqn (2) above may be expressed as

$$\overline{dF} = (\bar{I}\,\mathrm{dl} \times \bar{B}) = \mathrm{Q}\,(\bar{u} \times \bar{B}) \qquad (4)$$

4.4. Lorentz Force

If a charge Q moving with a velocity u is subjected to a combination of an electric field of intensity E and a magnetic field of flux density B, the charge will experience a force F expressed vectorially as,

$$\bar{\boldsymbol{F}} = \mathbf{Q}\,[\bar{\boldsymbol{E}} + (\bar{\boldsymbol{u}} \times \bar{\boldsymbol{B}})] \qquad (1)$$

Eqn (1) is referred to as the Lorentz Force.

4.5 The force between two parallel linear current-carrying conductors

Consider two parallel wires 1 and 2, carrying current I_1, I_2, respectively, in the same direction, as shown in Fig. 4.6. The direction of the magnetic induction B_1 produced by current I_1 in the position of conductor 2 is into the plane of the paper. Hence, conductor 2 experiences a force F_1 due to the effect of B_1 on I_2, as shown in Fig. 4.6.

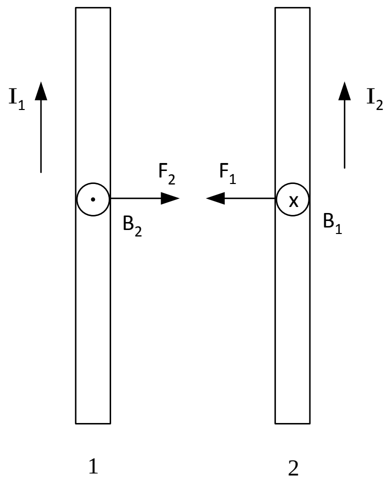


Fig. 4.6 Force between two parallel current-carrying conductors

Similarly, the force F_2 on conductor 1 due to current I_2 is as indicated. The two conductors are, therefore, subjected to **force of attraction**. Conversely, the two conductors will **repel** each other if the currents I_1, I_2 flow in opposite directions. *Note that this contrasts with electrostatics where like charges repel but opposite charges attract!*

Force $\overline{F_1}$ is given by

$$\overline{F_1} = \overline{I_2} \times \overline{B_1} \ (\mathrm{N}\, m^{-1}) \tag{1}$$

Or, $$F_1 = I_2 B_1 \ (\mathrm{N}\, m^{-1}) \tag{2}$$

Now, $$B_1 = \frac{\mu I_1}{2\pi d} \quad \text{from eqn (4.2.4)} \tag{3}$$

where d is the distance between the two conductors.

Therefore, $$F_1 = \frac{\mu I_1 I_2}{2\pi d} = F_2 \tag{4}$$

Observe that overhead high voltage power transmission lines carrying currents in same direction are normally kept separated by use of wooden spacers to prevent the force of attraction that might otherwise lead to sparking or short circuit.

4.6. Ampere's Law and H

From eqn (4.2.4) the flux density B at a distance R from a long straight conductor carrying current I is given by

$$B = \frac{\mu I}{2\mu R} \qquad (1)$$

If B is now integrated around a path of radius R enclosing the conductor once, we have

$$\oint \bar{B}.\bar{dl} = \frac{\mu I}{2\mu R} \oint dl = \frac{\mu I}{2\mu R} 2\pi R = \mu I \qquad (2)$$

Or, $$\oint \boldsymbol{\bar{B}}.\boldsymbol{\overline{dl}} = \boldsymbol{\mu I} \qquad (3)$$

Equation (3) may be made independent of the medium by introducing another vector,

$$\bar{H} = \frac{\bar{B}}{\mu} \qquad (4)$$

known as the magnetic field intensity, such that

$$\oint \boldsymbol{\bar{H}}\,.\,\boldsymbol{\overline{dl}} = \mathbf{I} \qquad (5)$$

Eqn (5) is a general rule, not only for a straight conductor as above, but in all other cases where the integration is taken over any singly closed path enclosing a current. **This is known as Ampere's Law**, which states that *the line integral of the magnetic field intensity H around* a *single closed path is equal to the current enclosed.* The application of this law greatly simplifies the calculation of the magnetic flux density B or the magnetic field intensity H. This is similar to the application of Gauss' law in electrostatics in the solution of the electric flux density D or electric field intensity E.

4.6.1 Application to a solid cylindrical conductor.

If, instead of a long thin conductor, we have a solid cylindrical conductor of radius R carrying a current I uniformly distributed within it, with uniform current density J, we can derive expression for H both inside and outside of the conductor by applying Ampere's law, as illustrated in Fig. 4.7.

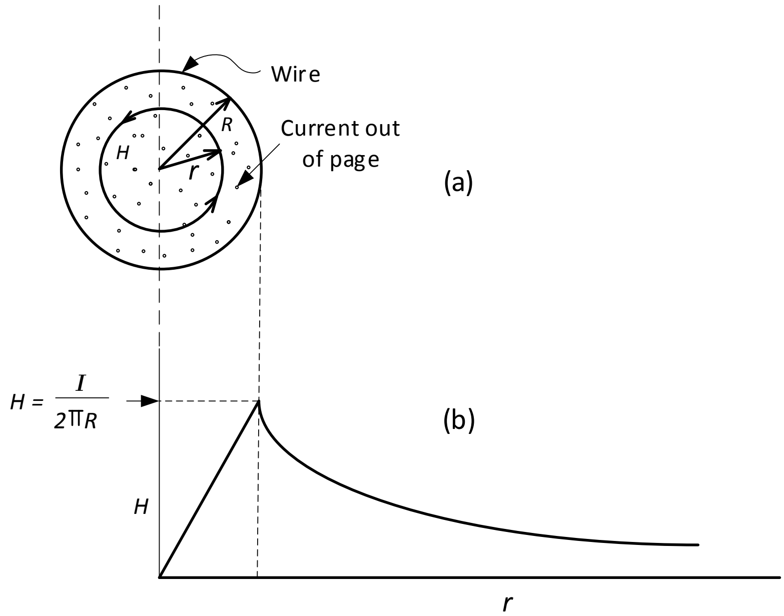


Fig. 4.7: H inside and outside of a solid current-carrying conductor.

The current density

$$J = \frac{I}{\pi R^2} \text{ A}m^{-2} \qquad (1)$$

Inside the conductor, the value of H at a distance r from the axis of the conductor is determined solely by the current inside the radius r.

Thus, for r $\leq$ R, let the current flowing inside be I_i, and H = H_i , then

$$I_i = \frac{I}{\pi R^2} (\pi r^2) = \frac{I r^2}{R^2} \tag{2}$$

$$H_i (2\pi r) = I_i = \frac{I r^2}{R^2} \tag{3}$$

$$\boldsymbol{H_i} = \frac{\boldsymbol{I}}{\boldsymbol{2\pi R^2}} \mathbf{r} \tag{4}$$

Outside the conductor, i.e. r$>$ R, we have

$$H (2\pi r) = I \quad \text{or} \quad \mathbf{H} = \frac{\boldsymbol{I}}{\boldsymbol{2\pi r}} \tag{5}$$

At the surface of the conductor, r = R, and H = $\frac{I}{2\pi R}$ (6)

A graph of the variation of H with r inside and outside the conductor is shown in Fig. 4.7

Table 4.6.1 gives the comparison between Gauss' law and Ampere's law.

Table 4.6.1: Comparison between Gauss's Law and Ampere's Law

Gauss's Law	**Ampere's Law**
Determines $\bar{E}$ for symmetric structures	Determines $\bar{B}$ for symmetric structures
Employs closed **surfaces**	Employs closed **loops**
$\oint \overline{\boldsymbol{E}} \cdot \overline{dA} = \frac{Q}{\varepsilon}$ $\oint \bar{D} \,.\, \overline{dA} = Q$	$\oint \overline{\boldsymbol{B}} \cdot \overline{dl} = \mu I$ $\oint \bar{H} \,.\, \overline{dl} = I$
Closed surface integral	Closed path integral

4.7 Divergence of $\overline{\boldsymbol{B}}$ and curl of $\overline{\boldsymbol{H}}$

4.7.1 Divergence of $\overline{\boldsymbol{B}}$

The flux tubes of a static electric field originate and end on electric charges. On the other hand, tubes of magnetic flux are continuous, they have no sources or sinks; there are no isolated magnetic poles. This is a fundamental difference between static electric and magnetic fields. To describe the continuous nature of magnetic flux tubes, we can write,

$$\oint \bar{B} \, . \, \overline{dS} \; = \; 0 \qquad (1)$$

i.e., as many magnetic flux tubes as enter any closed surface emerge from it.

Eqn (1) may be regarded as Gauss' law applied to magnetic fields.

By the definition of the divergence of a vector quantity, eqn (1) may be expressed as,

$$\nabla \, . \, \bar{B} \; = \; 0 \qquad (2)$$

Eqn (1) is the integral form, the macroscopic form, while eqn (2) is the differential form, the microscopic form, of Gauss' law applied to magnetic fields.

4.7.2 Curl of $\overline{\boldsymbol{H}}$

From eqn (4.6.5), Ampere's law in integral form is expressed as,

$$\oint \bar{H} \, . \, \overline{dl} \; = \; \mathrm{I} \; = \; \iint \bar{J} \, . \, \overline{dS} \qquad (1)$$

Consider an incremental plane area, ΔS, in a conducting medium through which an incremental current, ΔI, passes. The curl of, $\bar{H}$, is expressed as,

$$\text{Curl } \bar{H} \; = \; \nabla \times \bar{H} \; = \; \lim \Delta S \to 0 \; \frac{\oint \bar{H} \, .\overline{dl}}{\Delta S} \; = \; \lim \Delta S \to 0 \; \frac{\Delta I}{\Delta S} \; = \; \bar{J} \qquad (2)$$

Or, $$\textbf{Curl } \overline{\boldsymbol{H}} \; = \; \overline{\boldsymbol{J}} \qquad (3)$$

where, $\bar{J}$, is the current density ($\mathrm{A}m^{-2}$) at the point around which the area dS shrinks to zero.

Curl $\bar{H}$ is a vector at right angles to ΔS and in the direction of $\bar{J}$.

Eqn (1) is the integral form, the macroscopic form, while eqn (2) is the differential, microscopic form, of Ampere's law.

$\nabla \times \bar{H}$ is expressed in determinant form, in Cartesian coordinates as,

$$\nabla \times \bar{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \tag{4}$$

$$= \hat{x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{y}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \tag{5}$$

and $$\bar{J} = \hat{x}\, J_x + \hat{y}\, J_y + \hat{z}\, J_z \tag{6}$$

4.7.3 The concept of Displacement Current

The idea of a displacement current was introduced into the Ampere's law by James Clerk Maxwell to cater for the magnetic field existing in a region of space where there is no possibility of a conduction current taking place, such as in a vacuum or free space. An example is the vacuum or dielectric space between the two plates of a parallel-plate capacitor. A displacement current exists during the process of charging the capacitor. As shown in Fig. 4.8, the relation between the charge, Q, capacitance, C, and the potential, V, of a parallel-plate capacitor at an instant of time is given by,

$$Q = CV \tag{1}$$

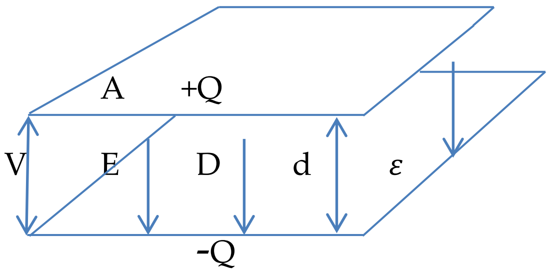


Fig. 4.8: Capacitance of a parallel-plate capacitor

Since current , $\mathrm{i} \;=\; \frac{dQ}{dt}$ (2)

We have the displacement current,

$$i_d \;=\; \frac{dQ}{dt} \;=\; C\,\frac{dV}{dt} \tag{3}$$

But $$C \;=\; \frac{\varepsilon\, A}{d} \tag{4}$$

where A is the area of the plate and d is the plate separation.

∴ $$i_d \;=\; \frac{\varepsilon\, A}{d}\,\frac{dV}{dt} \tag{5}$$

Also, $$V \;=\; E\,d \tag{6}$$

∴ $$i_d \;=\; \frac{\varepsilon\, A}{d}\,\frac{d}{dt}(\mathrm{Ed}) \;=\; \varepsilon\, A\,\frac{dE}{dt} \tag{7}$$

Again, the displacement current density D is related to the electric field intensity E by,

$$D \;=\; \varepsilon\, E \tag{8}$$

∴ $$i_d \;=\; A\,\frac{d\,D}{dt} \tag{9}$$

But, $$i_d \;=\; \iint J_d \, dS \;=\; J_d \; A \qquad (10)$$

where J_d is the displacement current density through the area A.

Combining eqns (9) and (10), we have,

$$\mathbf{J_d} \;=\; \frac{\boldsymbol{d\,D}}{\boldsymbol{dt}} \qquad \textbf{(11)}$$

Ampere's law is now modified, or generalized, to include conduction and displacement currents through a conducting dielectric medium, and is written as,

$$\oint \boldsymbol{\bar{H}} \,.\, \boldsymbol{\overline{dl}} \;=\; \iint \left(\boldsymbol{\bar{J}} \;+\; \frac{\boldsymbol{\partial \bar{D}}}{\boldsymbol{\partial t}} \right) . \, \boldsymbol{\overline{dS}} \qquad \textbf{(12)}$$

where $\boldsymbol{\bar{J}}$ is the conduction current density.

In differential form, Maxwell's equation derived from Ampere's law becomes,

$$\nabla \times \bar{H} \;=\; \bar{J} \;+\; \frac{\partial \bar{D}}{\partial t} \qquad (13)$$

or, $$\nabla \times \bar{H} \;=\; \sigma \, \bar{E} \;+\; \varepsilon \frac{\partial \bar{E}}{\partial t} \qquad (14)$$

where σ is the conductivity and ε the permittivity of the medium.

4.8 Inductors and Inductances

The resistor, capacitor and inductor, are the commonest electrical components employed in electrical circuits and systems. While the resistor is an energy dissipator, the capacitor and inductor are energy storage devices. An inductor is a magnetic counterpart of a capacitor. While a capacitor stores energy in an electric field, the inductor stores it in a magnetic field. Examples of inductors are wire loops, coils and solenoids.

Figure 4.8 is a typical solenoid inductor with a number of turns of wire wound round a solid former such as ceramics, plastics, glass, fiber, paper or wood. At times, the solenoid may not be wound round any solid material and air will be the medium inside it; the wire will, of course, be of reasonable stiffness.

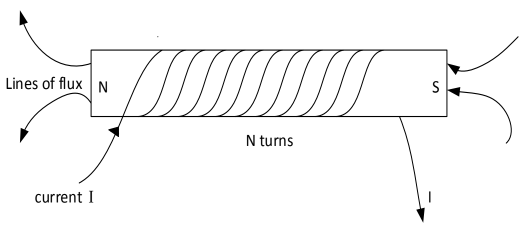


Fig. 4.8: solenoid with lines of magnetic flux

the lines link all the turns of wire, the total magnetic flux linkage, Λ, of the coil is equal to the total The magnetic lines of flux produced by a current in a solenoidal coil form closed loops. Each line that passes through the solenoid as shown in Fig. 4.8 links the current N times, where N is the number of turns of wire of the solenoid.

If all magnetic flux, ψ_m, through the coil times the number of turns,

$$\text{i.e.,} \quad \Lambda = N\,\psi_m \qquad (1)$$

By definition, the inductance, L, is the ratio of Λ to the current I,

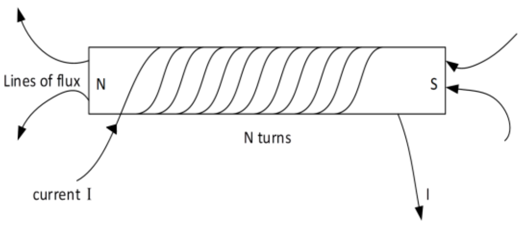
$$\text{i.e., } L = \frac{\Lambda}{I} = \frac{N\,\psi_m}{I} \qquad (2)$$

This definition is satisfactory for a medium with constant permeability, such as air. However, the permeability of a ferrous material such as iron or cobalt is not constant; in this case the inductance is defined as the ratio of the infinitesimal change in flux linkage to the infinitesimal change in current producing it,

$$\text{i.e., } L = \frac{d\Lambda}{dI} \qquad (3)$$

The inductance of an inductor can be calculated from the geometry of the inductor. The following are typical of the common inductors in use in electrical circuits and systems:

4.8.1 Long solenoid



$$L = \frac{\mu N^2 A}{l}$$

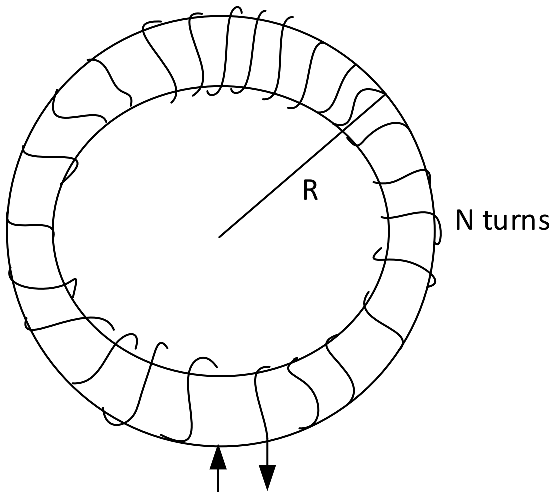
N = Number of turns

A = cross-sectional area of solenoid

μ = permeability of the medium

l = length of solenoid

4.8.2 Toroid



$$L = \frac{\mu N^2 r^2}{2R}$$

r = radius of the coil

R = radius of the toroid

4.8.3 Co-axial cable

$$L = \frac{\mu\, l}{2\pi} log_e \frac{b}{a}$$

a = radius of inner conductor

b = inside radius of outer conductor

l = length of cable

4.8.4 Two-wire transmission line

$$L = \frac{\mu_0 \, l}{\pi} \log_e \frac{D}{a}$$

a = radius of conductor

D= spacing between centres of conductors

5.0 TIME-VARYING ELECTRIC AND MAGNETIC FIELDS: FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION.

Ampere's law states that a current-carrying conductor produces a magnetic field. About 1831, Michael Faraday, an English physicist, discovered a reverse effect such that a changing magnetic field could produce a current in a closed circuit. Faraday's law explains the working principles of most of the electrical motors, generators, transformers and inductors in use in electrical engineering today.

A simple illustration of Faraday's law is a loop of wire with a bar magnet moving towards or away from the loop such that the magnetic flux of the bar magnet induces a current in the loop. When the bar magnet moves towards the loop, the induced current moves in one direction, but when the magnet moves away, the current flows in the opposite direction. In either case, the induced current flows in a direction such that the loop's magnetic flux opposes that of the magnet that produces it. By moving the magnet alternately towards or away from the loop, an alternating current (ac) is induced in the loop; which is the simple principle of the ac generator.

The fact that the induced current in the loop is always in such a direction as to oppose the change in flux producing it is a statement accredited to Heinrich Lenz who first established it in 1834, now known as Lenz's law.

Both Faraday's and Lenz's laws are combined to give the relation,

$$\mathbf{e} = -\frac{d\psi_m}{dt} \qquad (1)$$

where, e = emf induced in the loop, (V)

$\psi_m =$ magnetic flux producing e (Wb)

t = time (s)

If, instead of a single loop of wire, there are N turns linked by the same flux, the resulting emf is multiplied by N.

Consider a simple arrangement of a loop of wire linked by a magnetic induction, $\bar{B}$, as shown in Figure 4.9.

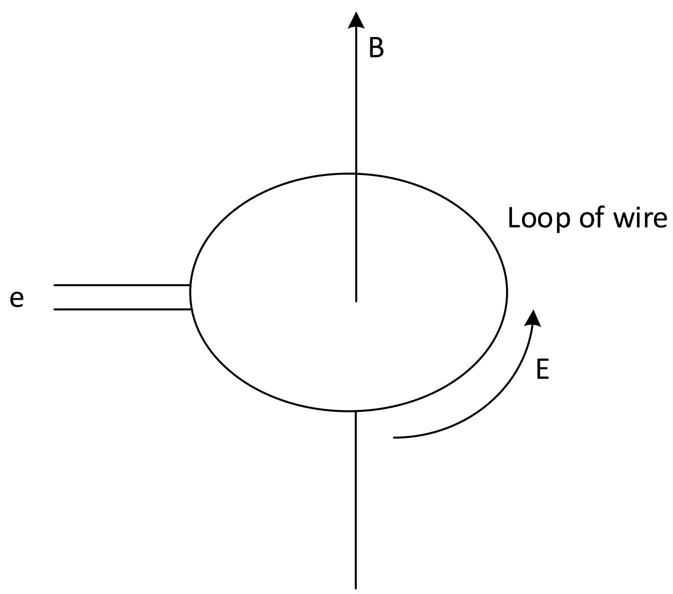


Fig. 4.9 Open-circuited loop with emf induced at its terminals

The induced emf, e, can be associated with an electric field, $\bar{E}$, whose line-integral over the circumference of the loop is equated to, e,

$$\text{i.e., } e = \oint \bar{E} \cdot \overline{dl} \qquad (2).$$

(Note: the gap separation on the loop is negligible).

The total magnetic flux through the loop is also expressed as ,

$$\psi_m = \iint \bar{B} \cdot \overline{dS} \qquad (3)$$

where the surface over which the integration is carried out is the surface bounded by the periphery of the circuit.

Combining eqns (1) and (3), we may write,

$$\mathbf{e = -\frac{d}{dt} \iint \bar{B} \cdot \overline{dS}} \qquad (4)$$

Now, consider the situation where the loop or closed circuit is fixed or stationary but the flux changes with time, eqn(4) reduces to,

$$e_t = -\iint \frac{\partial \bar{B}}{\partial t} \cdot \overline{\mathrm{dS}} \qquad (5)$$

which is sometimes referred to as *the transformer induction* equation.

Combining eqns (2) and (5), we have,

$$e_t = \oint \bar{E} \cdot \overline{dl} = -\iint \frac{\partial \bar{B}}{\partial t} \cdot \overline{\mathrm{dS}} \qquad (6)$$

Again, consider a situation where the flux, $\bar{B}$, is constant, but the loop or closed circuit is in motion, this also gives rise to an induced emf, which can be deduced from the Lorenz force equation, where the force, $\bar{F}$ on an electric charge, Q, moving with a velocity, $\bar{u}$, in a magnetic field of induction,$\bar{B}$, is given by,

$$\bar{F} = \mathrm{Q}(\bar{u} \times \bar{B}) \qquad (7)$$

Or, the electric field intensity, $\bar{E}$, is given by,

$$\bar{E} = \frac{\bar{F}}{Q} = (\bar{u} \times \bar{B}) \qquad (8)$$

Eqn (8) may be applied to determine the induced emf in a circuit moving through a a magnetic field of induction, $\bar{B}$, and we may write,

$$e_m = \oint \bar{E} \,.\, \overline{dl} = \oint (\bar{u} \times \bar{B}) \,.\, \overline{dl} \qquad (9)$$

which is termed *the motional induction* or *the flux-cutting induction.*

In a general sense, eqns (6) and (9) are combined to account for both kinds of changes occurring simultaneously, i.e, when the loop or circuit is in motion and B changes in time, to give the total induced emf as,

$$\mathrm{e} = e_m + e_t = \oint (\bar{u} \times \bar{B}) \,.\, \overline{dl} - \iint \frac{\partial \bar{B}}{\partial t} \,.\, \overline{\mathrm{dS}} \qquad (10)$$

Example 1.

Consider a rectangular loop of wire of area A and a magnetic flux density B at right angles to the plane of the loop, uniform over the area of the loop. Assume the magnitude of B varies sinusoidally with respect to time, i.e., $\mathrm{B} = B_0 \cos \omega t$, determine the emf induced in the loop.

Solution:

This is a case where the loop is constant or stationary, and the flux is varying. So, we apply eqn(6) where

$$\mathrm{e} = -\iint \frac{\partial \bar{B}}{\partial t} \,.\, \overline{\mathrm{dS}} = \mathrm{A}\omega B_0 \sin \omega t$$

Example 2

Consider the case where the flux density is constant over a rectangular loop of constant width but whose length is increased uniformly with time with a velocity, u.

Solution:

Here, eqn (9) is applicable, i.e.,

$$e = \oint(\bar{u} \times \bar{B}) \cdot \overline{dl}$$

Example 3:

Consider a rotating rectangular loop in a steady magnetic field. Let the loop rotate with a uniform angular velocity ω radians per second. This is a typical arrangement of a simple ac generator (see Fig. 8-8, p.331 of Kraus).

Solution:

Again, apply $e = \oint(\bar{u} \times \bar{B}) \cdot \overline{dl} = 2uBl\sin\omega t = 2\omega RBl sin\omega t$, where R is the radius of the rotating loop and l is its length. Note: $u = \omega R$.

6.0 MAXWELL'S EQUATIONS

6.1. GAUSS'S LAW: ELECTRIC FLUX

Integral Form

$$\varphi_e = \oint_s \bar{D}.\overline{ds} = \int_v \rho dv \qquad 1(a)$$

Differential Form

$$\nabla.\bar{D} = \rho \qquad 1(b)$$

6.2. GAUSS' LAW: MAGNETIC FLUX

Integral Form

$$\varphi_m = \oint_s \bar{B}.\overline{ds} = 0 \qquad 2(a)$$

Differential Form

$$\nabla.\bar{B} = 0 \qquad 2(b)$$

6.3. AMPERE'S LAW

Integral form

$$\boldsymbol{I} = \oint \bar{H}.\bar{dl} = \oint_s \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right).\overline{ds} \qquad 3(a)$$

Differential Form

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \qquad 3(b)$$

6.4. FARADAY'S LAW

Integral Form

$$e = \oint \bar{E}.\overline{dl} \quad = -\int_{s} \frac{\partial \bar{B}}{\partial t}.\overline{ds} \qquad 4(a)$$

Differential Form

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad 4(b)$$

6.5 OTHER EQUATIONS

Other equations and relations that are applicable for solving electromagnetic problems are:

$$\bar{J} = \sigma\bar{E} \qquad \text{(Ohm's law at a point)} \qquad (5)$$

$$\nabla \cdot \bar{J} = -\frac{\partial \rho}{\partial t} \qquad (6)$$

$$\bar{F} = q\bar{E} \qquad (7)$$

$$Force\text{(on a point charge, q, in an electric field)}$$

$$\bar{F} = (\bar{I} \times \bar{B})dl \qquad (8)$$

where $\bar{F}$ is the Force (on a current element, Idl, in a magnetic field of flux density $\bar{B}$)

$$\bar{D} = \varepsilon\bar{E} \qquad (9)$$

$$\bar{B} = \mu\bar{H} \qquad (10)$$

$$\text{Force, } \bar{F} = q(\bar{E} + \bar{v} \times \bar{B}) \qquad (11)$$

$(which\ is\ Lorentz\ force\ on\ a\ point\ charge\ placed\ in\ both\ electric\ and\ magnetic\ fields)$

$\bar{J}$ = electric current density, A/m^2

ρ = electric volume charge density, C/m^3

σ = conductivity of medium, S/m

ε = permittivity of medium = $\varepsilon_0\varepsilon_r$

μ = permeability of medium = $\mu_0\mu_r$

v = velocity of charge in electric and magnetic fields

7.0. WAVE PROPAGATION IN FREE SPACE, $\rho = 0$, $J = 0$

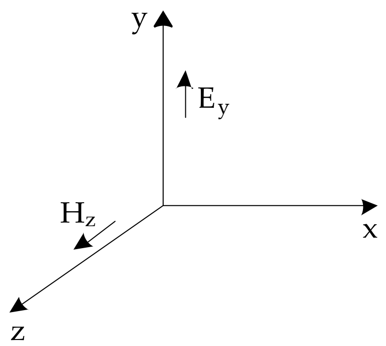


Fig. 7.1 Alignment of E and H fields

Assume $\bar{E}$ is aligned along y-direction and $\bar{H}$ is aligned along z-direction, with both varying with time along x-direction, as shown in Fig.7.1. Apply Ampere's law in differential form:

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} \qquad (1)$$

The components of $\nabla \times \bar{H}$ and $\frac{\partial \bar{D}}{\partial t}$ applicable are:

From Equation 4.7.2 (5)

$$\hat{x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{y}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$= \frac{\partial}{\partial t}(\hat{x} D_x + \hat{y} D_y + \hat{z} D_z) \tag{2}$$

and with

$$H_x = H_y = 0; \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0 \tag{3}$$

we have, $$\hat{y}\left(-\frac{\partial H_z}{\partial x}\right) = \hat{y}\,\frac{\partial D_y}{\partial t} \tag{4}$$

or

$$\frac{\partial H_z}{\partial x} = -\frac{\partial D_y}{\partial t} = -\varepsilon_0 \frac{\partial E_y}{\partial t} \tag{5}$$

Also, from $\nabla \times \bar{E}$ we have,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{6}$$

$$\hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial Ex}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$= -\frac{\partial}{\partial t}(\hat{x} B_x + \hat{y} B_y + \hat{z} B_z) \tag{7}$$

and with

$$E_x = E_z = 0; \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$$

$$\hat{z}\left(\frac{\partial E_y}{\partial x}\right) = -\ \hat{z}\ \frac{\partial B_z}{\partial t} \tag{8}$$

or

$$\frac{\partial E_y}{\partial x} = -\ \frac{\partial B_z}{\partial t} = -\mu_0 \frac{\partial H_z}{\partial t} \tag{9}$$

By differentiating Equation 5 with respect to *t* and Equation 9 with respect to *x*, we can eliminate H_z component: i.e.

$$\frac{\partial}{\partial t}\left(\frac{\partial H_z}{\partial x}\right) = \ -\varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \tag{10}$$

$$-\mu_0 \frac{\partial}{\partial x}\left(\frac{\partial H_z}{\partial t}\right) = \frac{\partial^2 E_y}{\partial x^2} \tag{11}$$

or

$$\frac{\partial}{\partial x}\left(\frac{\partial H_z}{\partial t}\right) = \ -\frac{1}{\mu_0}\frac{\partial^2 E_y}{\partial x^2} \tag{12}$$

The LHS of Equations 10 and 12 are the same, therefore:

$$-\varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} = \ -\frac{1}{\mu_0}\frac{\partial^2 E_y}{\partial x^2} \tag{13}$$

or

$$\boldsymbol{\frac{\partial^2 E_y}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0}\ \frac{\partial^2 E_y}{\partial x^2}} \tag{14}$$

Similarly, E_y component can be eliminated from equations 5 and 9 to give:

$$\boxed{\frac{\partial^2 H_z}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 H_z}{\partial x^2}} \tag{15}$$

Equations 14 and 15 are well-known **wave equations**, which have similar solutions in E_y and H_z, electric and magnetic field components, respectively.

Equations 5 and 9 show the interrelationship of the electric and magnetic fields. The wave equations describe the motion of the electromagnetic wave as a function of time and space, propagating along the x-direction.

Quiz: Given that $\boldsymbol{E_y = E_0 \sin \beta(x + ct)}$ is a solution of the wave equation (14), where $\boldsymbol{\beta}$ and **c** are constants, show that $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ and calculate the value of c, with $\mu_0 = 4\pi \times 10^{-7}$ Hm^{-1} and $\varepsilon_0 = 8.854 \times 10^{-12}$ Fm^{-1}.

A general solution for equation 14 is

$$\boxed{\boldsymbol{E_y = E_0 \sin \beta(x + ct) + E_1 \sin \beta(x - ct)}} \tag{16}$$

Where $\beta = \frac{2\pi}{\lambda}$, called the phase constant and λ = wavelength of the sinusoidal wave

Equation 16 may also be written as

$$E_y = E_0 \sin(\beta x + \omega t) + E_1 \sin(\beta x - \omega t) \tag{17}$$

where $\omega = \beta c = \frac{2\pi c}{\lambda} = \frac{2\pi (f\lambda)}{\lambda} = 2\pi f$, f = frequency of the wave

$$E_0 = \text{amplitude of the wave}$$

(The sine term could also be replaced by cosine, i.e.

$E_y = E_0 \cos(\beta x + wt) + E_1 \cos(\beta x - wt)$ is also a solution of the wave equation.)

Equation 16 represents two waves travelling in opposite directions; the first term is associated with a wave travelling in the –ve direction while the second term is a wave travelling in the +ve x-direction.

Another form of the solution can be written as an exponential function:

$$E_y = E_0 exp(\omega t \pm \beta x) = E_0 e^{j(\omega t \pm \beta x)} \tag{18}$$

Fig. 7.2 illustrates the wave nature of the electric and magnetic fields propagating in space

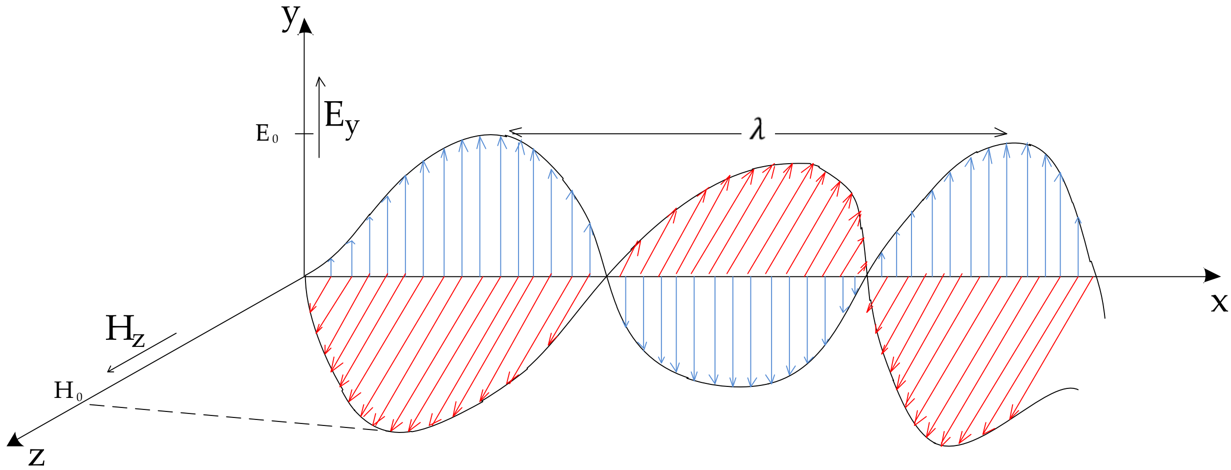


Figure 7.2: Forward travelling E and H waves

$$E_y = E_0 \sin(\omega t - \beta x)$$

$$H_z = H_0 \sin(\omega t - \beta x)$$

Both E_y and H_z are synchronized oscillations propagating at the speed of light through vacuum (or free space). The oscillations of the two fields are perpendicular to each other and perpendicular to the direction of motion (x-direction in this case).

Electromagnetic waves are, therefore, self-propagating transverse oscillating waves of electric and magnetic fields. The electric and magnetic fields are in phase with each other, reaching maxima values together.

7.1 CHARACTERISTIC IMPEDANCE

Consider the forward travelling E- and H- waves,

$E_y = E_0 \sin(\omega t - \beta x), \quad H_z = H_0 \sin(\omega t - \beta x)$

Apply Equation 7.0.(9), $\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t}$ to give

$$-\beta E_0 \cos(\omega t - \beta x) = -\mu_0 \omega H_0 \cos(\omega t - \beta x)$$

$$\therefore \quad \frac{E_0}{H_0} = \mu_0 \frac{\omega}{\beta} = \mu_0 c = \frac{\mu_0}{\sqrt{\mu_0 \varepsilon_0}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = Z_0 \qquad (1)$$

Z_0 is called the characteristic impedance or intrinsic impedance of the medium.

Z_0 (for air) = $\sqrt{\frac{4\pi \times 10^{-7}}{\frac{1}{36\pi \times 10^9}}} = \sqrt{144\pi^2 \times 10^2} = 120\pi$ or 377 Ω

$$E_0 = Z_0 H_0 \qquad (2)$$

7.2. TRANSMITTED POWER: POYNTING VECTOR

The vector quantity $\bar{S} = \bar{E} \times \bar{H}$ is the instantaneous power density (W/m^2) transmitted by the electromagnetic wave in the direction of propagation, (x-direction).

Average power density, $S_{av} = \frac{1}{2} E_0 H_0$ (1)

$$S_{av} = \frac{1}{2} E_0 H_0 = \frac{1}{2} \frac{{E_0}^2}{Z_0} = \frac{1}{2} {H_0}^2 Z_0 \quad \left(\frac{W}{m^2}\right) \tag{2}$$

Problem: Assignment 1

The earth receives 2.0×10^9 cal min^{-1} cm^{-2} of sunlight.

(a) What is the Poynting vector in W/m^2 ?

(b) What is the power output of the sun in sunlight assuming that the sun radiates isotropically?

(c) What is the rms electric field E at the earth assuming that the sunlight is all at a single frequency?

(d) How long does it take the sunlight to reach the earth?

(Take the earth-sun distance = 150×10^9 m, 1W = 14.3×10^9 cal min^{-1})

8.0 Wave Propagation in a conducting medium: The skin effect

The wave equation for an electromagnetic wave propagating in a conducting medium is derivable from Maxwell's equations. The relevant equations are as follows:

$$-\frac{\partial H_z}{\partial x} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \tag{1}$$

and

$$\frac{\partial E_y}{\partial x} = -\frac{\partial \bar{B}}{\partial t} \tag{2}$$

Noting $J = \sigma E$, $D = \varepsilon E$, B= μH and expressing the E and H components propagating in x-direction as $E_y = E_0 e^{j(\omega t \pm \beta x)}$, similarly for the H_z component, equations (1) and (2) become:

$$-\frac{\partial H_z}{\partial x} = \sigma E_y + \varepsilon \frac{\partial E_y}{\partial t} \tag{3}$$

or

$$\frac{\partial H_z}{\partial x} = -(\sigma + j\omega\varepsilon)E_y \tag{4}$$

and

$$\frac{\partial E_y}{\partial x} = -j\omega\mu H_z \tag{5}$$

Differentiating equation (5) with respect to x and substituting equation (4), we have

$$\frac{\partial^2 E_y}{\partial x^2} = -j\omega\mu\frac{\partial H_z}{\partial x} = j\omega\mu(\sigma + j\omega\varepsilon)E_y \quad (6)$$

$$\frac{\partial^2 E_y}{\partial x^2} = (j\omega\mu\sigma - \omega^2\mu\varepsilon)E_y \quad (7)$$

$$= \gamma^2 E_y \quad (8)$$

where γ is the propagation constant, a complex parameter with real and imaginary terms expressed as

$$\gamma = \alpha + j\beta \quad (9)$$

α = the attenuation constant in neper m^{-1} and β= the phase constant in radian m^{-1}.

A solution of equation (8) for a wave travelling in the positive x-direction is,

$$E_y = E_0 e^{-\gamma x} = E_0 e^{-\alpha x} e^{-j\beta x} \quad (10)$$

(suppressing the $e^{j\omega t}$ term)

For a wave travelling in a conducting medium where the conductivity $\sigma \gg \omega\varepsilon$, equation (7) is approximated to

$$\frac{\partial^2 E_y}{\partial x^2} = j\omega\mu\sigma E_y = \gamma^2 E_y \quad (11)$$

or $$\gamma^2 = j\omega\mu\sigma \tag{12}$$

$$\gamma = \sqrt{j\omega\mu\sigma} \quad = \alpha + j\beta \tag{13}$$

Using the fact that

$$\sqrt{j} = \frac{1+j}{\sqrt{2}}$$ (14) PROVE!!

we find that the real and imaginary parts of γ are obtained by writing,

$$\gamma = \sqrt{\frac{\omega\mu\sigma}{2}} + j\sqrt{\frac{\omega\mu\sigma}{2}} \tag{15}$$

$$\text{with} \quad \alpha = \sqrt{\frac{\omega\mu\sigma}{2}} \quad \text{and} \quad \beta = \sqrt{\frac{\omega\mu\sigma}{2}} \tag{16}$$

Equation (10) becomes

$$E_y = E_0 e^{-\sqrt{\frac{\omega\mu\sigma}{2}}x} e^{-j\sqrt{\frac{\omega\mu\sigma}{2}}x} \tag{17}$$

or

$$E_y = E_0 e^{-\frac{x}{\delta}} e^{-j\frac{x}{\delta}} \tag{18}$$

$$\text{where } \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \quad = \quad \sqrt{\frac{1}{\pi f\mu\sigma}} \tag{19}$$

8.1 Skin depth

Equation (18) implies that the amplitude of the wave decreases exponentially as it penetrates into a conducting medium by a factor $e^{-x/\delta}$.

When $x = \delta$, the wave decreases by a factor e^{-1} or $\frac{1}{e} = \frac{1}{2.718}$ or 0.37 or 37% of its value upon entering the conducting medium as illustrated in Figure 8.1 below:

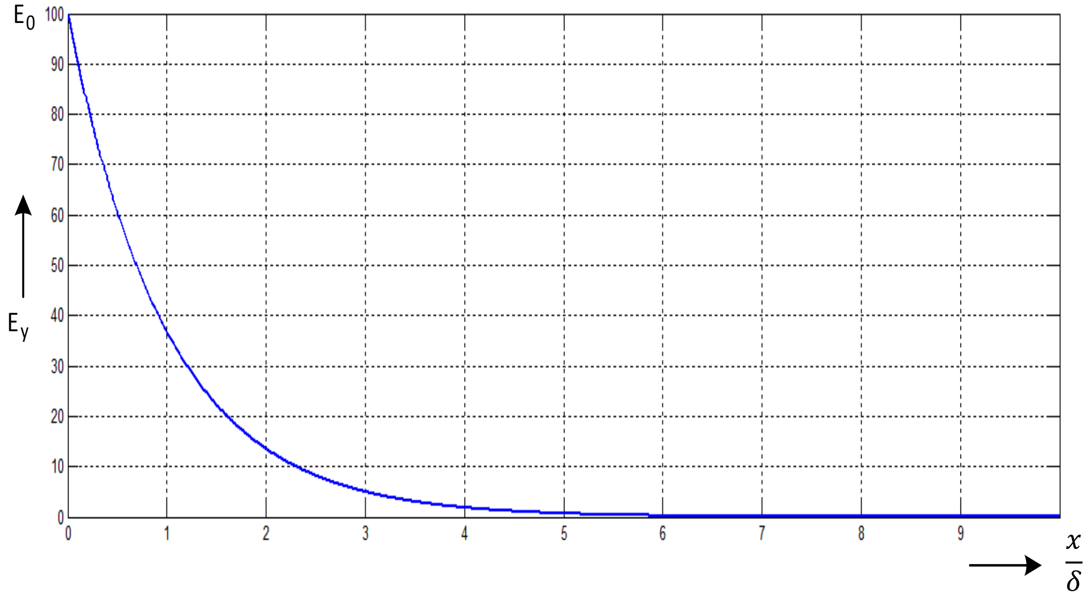


Fig. 8.1 Wave attenuation in a conducting medium

δ is called the **depth of penetration** or the **skin depth** of the wave inside the conductor.

To give a quantitative value of δ, consider a good conductor such as copper, and electromagnetic wave in the microwave range of frequency, say, 10 GHz.

For copper, $\mu_r = 1,\ \mu_0 = 4\pi \times 10^{-7} H\, m^{-1};\ \sigma = 5.8 \times 10^7 S\, m^{-1}$

At $= 10^{10} Hz$,

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2}{2\pi f\mu\sigma}} = \frac{1}{\sqrt{\pi f \mu_0 \sigma}} \tag{20}$$

$$= [\pi \times 10^{10} \times 4\pi \times 10^{-7} \times 5.8 \times 10^7]^{-\frac{1}{2}}$$

$$= 6.6 \times 10^{-7} m$$

$$= 0.66\ \mu\text{m}$$

Since $\delta \propto \frac{1}{\sqrt{f}}$, the depth of penetration is even smaller at higher frequencies and higher at lower frequencies. For instance, at $f = 100MHz,\quad \delta = 66 \times 10^{-7} m$.

This very low depth of penetration makes a conductor a good protective shield for any equipment inside a conducting enclosure from an external electromagnetic field. A practical illustration is the fact that a car with a built-in radio receiver will have practically no reception inside the car unless it is connected to an external whip antenna to intercept the radiowave from outside, since the receiver is practically inside a metal enclosure. Likewise, the outer conductor of a coaxial cable acts as a shield to the inner conductor and prevents an external interfering signal from entering the cable.

9.0. Transmission Lines

9.1. INTRODUCTION

A transmission line is a pair of electrical conductors carrying an electrical signal from one point to another, such as power line, coaxial cable, pair of wires (twisted or untwisted) used to connect domestic appliances or laboratory equipment, telephone lines, etc. The line can carry dc or ac voltages and currents at different frequencies. In many electrical circuits operating at low frequencies, the length of the wires connecting the components may be ignored; the voltage on the line at a given time may be regarded as constant along the wire. However, at very high frequencies, the wire length may become important and the behaviour of the wire may affect the signal being transmitted.

For the purpose of analysis, an electrical transmission line may be modeled as a two-port network, with one port (input port) connected to the source of power and the other (output port), connected to the load, as shown in Fig. 9.1

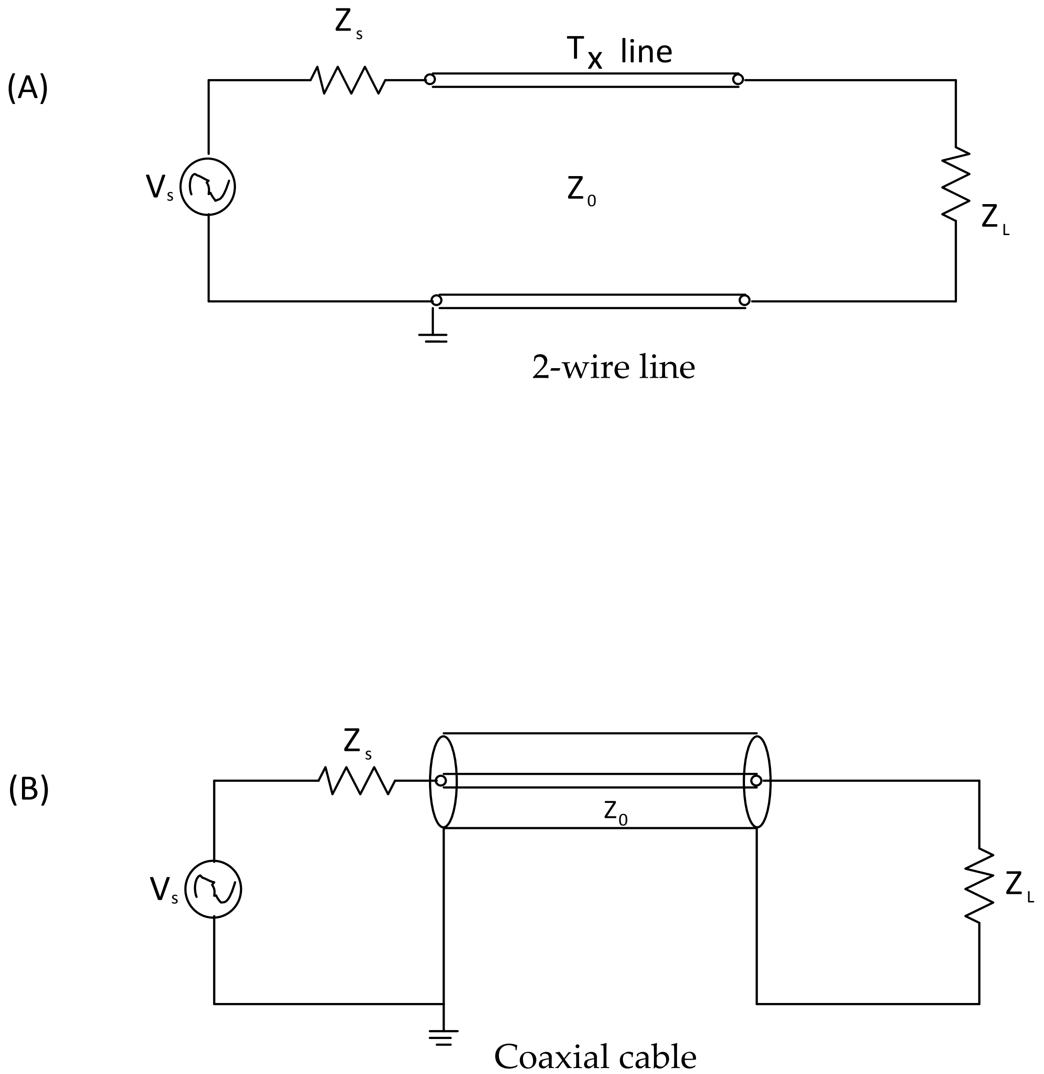


Figure 9.1 Transmission lines

In the simplest case, the network may be assumed to be linear, i.e. the **complex voltage** between the pair of wires is proportional to the **complex current** flowing in the wire. If the transmission line is uniform along its length, then its behaviour is largely described by a single parameter, called the **characteristic impedance**, Z_0, of the line. This impedance is the ratio of the complex voltage of a given signal to the complex current at any point on the line when there is **no reflection** at the load back to the source. Typical values of Z_0 are 50 Ω or 75 Ω for coaxial cable, about 100 Ω for a twisted pair or about 300 Ω for a common type of untwisted pair used in radio transmission.

When sending power down the line, it is usually desirable that as much power as possible will be absorbed by the load and as little as possible reflected back to the source. **When the load impedance is made equal to the characteristic impedance, Z_0, there will be no reflection; the line is then said to be matched.**

Some of the power fed into the line will, of course, be lost due to the resistance of the wire (resistive or ohmic loss). At high frequencies, some loss could occur through the dielectric material (dielectric loss) inside the transmission line.

9.2. TRANSMISSION LINE MODEL

The model used is to regard the transmission line to be made up of resistance and inductance uniformly distributed along the length of the line; capacitance and conductance also uniformly distributed within the dielectric material separating the conductors of the line.

A small length, dx, of the line may, therefore, be represented by a network as shown in Fig. 9.2:

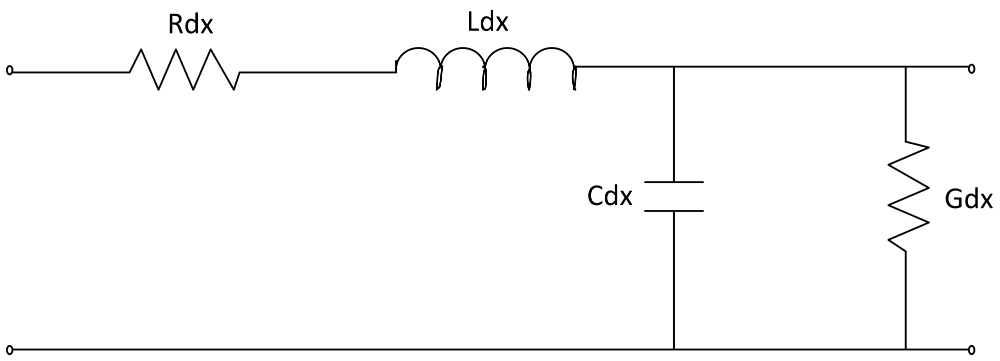


Figure 9.2 Network representation of transmission line

where R, L, C and G are the resistance, inductance, capacitance and conductance, per unit length, respectively. i.e. $R(\Omega\, m^{-1})$, $L(H\, m^{-1})$, $C(\mathrm{F}\, m^{-1})$ and $G(S\, m^{-1})$ or ohm m^{-1}, henry m^{-1}, farad m^{-1} and siemen m^{-1}, respectively. These are called the primary constants of the transmission line.

9.3. WAVE EQUATIONS

Consider a length, dx, of the line with voltage, V, across the line and current I, on the line, as shown in Fig. 9.3:

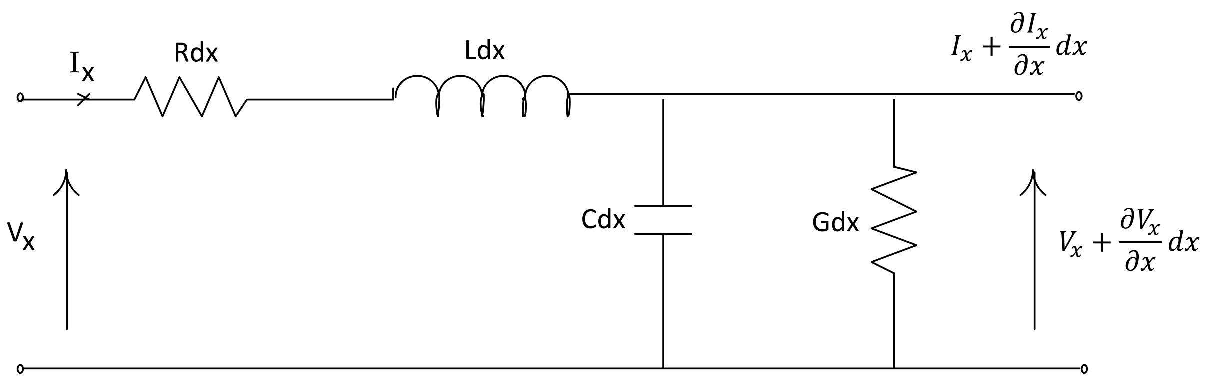


Figure 9.3 Voltage and Current variation on the line

Assuming V and I vary with time, t, according as $e^{j\omega t}$, the relationship between the voltage and current, applying Kirchhoff's laws, can be expressed as:

$$\frac{\partial V_x}{\partial x} = -(R + j\omega L)I_x \tag{1}$$

$$\frac{\partial I_x}{\partial x} = -(G + j\omega C)V_x \tag{2}$$

Differentiate equation (1) with respect to x and substitute equation (2), we have, (*dropping the subscript x for simplicity),*

$$\frac{\partial^2 V}{\partial x^2} = (R + j\omega L)(G + j\omega C)V \tag{3}$$

Similarly, by differentiating equation (2) with respect to x and substituting equation (1), we have,

$$\frac{\partial^2 I}{\partial x^2} = (R + j\omega L)(G + j\omega C)I \tag{4}$$

Equations (3) and (4) may be rewritten, respectively, as:

$$\frac{\partial^2 V}{\partial x^2} = \gamma^2 V \tag{5}$$

$$\frac{\partial^2 I}{\partial x^2} = \gamma^2 I \tag{6}$$

where

$$\gamma^2 = (R + j\omega L)(G + j\omega C), \tag{7}$$

or

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{8}$$

γ is called the **propagation constan**t of the line. It is a complex quantity which may be expressed as:

$$\gamma = \alpha + j\beta \tag{9}$$

where α is called the **attenuation constant** in neper m^{-1}, and β, the **phase constant** in radian m^{-1}, is the imaginary part of the propagation constant.

9.4. LOSSLESS LINE

When R and G are negligibly small (i.e. $R \ll j\omega L$, $G \ll j\omega C$ at high frequencies), or $R = G = 0$ (ideal situation), the line is regarded as lossless. No energy is dissipated by way of heating on the line.

Equations (9.3.3) and (9.3.4) become:

$$\frac{\partial^2 V}{\partial x^2} = (j\omega)^2 LC\, V \quad = \quad \gamma^2 \; V \tag{1}$$

$$\frac{\partial^2 I}{\partial x^2} = (j\omega)^2 LC\, I \quad = \quad \gamma^2\, I \tag{2}$$

$$\text{where} \quad \gamma = \alpha + j\beta \quad = \quad j\omega\,\sqrt{LC} \tag{3}$$

i.e

$$\alpha \quad = \quad 0 \qquad and \qquad \beta = \omega\sqrt{LC} \tag{4}$$

9.5 PHASE VELOCITY, v

To investigate the velocity at which a signal propagates down the line, we may consider a solution to the wave equation (9.4.1) as:

$$\mathrm{V} = \mathrm{V}^{+} sin(\omega t - \beta x) \tag{1}$$

For a given point on the wave to maintain its constant phase angle, we take,

$$\omega t - \beta x = \text{constant} \tag{2}$$

Differentiating equation (2) with respect to time, t, we have

$$\frac{d}{dt}(\omega t - \beta x) = 0 \tag{3}$$

or

$$\omega - \beta \frac{dx}{dt} = 0 \tag{4}$$

The phase velocity, v $= \frac{dx}{dt} = \frac{\omega}{\beta}$

$$\mathrm{v} = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} \tag{5}$$

$$Note\text{: v} = \frac{\omega}{\beta} = \frac{2\pi f}{2\pi/_{\lambda}} = f\lambda$$

9.6. GENERAL SOLUTION TO THE WAVE EQUATIONS (9.4.1) AND (9.4.2)

If we assume a solution to equations (9.4.1) and (9.4.2) in phasor notation as:

$$V = V^{+}e^{-j\beta x} + V^{-}e^{+j\beta x} \tag{1}$$

and

$$I = I^{+}e^{-j\beta x} + I^{-}e^{+j\beta x} \tag{2}$$

(suppressing $e^{j\omega t}$ factor),

we can determine I^+, I^- in terms of V^+, V^- by differentiating equation (1) to give:

$$\frac{\partial V}{\partial x} = -j\beta V^+ e^{-j\beta x} + j\beta V^- e^{\beta x} \tag{3}$$

From equation (9.3.1), we have

$$I = -\frac{1}{j\omega L}\frac{\partial V}{\partial x} \qquad \text{(with } R = 0\text{)} \tag{4}$$

$$\therefore \quad I = -\frac{1}{j\omega L}\left\{-j\beta V^+ e^{-j\beta x} + j\beta V^- e^{j\beta x}\right\} \tag{5}$$

$$= \frac{j\beta}{j\omega L} V^+ e^{-j\beta x} - \frac{j\beta}{j\omega L} V^- e^{j\beta x} \tag{6}$$

$$= I^+ e^{-j\beta x} + I^- e^{j\beta x} \tag{7}$$

$$\text{i.e.} \qquad I^+ = \frac{\beta}{\omega L} V^+ \tag{8}$$

$$I^- = -\frac{\beta}{\omega L} V^- \tag{9}$$

Equations (1) and (2) or (6) represent two waveforms for voltage and current, respectively, travelling on the line. The first term, $V^+ e^{-j\beta x}$, or $I^+ e^{-\beta x}$ travels from the generator to the load, while the second term represents the reflected wave from the load to the generator.

At any given point x on the line the two waves, incident and reflected, add up to produce a standing wave on the line.

9.7. CHARACTERISTIC IMPEDANCE, Z_0

Focusing on the forward-travelling wave,

$$V_i = V^+ e^{-j\beta x} \qquad \text{(incident voltage wave)} \tag{1}$$

$$I_i = I^+ e^{-j\beta x} = \frac{\beta}{\omega L} V^+ e^{-j\beta x} \quad \text{(incident current)} \tag{2}$$

The characteristic impedance, Z_0, is defined as,

$$Z_0 = \frac{V_i}{I_i} = \frac{\omega L}{\beta} = \frac{\omega L}{\omega\sqrt{LC}} = \sqrt{\frac{L}{C}} \tag{3}$$

Rewrite equations (1) and (2) to include Z_0;

$$V_i = V^+ e^{-j\beta x} \tag{4}$$

$$I_i = \frac{V^+}{Z_0} e^{-j\beta x} \tag{5}$$

9.8. VOLTAGE REFLECTION COEFFICIENT, Γ_v, AND CURRENT REFLECTION COEFFICIENT, Γ_i

Figure 9.4 represents forward-travelling or incident as well as backward-travelling or reflected voltage and current wave.

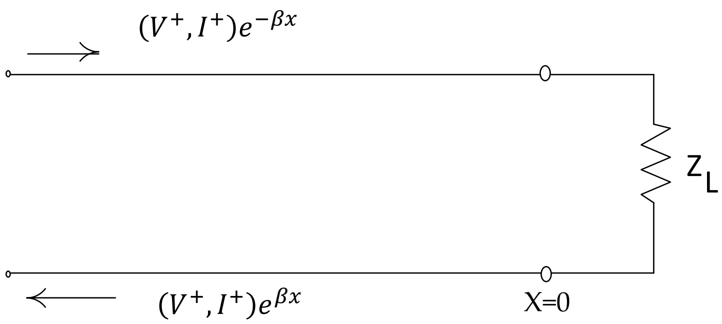


Figure 9.4 Incident and Reflected waves

Take distance measurement from the load end, where $x = 0$.

Incident voltage = V^+ ; reflected voltage = V^-

Incident current = $I^+ = \frac{V^+}{Z_0}$; reflected current = $I^- = -\frac{V^-}{Z_0}$

The load impedance, Z_L, is given by:

$$Z_L = \frac{V^+ + V^-}{\frac{V^+}{Z_0} - \frac{V^-}{Z_0}} = \left(\frac{V^+ + V^-}{V^+ - V^-}\right) Z_0 \qquad (1)$$

The voltage reflection coefficient, Γ_v is defined by

$$\boldsymbol{\Gamma_v = \frac{V^-}{V^+}} \qquad (2)$$

Divide the numerator and denominator of equation (1) by V^+, to give

$$\frac{Z_L}{Z_0} = \frac{1 + \frac{V^-}{V^+}}{1 - \frac{V^-}{V^+}} = \frac{1 + \Gamma_v}{1 - \Gamma_v} \tag{3}$$

or

$$\boldsymbol{\Gamma_v} \;=\; \frac{\boldsymbol{Z_L - Z_0}}{\boldsymbol{Z_L + Z_0}} \tag{4}$$

The Current Reflection Coefficient, Γ_i , is:

$$\boldsymbol{\Gamma_i} = \frac{I^-}{I^+} \;=\; -\frac{V^-}{V^+} = \;\boldsymbol{-\Gamma_v} \tag{5}$$

i.e. the Current Reflection Coefficient is of the same magnitude, but 180^0 out of phase with the Voltage Reflection Coefficient, Γ_v .

Note the following special situations:

(a) When $Z_L = Z_0$, $\Gamma_v = 0$, no signal is reflected from the load; the line is said to be **matched.**

(b) If the line is open-circuited, $Z_L = \infty$, $\Gamma_v = 1$; the reflected voltage at the load is equal to the incident voltage in magnitude and phase.

(c) If the line is short-circuited, $Z_L = 0$; $\Gamma_v = -1$. The reflected voltage is of the same magnitude as the incident voltage, but 180^0 out of phase with it.

9.9. THE IMPEDANCE, Z_x, AT ANY POINT x ON THE LINE

$$Z_x = \frac{V_{ix}}{I_{ix}} = \frac{V^+e^{-j\beta x} + V^-e^{j\beta x}}{\frac{V^+}{Z_0}e^{-j\beta x} - \frac{V^-}{Z_0}e^{j\beta x}} \tag{1}$$

or

$$Z_x = Z_0\left[\frac{V^+(cos\beta x - jsin\beta x) + V^-(cos\beta x + jsin\beta x)}{V^+(cos\beta x - jsin\beta x) - V^-(cos\beta x + jsin\beta x)}\right] \tag{2}$$

Divide numerator and denominator by V^+ and $\cos\beta x$, we have,

$$Z_x = Z_0\left[\frac{(1 - jtan\beta x) + \frac{V^-}{V^+}(1 + jtan\beta x)}{(1 - jtan\beta x) - \frac{V^-}{V^+}(1 + jtan\beta x)}\right] \tag{3}$$

Noting equation (9.8.3) that

$$\frac{Z_L}{Z_0} = \frac{1 + \frac{V^-}{V^+}}{1 - \frac{V^-}{V^+}}$$

equation (3), after a few simplifying steps and expressing distance x from the load end, results in a simpler, more familiar expression,

$$\boldsymbol{Z_x = Z_0\left[\frac{Z_L + jZ_0tan\beta x}{Z_0 + jZ_Ltan\beta x}\right]} \tag{4}$$

9.10. VOLTAGE STANDING WAVE RATIO, (VSWR).

The incident and reflected voltage waveforms add up to give a standing wave on the line, with maximum and minimum values as shown in Figure 9.5

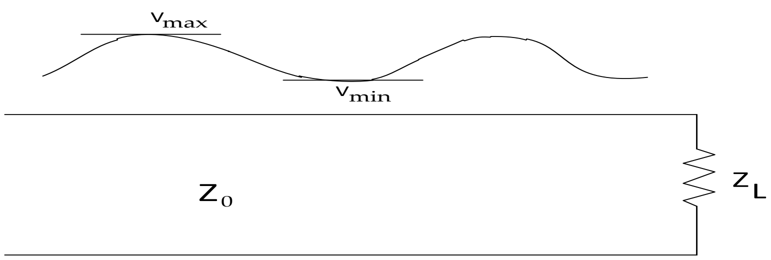


Fig. 9.5 Voltage Standing Wave on the line

$$V_{max} = V^+ + V^- \tag{1}$$

$$V_{\min} = V^+ - V^- \tag{2}$$

The voltage standing waves ratio, S, is defined as

$$\boldsymbol{S = \frac{V_{max}}{V_{min}} = \frac{V^+ + V^-}{V^+ - V^-} = \frac{I_{max}}{I_{min}}} \tag{3}$$

Divide the numerator and denominator by V^+, we have

$$\boldsymbol{S} = \frac{1 + \frac{V^-}{V^+}}{1 - \frac{V^-}{V^+}} = \boldsymbol{\frac{1 + |\Gamma_v|}{1 - |\Gamma_v|}} \tag{4}$$

or

$$\boldsymbol{|\Gamma_v| = \frac{S - 1}{S + 1}} \tag{5}$$

Example 9.10.1 A transmission line having a characteristic impedance of 50 Ω is terminated with a load of 100 + j 100 Ω. Calculate (a) the voltage reflection coefficient (b) the VSWR.

$$\boldsymbol{Solution}: \quad \Gamma_v = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 + j100 - 50}{100 + j100 + 50} = \frac{50 + j100}{150 + j100}$$

$$= \frac{111.8 < 63.43^0}{180.3 < 33.69^0} = 0.62 < 29.74^0$$

$|\Gamma_v| = 0.62$, VSWR = $\frac{1 + |\Gamma_v|}{1 - |\Gamma_v|} = \frac{1.62}{0.38} = 4.2$

Example 9.10.2 A lossless transmission line has characteristic impedance of 75 Ω and phase constant of 3 rad m^{-1} at 100 MHz frequency. Calculate the inductance and capacitance per metre of the line.

Solution: $Z_0 = \sqrt{\frac{L}{C}}$ $\quad \beta = \omega\sqrt{LC} = 2\pi f\sqrt{LC}$ $\quad \frac{Z_0}{\beta} = \frac{\sqrt{\frac{L}{C}}}{2\pi f\sqrt{LC}} = \frac{1}{2\pi fC}$

$\frac{Z_0}{\beta} = \frac{75}{3} = \frac{1}{2\pi fC}$ $\quad$ f $= 10^8$ Hz

C = 6.37×10^{-11} F m^{-1} ; $\quad$ L = 358×10^{-9} H m^{-1}

9.11 THE QUARTER WAVE ($\lambda/_4$) TRANSFORMER

When a transmission line is terminated by a load Z_L whose value is different from the characteristic impedance, Z_0, of the line, there is a mismatch, and reflection of voltage occurs at the load.

By inserting another line of characteristic impedance Z_1 which is $\lambda/_4$ (one-quarter wavelength) long between the line and the load, a matching condition can be provided. See Figure 9.6:

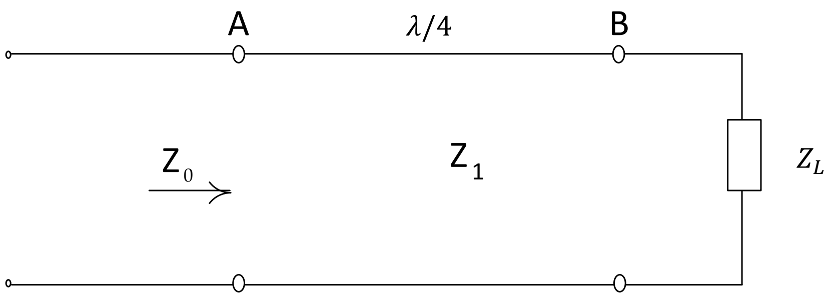


Fig. 9.6 $\lambda/_4$ – wave matching transformer

To calculate the value of Z_1, we make use of equation (4-9-4):

$$\boldsymbol{Z_0 = Z_1\left[\frac{Z_L + jZ_1 tan\beta x}{Z_1 + jZ_L tan\beta x}\right]} \tag{1}$$

(Note that Z_0 above corresponds to Z_x of equation (4.9.4) and Z_1 corresponds to Z_0)

Now, $tan\beta x = tan\left(\frac{2\pi}{\lambda}\frac{\lambda}{4}\right) = tan\frac{\pi}{2} = \infty$

By dividing the numerator and denominator by tan $\frac{\pi}{2}$, we have,

$$Z_0 = Z_1 \frac{(jZ_1)}{jZ_L} = \frac{{Z_1}^2}{Z_L} \qquad (2)$$

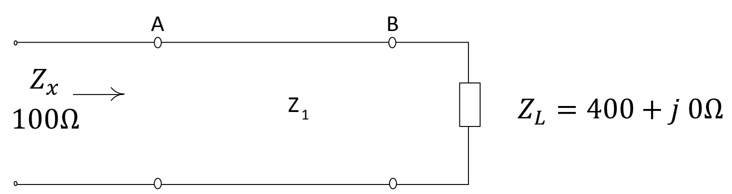
or $\boldsymbol{Z_1 = \sqrt{Z_0 Z_L}}$ (3)

i.e. the $\frac{\lambda}{4}$ line has a characteristic impedance value which is the geometric mean of the impedances at its input and output ends.

Suppose the signal on the line is of frequency 100 MHz, the wavelength, λ, is calculated from $c = f\lambda$, or $\lambda = \frac{c}{f} = \frac{3\times10^8}{1\times10^8} m = 3\ m$. or $\frac{\lambda}{4} = 0.75\ m$

Any length, shorter or longer than 0.75 m will be a mismatch resulting in some reflected wave.

Example 9.11.1: Let a $\frac{\lambda}{4}$ – line be inserted between a line of characteristic impedance of 100 Ω and the load of 400 + j 0 Ω.



$$Z_1 = \sqrt{Z_x Z_L} = Z_1 = \sqrt{100 \times 400} = 200\ \Omega$$

Example 9.11.2: A $\frac{\lambda}{4}$ – line of characteristic impedance of 60 Ω is terminated with Z_L. Determine Z_{in} when (i) Z_L = 0 (ii) Z_L = ∞ (iii) Z_L = 60 Ω .

Comment on the results obtained.

Solution: From equation 9.9.1, we find that for a $\frac{\lambda}{4}$ – line,

$$tan\beta x = tan\left(\frac{2\pi}{\lambda}\frac{\lambda}{4}\right) = tan\frac{\pi}{2} = \infty.$$

Therefore, when

(i) $Z_L = 0,\ Z_{in} = Z_0\left(\frac{jZ_0}{jZ_L}\right) = \frac{Z_0^2}{Z_L} = \infty;$

(ii) $Z_L = \infty,\ ,\ Z_{in} = 0;$

(iii) $Z_L = 60\ \Omega,\quad Z_{in} = 60\ \Omega,$

Comment: Under conditions (i) and (ii), we find that a short-circuited line appears at the input of the line as an open-circuited line, while an open-circuited line appears as a short-circuit.

Under condition (iii) a line terminated by its characteristic impedance has its input impedance of the same value; a condition for a matched line, whether or not the line is $\frac{\lambda}{4}$ in length.

Note also that for a short-circuited line,

$$(Z_{in})_{s/c} = \frac{j\,Z_0^2\ \tan\,\beta x}{Z_0} = \text{j}\ Z_0\ tan\beta x$$

$$(Z_{in})_{o/c} = Z_0\left(\frac{1}{jtan\beta x}\right) = -\text{j}\ Z_0\cot\beta x$$

$$(Z_{in})_{s/c}\ (Z_{in})_{o/c} = (\text{j}\ Z_0\ tan\beta x)(-\text{j}\ Z_0\cot\beta x) = Z_0^2$$

$$Z_0 = \sqrt{(Z_{in})\text{s/c}\ \ (Z_{in})\text{o/c}}$$

This is a way of determining the characteristic impedance of a transmission line.

9.12. THE SMITH CHART IN SOLVING TRANSMISSION LINE PROBLEMS

The solution of transmission line problems often involves lengthy numerical calculations, especially when complex numbers are involved. P.H. Smith has devised a simple and quick graphical solution to such problems to ease the rigour of calculation.

The Smith chart consists of a family of circles representing **normalized impedances (or admittances),** both real and imaginary values, plotted on a polar coordinate chart. The normalization is done by dividing the impedances by the characteristic impedance. The normalized impedance is located at a point within the chart, while the radial distance from the centre of the chart to that point gives the magnitude of the reflection coefficient, $|\Gamma_v|$, and also the VSWR relating to Γ_v. The angle which the radial line makes with horizontal line represents the phase angle of Γ_v. The distance from the load of a given input impedance is simply read off the chart circumference which is labeled in wavelengths.

When the line is not matched, a matching line stub is often attached to the line at a given distance from the load and the length of the stub adjusted to provide the matching. The chart is used to determine the point of attachment of and the length of the stub.

The following problem will illustrate the usefulness of the chart.

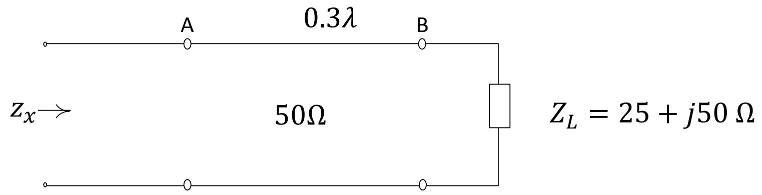
Example 9.12.1: A lossless 50Ω transmission line is terminated in $25 + j50\ \Omega$. Find

(a) the voltage reflection coefficient

(b) VSWR

(c) the impedance 0.3λ from the load.

Solution: By calculation:



(a)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{25 + j50 - 50}{25 + j50 + 50}$$

$$= \frac{-25 + j50}{75 + j50}$$

$$= \frac{-25 + j50}{75 + j50} \times \frac{75 - j50}{75 - j50}$$

$$= \frac{625 + j5000}{8125} = \mathbf{0.077 + j0.615}$$

$$= \sqrt{(0.077)^2 + (0.615)^2}\, tan^{-1}\frac{0.615}{0.077}$$

$$= \mathbf{0.62\angle 83^0}$$

(b) VSWR = $\frac{1+0.62}{1-0.62} = \frac{1.62}{0.38} = 4.26$

(c) $Z_x = Z_0 \left[\frac{Z_L + jZ_0 tan\beta x}{Z_0 + jZ_L tan\beta x}\right]$

$$\beta x = \frac{2\pi}{\lambda}(0.3\lambda) = 0.6\pi = 1.885\, rad = 108^0$$

$$Z_x = 50\left[\frac{25 + j50 + j(50)tan108^0}{50 + j(25 + j50)tan108^0}\right]$$

$$= \frac{25 + j50 + j(50)(-3.08)}{1 + j(0.5 + j1.0)(-3.08)}$$

$$= \frac{25 + j50 - j(154)}{1 + 3.08 - j(1.54)}$$

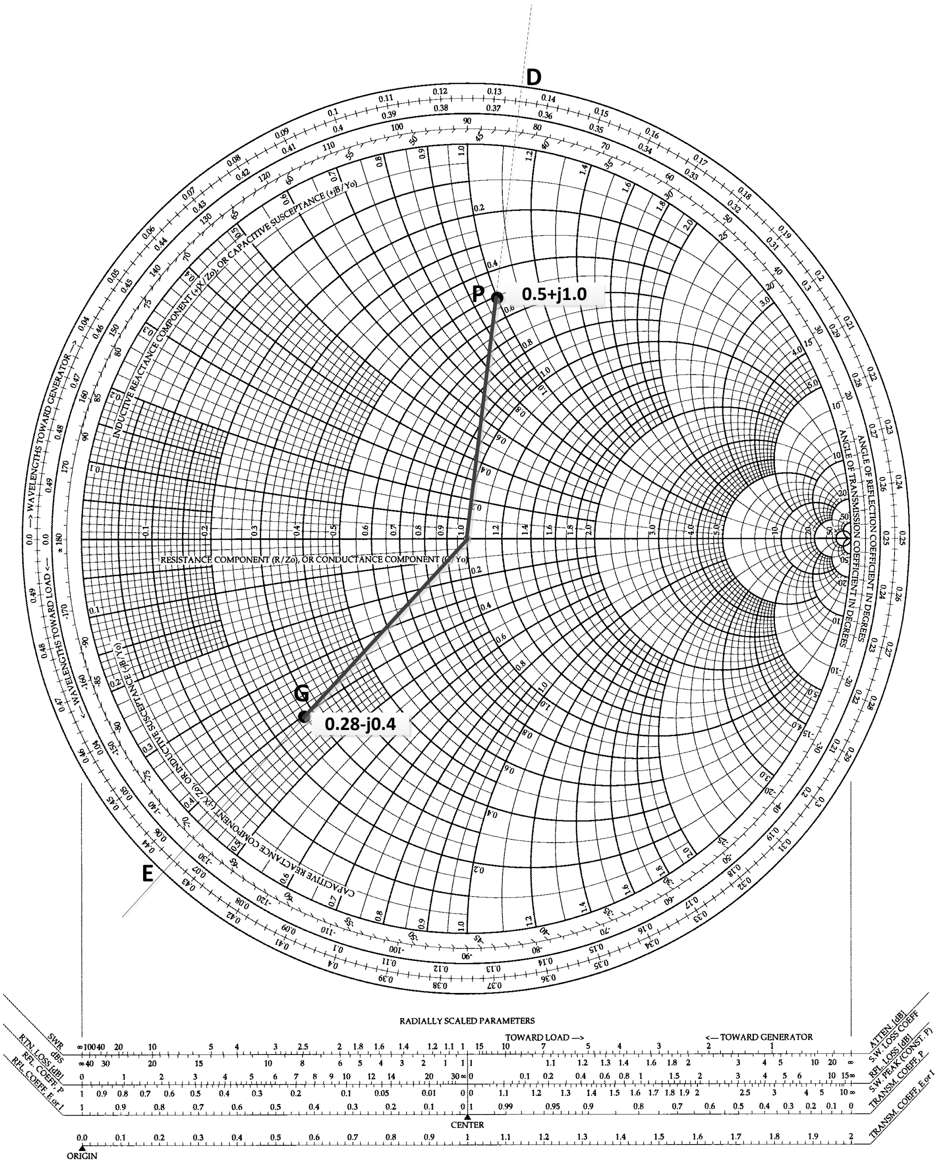
$$= \frac{25 - j(104)}{4.08 - j(1.54)} \times \frac{4.08 - j(1.54)}{4.08 - j(1.54)}$$

$$= \frac{(102 + 160.2) - j385.8}{19.02}$$

$$= \mathbf{13.8 - \mathit{j}20.3\ \Omega}$$

9.12.1 USING SMITH CHART

Normalized load impedance $= \frac{25}{50} + j\frac{50}{50} = 0.5 + j1.0$



Locate point P (0.5+j1.0) on the chart

Join the centre of circle, point O, to *P* and project to cut the circumference of the circle. Use scale on ruler to determine the radius, *R* of the circle and the length OP. Length OP is called the Load Line.

Read the angle OP makes the horizontal axis, on the right hand side.

$|\Gamma| = \frac{OP}{R} = \frac{5.2\ mm}{8.5\ mm} = \mathbf{0.61}$ $\angle\,\mathbf{\Gamma} = \mathbf{82^0}$ $\mathbf{\Gamma = 0.61 \angle 82^0}$

Measure length *OP* on the horizontal axis. It cuts the real axis at point C (= 4.2) which is the value of the VSWR . **VSWR= 4.2**

OP projected cuts the circumference at point *D*, which is at 0.135λ from the negative horizontal line. Add 0.30λ to 0.135λ (clockwise, i.e. toward the generator) to obtain 0.435λ. Locate 0.435λ on the circumference, (point *E*). Join *O* to *E*, and measure length *OP* along *OE*, to reach point *G*. Read the value of the normalized impedance at G (= 0.28 – j0.40).

The impedance, $\boldsymbol{Z_x} = (0.28 - j0.40) \times 50\ \Omega$ = ***14-j20* Ω.**

[Note, instead of measuring along *OP, OC*, OG, etc, a protractor could be used to draw a circle, centre *O*, radius *OP*, to join points *P, C* and *G*. This circle is called the VSWR-circle].

You can now see how much easier it is to use the Smith Chart for the solution of transmission line problems. **Get used to it!!**

9.13 STUB MATCHING ON TRANSMISSION LINE

A given transmission line with characteristic impedance may be matched to a load Z_L (= R + jX) by inserting a single stub between the line and the load, as shown in Fig. 9.6:

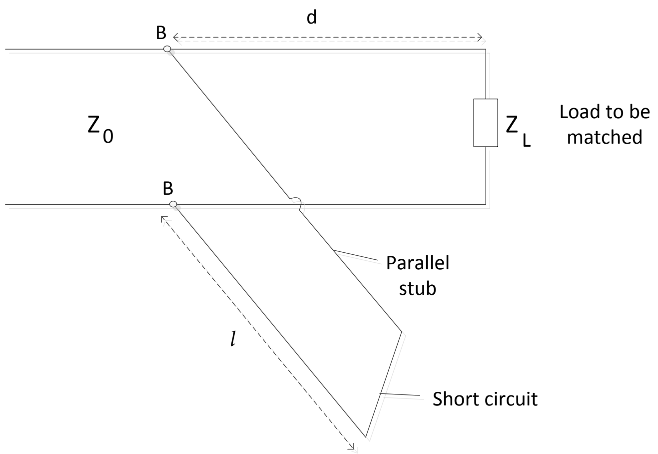


Fig. 9.6 Stub matching on transmission line

A stub is a short section of transmission line (short-circuited or open-circuited at the far end) whose input impedance at the connection point (BB as shown) can be changed by varying its length, *l*. The matching position, d, from the load is also changed appropriately to obtain the matching condition.

For parallel stub-matching, it is more convenient to use **admittance** rather than **impedance** for ease of calculation. Note that admittances in parallel are added to obtain equivalent admittance, as for impedances in series.

In the Smith chart, any point reflected through the centre point converts an impedance to an admittance, and vice versa.

For example, consider a normalized impedance z = 1.8 + j 2.0. Its corresponding normalized admittance is,

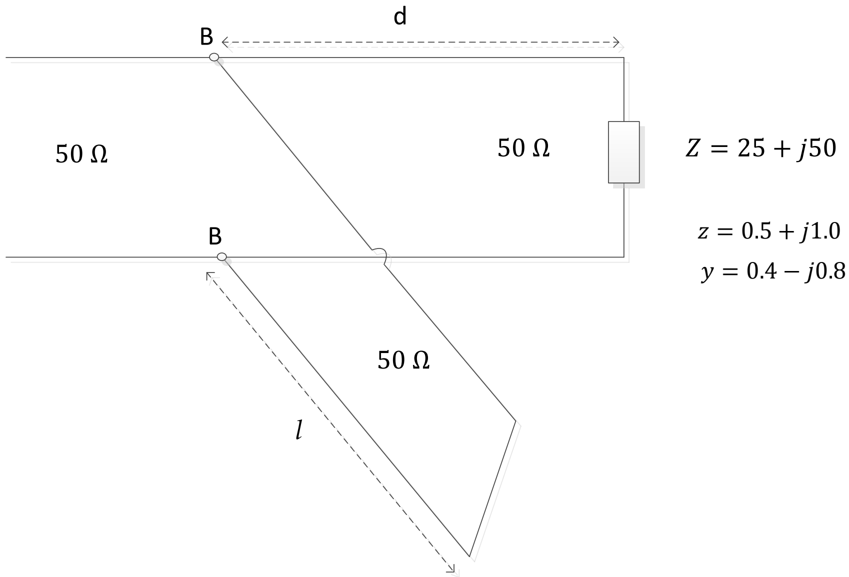
$$y = \frac{1}{z} = \frac{1}{1.8+j\,2.0} = \frac{1}{1.8+j\,2.0} \times \frac{1.8-j\,2.0}{1.8-j\,2.0} = 0.25 - j\,0.28$$

Check these two values of z and y on the Smith chart. The y-position is the reflection of the z-position through the centre point of the chart!

Transmission line matching by means of a single short-circuited stub is better explained by a worked example, as follows:

Example 9.13.1 A lossless 50 Ω transmission line is terminated in 25 + j 50 Ω load. Determine the point from the load at which a single 50 Ω short-circuited stub is to be attached and the length of the stub to provide matching of the line to the load.

Solution



Locate the position of z (normalized value of Z) on the Smith chart, $P_z = 0.5 + j1.0$

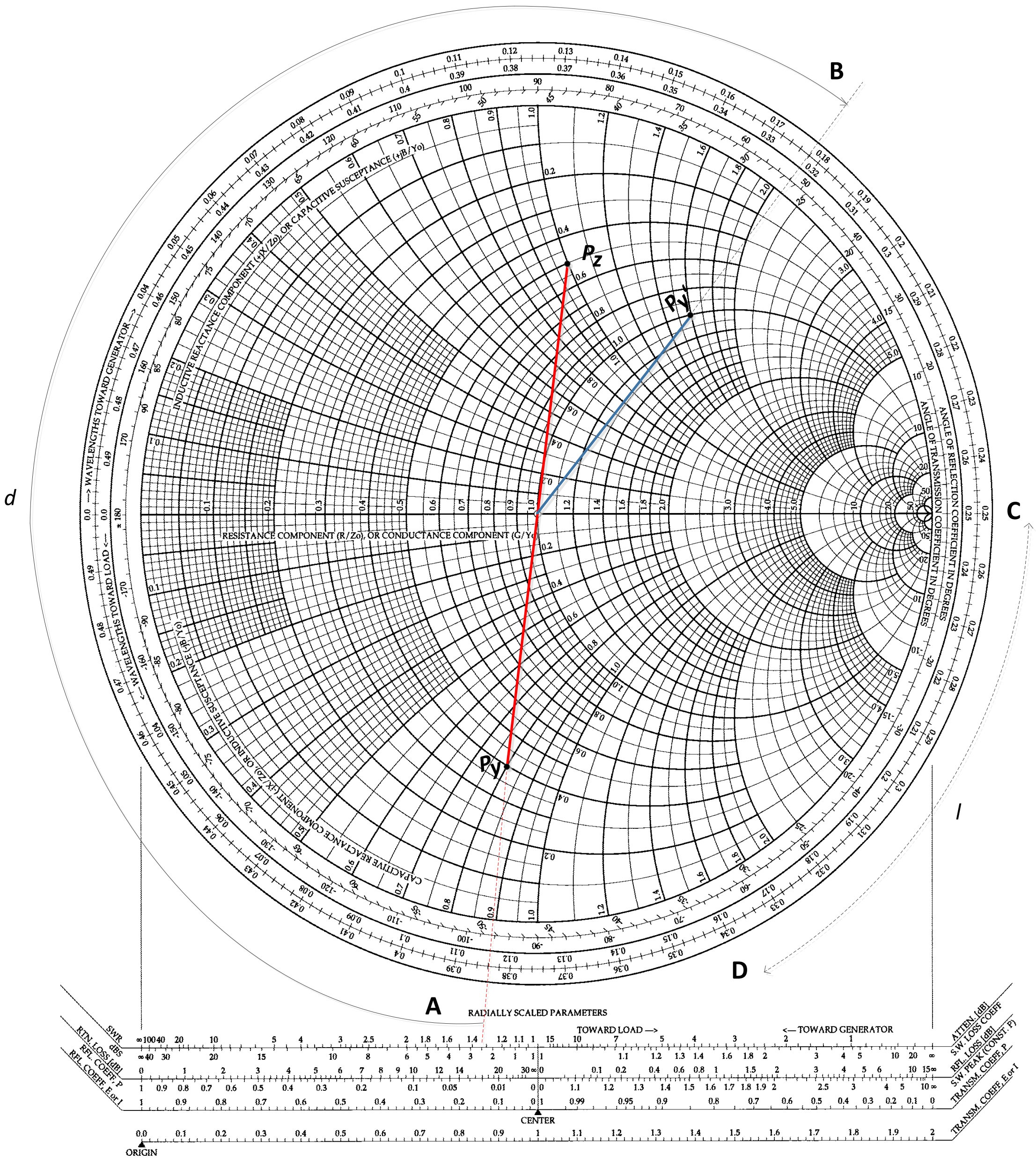
Reflect this point through the centre to reach $P_{y,}$ (the normalized admittance point) = 0.4 – j0.8

[Check the result by calculating

$$y = \frac{1}{z} = \frac{1}{0.5 + j1.0}$$

$$y = \frac{1}{0.5 + j1.0} \times \frac{0.5 - j1.0}{0.5 - j1.0} = 0.4 - j0.8 \quad]$$

With radius $OP_z = OP_y$, draw the line OP_y' from the centre which intercepts the circle x = 1.



With radius $OP_z = OP_y$, draw the line OP'_y from the centre which intercepts the circle x = 1 at P'_y

The radial distance A to B (which are the projections of OP_y and OP'_y respectively, to the periphery of the chart) in a clockwise direction gives the value of d, in wavelengths.

$$d = (0.5 – 0.384) + 0.178\ \lambda\ =\ 0.294\ \lambda$$

The admittance value at P'_y = y = 1.0 +j1.6, is the admittance of the line at point BB, where the stub is connected, due to the load.

If the stub length, *l*, is now adjusted to give a normalized admittance of

-j1.6 at BB, the total admittance at BB will now be 1.0 + j1.6 – j1.6 = 1.0+j0 and the line is matched. This corresponds to a move from the short-circuit end of the stub where the admittance is infinite, point C, to the point P''_y, projected to point, D, at the periphery of the chart where y = -j1.6. This gives the length of the stub as *l* = length CD on the chart.

$$l = (0.34 – 0.25)\ \lambda\ = 0.09\ \lambda$$

which corresponds to the distance C to D on the chart.

The values of, d, and, *l*, in metres can be calculated if the frequency of the signal is given, from: $\lambda\ (\text{m}) = \frac{3\ x\ 10^8}{f\ (Hz)}$

Suppose the frequency is 1 GHz (= 10^9 Hz), $\lambda = 0.3$ m;

d = 0.294 x 0.3 = 0.088 m, or 88 mm

and l = 0.09 x 0.3 = 0.027 m, or 27 mm.

10. ANTENNAS

10.1 INTRODUCTION

The transmission and reception of electromagnetic waves for radio communication are provided by radiating elements known as antennas. The antenna is an efficient transformer between free space and a transmission line. In effect, the antenna is an extension of the transmission line. The transmission line is used to guide electromagnetic waves from one location to another. In twin lines or coaxial cable, the waves are confined to the space between the conductors and little or no radiation of energy outside the conductors takes place, especially when the spacing between the conductors is a small fraction of the wavelength of the waves. If, however, the ends of a twin-line transmission line are flared out, the waves tend to be radiated out into free space. When the separation between the lines approaches the order of a wavelength or more, the opened-out lines act like an antenna. Initially, close to the antenna, the wave front is like an arc, changing gradually to a spherical shape as the distance from the antenna increases. See Figs. 10.1 (a) and (b).

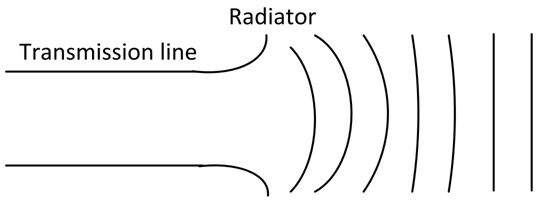


Figure 10.1 (a)

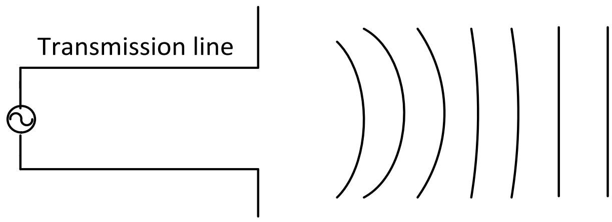


Figure 10.1 (b)

There is a region of transition between the near-field region close to the antenna and the far-field, free-space or radiation field region. From the circuit point of view, the antenna appears to the transmission line as a resistance, called the **radiation resistance,** not a physical but a virtual resistance, coupling the transmission line to the free space in the case of a transmitting antenna, or coupling the free space to the transmission in a receiving antenna. The behavior of transmitting and receiving antennas is essentially reciprocal in nature except in their power handling capability. Most wireless applications employ the far-field or radiation field which begins at about ten wavelengths (10 λ) away from the antenna. The near-field is hardly employed except for some special applications such as Radio Frequency Identification (RFID), or Near-Field Communication (NFC) which are somewhat new applications. Some manufacturers now build short-range near-field radio for applications such as wireless building access, ticket purchases or automotive functions.

In most communications systems, the same antenna is used for both transmitting and receiving signals as in the transceiver systems.

10.2 RADIATION FROM A SHORT DIPOLE

A short dipole is, more or less, the "building block" of all antennas, since a practical antenna can be regarded as an assemblage of short dipoles. A short dipole is a short wire, compared with the wavelength of the radiation, energized at its centre and terminated in large capacitance into which current can flow. See Fig. 10.2.

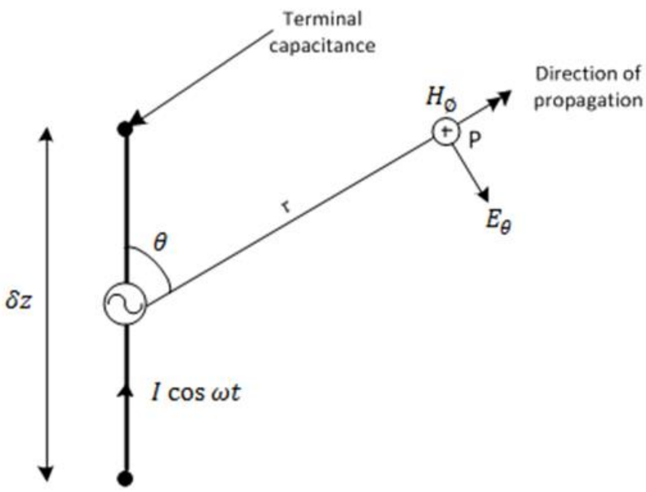


Figure 5.2: Radiation from a doublet

At a distance, r, which is large compared to the wavelength, λ, the only field components are the electric field, E_θ (at right angles to, r) and the magnetic field, $H_\emptyset$, (at right angles to E_θ and r). Both E_θ and $H_\emptyset$ are proportional to sinθ. E_θ and $H_\emptyset$ are at right angles in space and in time phase, leading to a spherical wave propagated in r-direction.

The ratio $\frac{E_\theta}{H_\emptyset}$ (similar to the ratio $\frac{volt}{ampere}$ for a resistor) is the characteristic or intrinsic impedance, Z_0, of free space, which is 120π or 377 Ω.

The power radiated by the short dipole is given by,

$$P = I^2 R_r \qquad (1)$$

where , $R_r = 80\,\pi^2 \left(\frac{dz}{\lambda,}\right)^2 \qquad (2)$

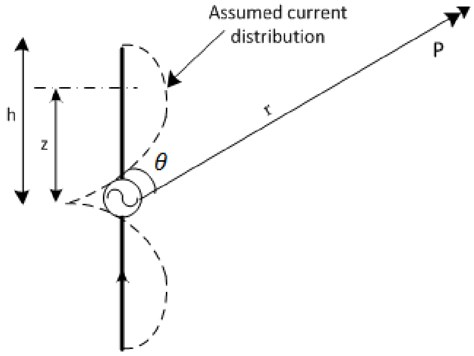
is the radiation resistance of the dipole antenna.

Eqn (2) indicates that the radiation resistance is proportional to the square of the length of the dipole. For instance, with dz = $\frac{1}{100}\lambda$, R_r = 0.079 Ω, and with dz = $\frac{1}{10}\lambda$, R_r = 7.9 Ω.

To achieve a reasonable value of the radiation resistance, and, therefore, of radiated power, the antenna length must be a significant fraction of a wavelength.

10.3 RADIATION FROM A $\frac{\lambda}{2}$ – DIPOLE

Consider a dipole with a length h = $\frac{\lambda}{2}$ fed in the middle as shown in Fig. 5.3(a).



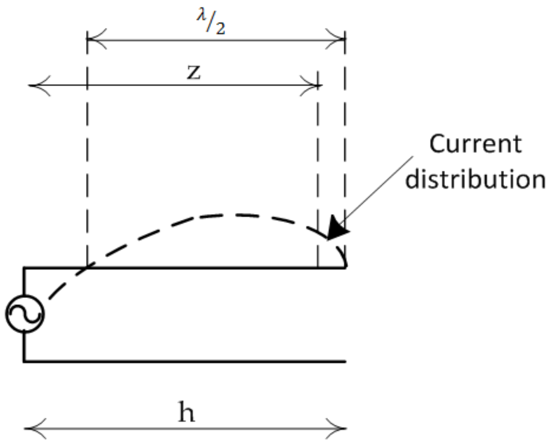


Fig. 10.3

To determine the electric field strength at a distant point P(r, θ) from the source point, we need to consider the distribution of the current along the dipole. This may be compared with a transmission line of length h = $\frac{\lambda}{4}$ which is open-circuited at the end, as shown in Fig. 5.3(b). the current distribution on the transmission line is of a sinusoidal standing wave pattern.

At a point, distance, z, from the input end, the current I_z can be expressed as,

$$I_z = I_0 \sin \beta(h - z) \cos \omega t \tag{3}$$

where $\beta = \frac{2\pi}{\lambda}$ (4)

The electric field at point $P(r, \theta)$ from the current distribution of eqn (3) is found to be,

$$E_\theta = \frac{j60I_0}{r}\left\{\frac{\cos(\beta hh\cos\theta - \cos\beta h)}{\sin\theta}\right\}\cos(\omega t - \beta r) \quad (5)$$

$$H_\emptyset = \frac{E_\theta}{Z_0} \quad (6)$$

The radiation pattern, i.e., the plot of E_θ as a function of θ, is given by the factor in the curly brackets { }. This pattern, plotted in Cartesian coordinates is as shown in Fig. 10.4.

Knowing the radiation pattern, the power radiated by the dipole can be calculated.

The resulting Poynting vector, S_{av}, is given by,

$$S_{av} = \frac{1}{2} E_\theta H_\emptyset^* \quad Wm^{-2},$$

where $H_\emptyset^*$ is the complex conjugate of $H_\emptyset$.

By taking the surface integral of S_{av} over the surface enclosing the antenna, we obtain the total power radiated by the antenna as,

$$P = \iint S_{av}\, dS = \frac{1}{2} \iint E_\theta\, H_\emptyset^*\, dS \tag{7}$$

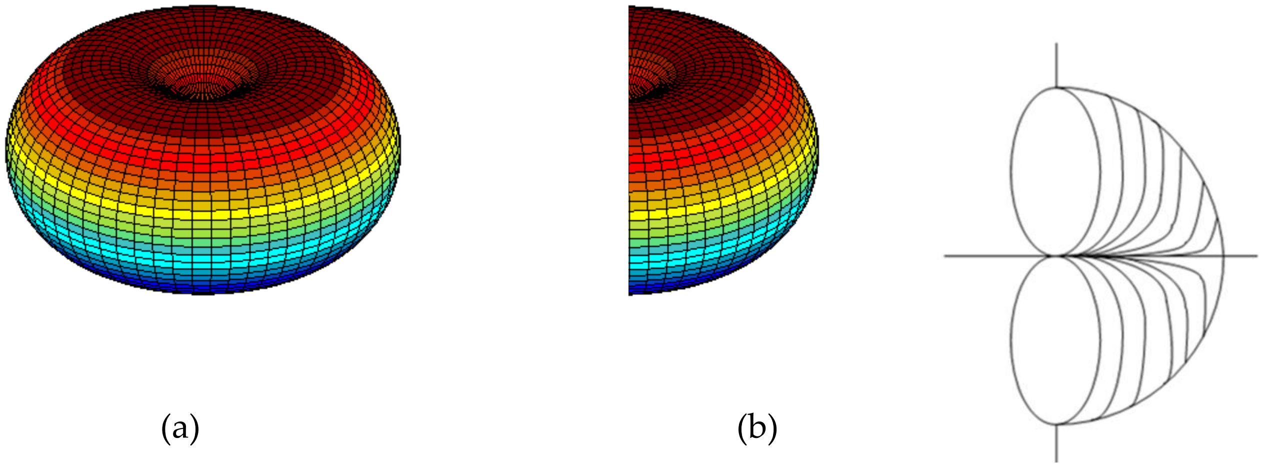
10.4 PRACTICAL ANTENNA PATTERNS FOR MEDIUM WAVE BROADCASTING: MEDIUM FREQUENCY ANTENNAS (300 HZ TO 3 MHZ FREQUENCY RANGE).

Vertical grounded antennas are usually employed for medium wave broadcasting at frequencies below 3 MHz. The radiation pattern characteristics should be such as to confine the radiation to small angles close to the horizon. This is achieved when the ground surface in the neighbourhood of the antenna base is of good conductivity, so as to limit the amount of energy lost to the ground. Grounded galvanized steel towers are usually erected on the ground. Artificial grounding system consisting of buried radial wires at the base of the tower is employed where the ground is of poor conductivity. Vertical radiation patterns for such antenna are as shown in Fig. 10.5.

It is observed that as the electrical length of the uniform vertical conductor is increased, larger and larger portions of a complete sine wave of current is distributed on the antenna. The integrated electrical fields from all parts of the antenna, together with the ground reflected waves interfere to varying degrees to produce electrical field strengths that cover longer and longer path lengths on the ground surface, until elevated radiation patterns start to emerge, when the electrical length is slightly over half a wavelength. Between $\frac{\lambda}{4}$ and $\frac{\lambda}{2}$ the patterns flatten out on the ground with increasing coverage. The greatest field intensity along the ground occurs at $\frac{5\lambda}{8}$. Beyond this, ground coverage begins to shrink and high-angle lobes start to emerge. This is undesirable as the lobes produce sky-wave interference with the ground wave and is unsuitable for broadcasting.

10.5 DIRECTIVITY, GAIN AND EFFICIENCY OF AN ANTENNA

The radiation pattern of an antenna is the shape of the electromagnetic energy radiated from or received by the antenna. Most antennas have directional characteristics which cause them to radiate or to receive energy in a specific direction. The radiation pattern of a half-wave vertical dipole, for instance, has a figure-8 pattern in the plane of the antenna, or a doughnut shape in three dimensions with the maximum radiation at right angles to the antenna, as illustrated in Figs. 10.6 (a) and (b), respectively.



(a) (b)

Fig. 10.6 Radiation Pattern of a half-wave vertical dipole

The term, **Directivity,** or the Directive Gain **D** $(\boldsymbol{\theta}, \emptyset)$**,** of an antenna is defined as the ratio of the power density radiated in the $(\theta, \emptyset)$ direction at some distance in the far field of the antenna to the power density at the particular point if the total power were radiated **isotropically.** An isotropic radiator is an ideal situation; it is the reference antenna that radiates equally in all directions. The above definition implies that the directivity of an isotropic antenna is unity.

Mathematically,

$$D(\theta, \emptyset) = \frac{S_{av}}{P_{rad}/4\pi r^2} \tag{1}$$

where S_{av} is the average Poynting vector in the direction $(\theta, \emptyset\ )$ of the antenna.

The Gain, G ($\boldsymbol{\theta}, \emptyset$), of an antenna is defined similarly as the directivity. It is the ratio of the power density radiated at some point in the far field of the antenna to the power, P_{in}, of the antenna when radiated isotropically.

Mathematically,

$$G(\theta, \emptyset) = \frac{S_{av}}{P_{in}/4\pi r^2} \tag{2}$$

Radiation Efficiency, $\boldsymbol{\eta}$, is defined as,

$$\eta = \frac{P_{rad}}{P_{in}} \tag{3}$$

Combining eqns (1), (2) and (3) gives,

$$\eta = \frac{\mathbf{G}\ (\boldsymbol{\theta}, \emptyset\ )}{\mathrm{D}\ (\theta, \emptyset\ )} \tag{4}$$

The gain of an antenna includes the effects of losses in the antenna and other surrounding structures, thereby making the efficiency to be less than unity. The directivity is determined solely by the shape of the radiation pattern of the antenna.

In actual fact,

$$\eta = \frac{R_r}{R_r + R_{loss}} \tag{5}$$

where R_r is the radiation resistance of the antenna and R_{loss} is the ohmic loss in the antenna material and any other losses in the surrounding structures, if present, such as loss to the ground in a grounded vertical antenna, for example.

Note that the definition of "gain" in antenna theory should not be confused with the definition in circuit theory where signal amplification is implied. There is no "real gain" associated with antennas, since they are made of metals which are passive materials that dissipate rather than amplify energy. Antenna gain refers to the focusing or directional properties of the antenna compared to that of a lossless, isotropic radiator.

Example 10.5.1 A directional antenna radiating a total power of 180 W produces a power density of 1 mWm^{-2} in a given direction at a point A, a distance $d = 2\ km$ away. Calculate

(a) the power density at A if an isotropic antenna radiates the same power of 180 W from the location of the directional antenna;

(b) the directivity in dB of the directional antenna.

Solution: Power density at A due to the isotropic antenna is,

$$(a)\ P_i = \frac{180}{4\pi d^2} = \frac{180}{4\pi \times 4 \times 10^6} = 3.58 \times 10^{-6}\ Wm^{-2}$$

$$\text{(b) Directivity} = \frac{1\ x\ 10^{-3}}{3.58 \text{ x } 10^{-6}} = 279.3 = 10 \log 279.3 = 25.5dB$$

Example 10.5.2 To produce a power density of $5\ mWm^{-2}$ in a given direction, at a distance of 5 km an antenna radiates a total power of 10 kW. Calculate the power radiated from an isotropic antenna to produce the same power density at the same point, if the directive power of the antenna is 30 dB.

Solution: The directive power of the antenna of 30 dB corresponds to a power ratio of 1000. That is, the ratio of the power radiated by the isotropic radiator to produce the same power density of the same value, at the same point, as the directive antenna, is 1000. Hence, the power of the isotropic antenna = 10 kW × 1000 = 10 MW.

Example 10.5.3 An antenna has a radiation resistance of 73 Ω, a loss resistance of 10 Ω and a power gain of 20. Calculate its directivity.

Solution:

$$\text{Efficiency of the antenna}, \eta = \frac{R_r}{R_r + R_{loss}} = \frac{73}{73 + 10} = 0.88$$

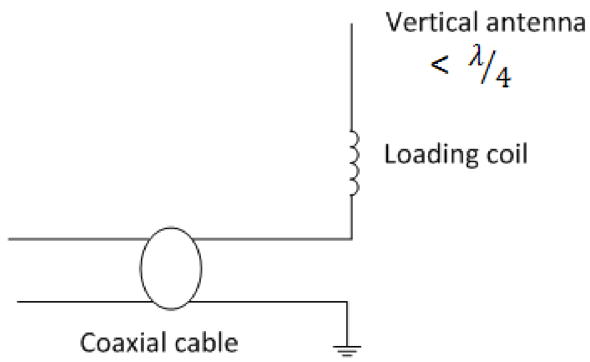
$$\text{Therefore Directivity} = \frac{Gain}{\eta} = \frac{20}{0.88} = 22.7$$

10.6 OTHER TYPES OF ANTENNAS

Antenna shapes and sizes exist depending on the frequency bands and services required. Antennas for high frequency (HF) broadcasting service are different in shapes and sizes from those of VHF/UHF, FM and TV or DSTV broadcasts.

Most low-frequency transmitting antennas use $\frac{\lambda}{4}$ vertical antennas. AM broadcast stations in the 535 to 1635 kHz frequency range use $\frac{\lambda}{4}$ vertical antennas since they are short, inexpensive and not offensive to sight. They provide omni-directional radiation patterns which is ideal for broadcasting.

Note that a $\frac{\lambda}{4}$ antenna at a frequency of 1 MHz is 75 m in length. For many applications, for instance, portable or mobile equipment, it is not feasible practically to have a full $\frac{\lambda}{4}$ antenna. A cordless telephone operating in 46 to 49 MHz range of frequency would require a $\frac{\lambda}{4}$ of about1.5 m long antenna. A whip antenna or even a telescopic type of this dimension would be impracticable to hold to one's ear. To overcome the problem, much shorter antennas are used but with a lumped electrical component, such as a loading coil, incorporated to compensate for the shortened length, as shown in Fig. 5.7.



10.7 LINEAR ARRAY OF ANTENNAS

The radiation pattern of a single $\frac{\lambda}{2}$ dipole antenna is a figure-8, which has a rather low directivity (or gain). In long distance communications, antennas with very high directivity are desirable, first, to provide high gain in a desired direction and, secondly, to reject signals from undesirable directions where signal interference could occur to or from other sources.

By constructing an assembly of radiating elements in a proper electrical and geometrical configuration, referred to as an antenna array, it is possible to produce a radiation pattern of high directivity or gain in a desired direction. If the elements are arranged in a straight line, the arrangement is described as a linear array.

Usually, the array elements are identical, though not a necessary condition, but a more practical construction, simple and convenient for design and fabrication. The individual elements could be made of wire dipoles, loops or apertures. The total radiation field is a vector superposition of the fields radiated by individual elements. The larger the number of array elements with appropriate spacing and excitation phase of the current in each element, the better the directivity, especially when the partial fields generated by individual elements interfere constructively in the desired direction.

Consider two linear arrays, each of four elements as shown in Figs. 5.8 (a) and (b), where $1<\alpha$ indicates that the current in each element is of unit amplitude and phase angle α. In Fig. 10.8 (a), the elements are energized with equal in-phase currents, while in Fig. 10.8 (b), the currents are of unit amplitude but in progressive phase lag of α radians from left to right.

In Fig. 10.8 (a), the contribution from each element at some distance from the array add up in phase in the two directions at right angles to the line of the elements to produce a **broadside array** pattern. In Fig 10.8 (b), if the progressive phase lag α between adjacent elements is made equal to βd (where β = $\frac{2\pi}{\lambda}$ and, d, is the spacing between the elements), the radiation pattern will produce a maximum value along the line of the elements to produce an **end-fire array.** If the phase angle is reversed, i.e., a phase lead rather than phase lag, the beam direction is also reversed.

By choosing a progressive phase delay between adjacent elements lying between +βd and −βd radians, the resulting beam can be made to swing in any desired direction; thus, the beam can be made to scan in all directions without changing the positions or orientations of the antenna elements physically. We then have electrical beam swinging or scanning.

Consider a linear array shown in Fig. 10.9, where there are n elements equally spaced by distance, d, apart, with equal current amplitudes in each element, but with a phase lead of α between adjacent elements. If the field strength at a given far-field location, P, and angle, ϕ with the line of the elements is E_0, it can be shown that the resultant field strength at P due to the linear array is given by,

$$E = E_0 \left\{ \frac{\sin(nu)}{\sin u} \right\} \tag{1}$$

where

$$u = \frac{\beta d\ cos\phi + \alpha}{2}$$

The factor in the curly brackets in eqn (1) is the radiation pattern of the array. It has a maximum value of, n, when, u = 0.

If α = 0, the maximum corresponds to cos ϕ = 90^0 or 270^0 which produces the broadside array pattern of Fig. 5. 8(a) with maximum field strength of nE_0 .

When α = -βd, the maximum corresponds to cos ϕ = 1, or ϕ = 0, which produces the end-fire array of Fig. 10.8 (b).

The maximum of the radiation pattern can be directed at any required angle ϕ to the line of elements by making α = -βd cos ϕ.

In addition to the main lobe(s), there are sidelobes, with intermediate zeros between them as in Fig 10.8 (a). The zeros become minima if the amplitudes of the currents in the elements are not equal. The longer the array, or the greater the number of elements, the narrower is the main beam or the greater is the directivity or gain of the array and the larger the number of sidelobes also.

10.8 THE YAGI-UDA ARRAY OR THE YAGI ANTENNA

The Yagi antenna is a commonplace antenna mounted on rooftops of buildings to receive TV signals in the VHF/UHF range of frequencies. It was invented by Uda, a Japanese professor an d developed by his co-worker, Yagi.

The basic principle is explained as follows:

Suppose we have two $\frac{\lambda}{2}$ dipole antennas, set up parallel to each other and separated by a distance $\frac{\lambda}{4}$. If only one of them is energized, called the driven antenna, it will be observed that the field created by the driven (or active) antenna will induce a current to flow in the second antenna as a result of radiation coupling. The second undriven (or passive) antenna is called a “parasite” or a “parasitic” element. See Fig. 10. 10.

The effect of the coupled elements is to produce a radiation pattern with a greater directivity, in the direction of the driven element, than that of the driven element acting alone, as illustrated in Fig. 10. 11.

Fig. 10 .11 Horizontal radiation pattern for two parallel $\frac{\lambda}{2}$ dipoles

The circle shows the field strength of the driven element alone.

If more parasitic elements are placed in front of the driven element as shown in Fig. 10.12, the radiation pattern gets narrower and narrower as the number of elements are increased.

Fig. 10.12 Multielement Yagi Antenna

The mechanical structure of the Yagi antenna appears quite simple, but the design details in terms of the lengths of the radiating elements, their spacing and the diameter of the rods used as radiators, are somewhat complex in order to achieve the required radiation beamwidth.

The **beamwidth** is defined as the angular separation between the two half-power points on the power density radiation pattern. It is also the angular separation between the two 3-dB down points on the field strength radiation pattern, as illustrated in Fig. 10.13.

Fig. 10.13 Beamwidth of the radiation pattern

10.9 PARABOLIC MICROWAVE ANTENNA

Achieving high gain is the main reason for using a parabolic microwave antenna for transmitting and receiving signals in the microwave range of frequencies of 1 to 100 GHz. Other important reasons are the following:

(a) The fact that broadcasting is not carried out at these frequencies, but for point-to-point communications, there is no need for omni-directional antennas;

(b) Receivers are usually much noisier at this frequency range than at lower frequencies, necessitating high gain antennas;

(c) Special microwave applications such as radar, for direction finding, satellite communications or space exploration systems demand high gain directional antennas.

The term, parabolic, comes from the fact that a metallic reflector whose shape is in the form of a geometric parabola is the main component in the beam-forming design of the antenna.

The parabola is a plane curve, defined as the locus of a point which moves in a way that its distance from a point, called the *focus*, plus its distance from a straight line, called the *directrix,* is always a constant, as illustrated in Fig. 10. 13.

Fig. 10. 13 Geometry of the parabola

The curve GCH describes a parabola whose focus is at F and the line GH is the directrix. By the definition of the parabola, we have FA + AA' = FB + BB' = FD + DD' = FE + EE' = constant, k. The value of k may be changed if a different parabolic shape is desired.

The ratio of the focal length FC to the mouth diameter GH is called the *aperture,* $\frac{FC}{GH}$, of the parabola.

If a source of radiation is placed at the focus, all waves emanating from the source and reflected by the parabolic surface will have travelled the same distance by the time they arrive at the directrix, irrespective of the point of reflection on the reflecting surface. All such waves will, therefore, arrive in phase at the directrix, resulting in a strong and concentrated beam parallel to the axis CC'.

A practical parabolic reflector is a three-dimensional surface, like a dish; the reason why it is often referred to as a *microwave dish*.

The reflector is directional for both transmitting and receiving; that is, all rays that arrive on the surface parallel to the axis CC' will be reflected from the surface and arrive at the focus in phase. Hence, the principle of reciprocity, which states that the properties of an antenna are independent of whether the antenna is used for transmitting or receiving a signal, holds in this case, also. The parabolic antenna is, therefore, a high-gain directional antenna because it collects radiation from a large area and concentrates them at the focal point.

The microwave dish is commonly seen mounted on radio masts at intervals on a long-distance terrestrial microwave telecommunications system in a relay fashion, or as digital satellite television (DSTV) receivers mounted of roof tops.

11. THE HOLLOW RECTANGULAR WAVEGUIDE

The name *waveguide* is customarily reserved for specially constructed hollow metallic pipes, though any system of conductors such as twin wires, coaxial cable or optical fibre (dielectric) capable of transmitting electromagnetic waves could be called a waveguide. Metallic waveguides which are of constant rectangular, circular or elliptical cross-sectional shapes are the most popular types in practice. They are used at microwave frequencies where transmission lines or coaxial cables are ineffective because of their high transmission loss. Because the cross-sectional dimensions of a waveguide are of the order of magnitude of the wavelength of the electromagnetic wave, the frequency of use below about 1 GHz is not normally considered, but the frequency of up to about 300 GHz is quite possible. Within this range, waveguides are generally superior to coaxial cables for a wide range of applications, for either low or high power levels. The maximum operating frequency of a coaxial cable is about 18 GHz.

Other advantages of waveguides are:

(i) They are simpler to manufacture than coaxial cables. In appearance, a circular waveguide looks like a coaxial cable, but it is hollow inside, devoid of the inner conductor of the coaxial cable. As a result, there is less likelihood of flashover in a waveguide which is filled with air instead of an inner conductor with supporting dielectric of the coaxial cable. The power handling capability of the waveguide is thereby improved.

(ii) Since the waveguide is filled with air, and the wave propagation inside the guide is by reflection from the walls of the guide instead of conduction along them, power losses in waveguides are much lower than in comparable coaxial cables. For example, a 4 cm air-dielectric coaxial cable has an attenuation of about 4 dB/100m at 3 GHz, which is quite good for the cable, which rises to about 11 dB/100 m for a similar foam-dielectric flexible cable. The figure for a similar dimension of waveguide produces about 2.3 to 2.6 dB/100m depending on whether the waveguide is made of aluminium or brass.

11.1 SOLUTION OF THE MAXWELL'S EQUATIONS FOR THE WAVE PROPAGATION WITHIN THE WAVEGUIDE

The hollow rectangular waveguide will be considered here because of the simplicity of applying the boundary conditions. Assume the wave travels in the x-direction which is the direction of the guide, with a harmonic variation with respect to time. There are two major modes of transmission, referred to as (a) the Transverse Electric (TE) and (b) the Transverse Magnetic (TM) modes.

As the names connote, the TE mode refers to a situation where the electric field component of the wave is entirely transverse to the direction of propagation, and has no component along the direction, i.e $E_x = 0$. Similarly, TM mode connotes the magnetic field component is transverse, and $H_x = 0$.

Let us consider the TE mode of transmission first. The procedure to be followed in solving for each of the field components as a function of time and space may be outlined by the following eight steps:

1. Start with Maxwell's equations.
2. Apply restriction of harmonic variation with respect to time.
3. Apply restriction of harmonic variation and attenuation with respect to x.

4. Select the TE mode of transmission, $E_x = 0$, but $H_x \neq 0$.
5. Find equations for the other four field components (E_y, $E_{z,}$, H_y , H_z) in terms of H_x.
6. Develop scalar wave equation for H_x.
7. Solve this wave equation for H_x subject to boundary conditions of the waveguide.
8. Substitute H_x back into equations of step 5, giving a set of equations expressing each field component as a function of space and time.

   Step 8 above constitutes the complete solution of the problem.

Starting with step 1 of the procedure:

- The Maxwell's divergence equations in rectangular coordinate systems are as follows:

$$\frac{\partial Ex}{\partial x} + \frac{\partial Ey}{\partial y} + \frac{\partial Ez}{\partial z} = 0 \tag{1}$$

$$\frac{\partial Hx}{\partial x} + \frac{\partial Hy}{\partial y} + \frac{\partial \mathrm{Hz}}{\partial \mathrm{z}} = 0 \tag{2}$$

- The curl expressions, are as follows:

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \tag{3}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad (4)$$

which, in rectangular coordinates , are expressed by the following six scalar equations:

$$\frac{\partial \mathrm{H_z}}{\partial \mathrm{y}} - \frac{\partial \mathrm{Hy}}{\partial \mathrm{z}} - \sigma E_x - \epsilon \frac{\partial E_x}{\partial t} = 0 \quad (5)$$

$$\frac{\partial Hx}{\partial z} - \frac{\partial Hz}{\partial x} - \sigma E_y - \epsilon \frac{\partial E_y}{\partial t} = 0 \quad (6)$$

$$\frac{\partial Hy}{\partial x} - \frac{\partial Hx}{\partial y} - \sigma E_z - \epsilon \frac{\partial Ez}{\partial t} = 0 \quad (7)$$

$$\frac{\partial \mathrm{E_z}}{\partial \mathrm{y}} - \frac{\partial \mathrm{Ey}}{\partial \mathrm{z}} + \mu \frac{\partial H_x}{\partial t} = 0 \quad (8)$$

$$\frac{\partial Ex}{\partial z} - \frac{\partial Ez}{\partial x} + \mu \frac{\partial H_y}{\partial t} = 0 \quad (9)$$

$$\frac{\partial Ey}{\partial x} - \frac{\partial Ex}{\partial y} + \mu \frac{\partial Hz}{\partial t} = 0 \quad (10)$$

Assume now that any of the field component varies harmonically with time and distance and also attenuates with distance (steps 2 and 3), and that the waves travel in the positive x-direction, then we may express the field component, E_y as,

$$E_y = E_1 \, e^{j\omega t - \gamma x} \quad (11)$$

where γ = propagation constant = α + jβ

α = attenuation constant, and

β = phase constant.

When the restriction, eqn (11), is introduced into eqns (1) and (2) as well as eqns (5) to (10), we have the following equations:

$$-\gamma E_x + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \qquad (12)$$

$$-\gamma H_x + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \qquad (13)$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - (\sigma + j\omega\epsilon) E_x = 0 \qquad (14)$$

$$\frac{\partial H_x}{\partial z} + \gamma H_z - (\sigma + j\omega\epsilon) E_y = 0 \qquad (15)$$

$$-\gamma H_y - \frac{\partial H_x}{\partial y} - (\sigma + j\omega\epsilon) E_z = 0 \qquad (16)$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + j\omega\mu H_x = 0 \qquad (17)$$

$$\frac{\partial E_x}{\partial z} + \gamma E_z + j\omega\mu H_y = 0 \qquad (18)$$

$$-\gamma E_y - \frac{\partial E_x}{\partial y} + j\omega\mu H_z = 0 \qquad (19)$$

These equations can be simplified by introducing a series impedance Z and shunt admittance Y as in the case of transmission line, where

$$Z = -j\omega\mu \quad (\Omega\, m^{-1}) \qquad (20)$$

$$Y = \sigma + j\omega \epsilon \ (S\, m^{-1}) \qquad (21)$$

Substituting these relations in eqns (12) to (19), we have the following equations:

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$$-\gamma E_x + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \qquad (22)$$

$$-\gamma H_x + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \qquad (23)$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial Hy}{\partial z} - YE_x = 0 \qquad (24)$$

$$\frac{\partial H_x}{\partial z} + \gamma H_z - YE_y = 0 \qquad (25)$$

$$-\gamma H_y - \frac{\partial H_x}{\partial y} - YE_z = 0 \qquad (26)$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial Ey}{\partial z} - ZH_x = 0 \qquad (27)$$

$$\frac{\partial E_x}{\partial z} + \gamma E_z - ZH_y = 0 \qquad (28)$$

$$-\gamma E_y - \frac{\partial E_x}{\partial y} - Z H_z = 0 \qquad (29)$$

Eqns (22) to (29) are the general equations for the steady-state field of a wave travelling in the x-direction without any restrictions yet on the mode of propagation of the wave within the guide or on the shape of the guide.

We can now go to step 4 and select the mode of propagation, TE, for which E_x = 0, but $H_x \neq 0$.

$$\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \qquad (30)$$

$$-\gamma H_x + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \qquad (31)$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = 0 \qquad (32)$$

$$\frac{\partial H_x}{\partial z} + \gamma H_z - YE_y = 0 \qquad (33)$$

$$-\gamma H_y - \frac{\partial H_x}{\partial y} - YE_z = 0 \qquad (34)$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} - ZH_x = 0 \qquad (35)$$

$$\gamma E_z - ZH_y = 0 \qquad (36)$$

$$-\gamma E_y - Z H_z = 0 \qquad (37)$$

Consider eqns (36) and (37), the ratio

$$\frac{E_y}{H_z} = -\frac{E_z}{H_y} = \frac{-Z}{\gamma} = \frac{j\omega\mu}{\gamma} = Z_{yz}. \qquad (38)$$

has the dimension of impedance. Since these the field components are transverse in nature, the resulting impedance is referred to as the *transverse-wave impedance*, Z_{yz}.

Introducing this new impedance into eqn (34) and solving for H_y in terms of H_x gives

$$H_y = \frac{-1}{\gamma - YZ_{yz}} \frac{\partial H_x}{\partial y} \quad (39)$$

Similarly, the other field components, H_z , E_y , E_z are expressed in terms of H_x as follows:

$$H_z = \frac{-1}{\gamma - YZ_{yz}} \frac{\partial H_x}{\partial z} \quad (40)$$

$$E_y = \frac{Z_{yz}}{\gamma - YZ_{yz}} \frac{\partial H_x}{\partial z} \quad (41)$$

$$E_z = \frac{-Z_{yz}}{\gamma - YZ_{yz}} \frac{\partial H_x}{\partial y} \quad (42)$$

Eqns (39) to (42) express the four transverse field components in terms of H_x.

This completes step 5 of the eight-step procedure.

To develop the wave equation for H_x (step 6), take the y-derivative of eqn (39) and the z-derivative of eqn (40) and substitute both in eqn (31) to obtain

$$-\gamma H_x - \frac{1}{\gamma - YZ_{yz}} \left(\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} \right) = 0 \quad (43) \text{ or}$$

$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} + \gamma (\gamma - YZ_{yz}) H_x = 0 \quad (44)$$

Putting $k^2 = \gamma (\gamma - YZ_{yz})$ (45)

reduces eqn (44) to

$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} + k^2 H_x = 0 \qquad (46)$$

This is a partial differential equation of the second order and first degree, which is the scalar wave equation for H_x. The equation applies to a TE wave in a guide of any cross-sectional shape. **This completes step 6.**

This wave equation is now to be solved subject to the boundary conditions of the waveguide, (step 7). The waveguide under consideration is a hollow rectangular type as shown in Fig. 6.1

Fig. 11.1 Hollow rectangular waveguide

(see Fig. 13-35, p 537 , sect 13-15, Kraus Carver)

The height and width of the rectangular waveguide are y_1, z_1, along the y- and z- axes, respectively. Assuming that the walls are perfectly conducting, the tangential component of **E** must vanish at the surface of the guide. Thus, at the sidewalls, E_y must be zero, and at the top and bottom surfaces, E_z must be zero. Eqn (46) is now to be solved subject to these boundary conditions.

The solution to be adopted is by the method of separation of variables. Since H_x is a function of y and z, we may seek a solution of the form

$$H_x = YZ \qquad (47)$$

where Y is a function of y only, i.e., Y $= f$ (y), and Z is a function of z only.

[Caution! *Y and Z in this section must not be confused with admittance and impedance of previous sections*]

Substituting eqn (47) into eqn (46) we have

$$Z\frac{\partial^2 Y}{\partial y^2} + Y\frac{\partial^2 Z}{\partial z^2} + k^2 YZ = 0 \qquad (48)$$

Dividing by YZ to separate the variables gives

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -k^2 \qquad (49)$$

The first term is a function of y only, the second term is a function of z only, while k^2 is a constant. For the two terms, each involving a different independent variable, to sum up to a constant, requires that each term must be a constant. Therefore, we may write

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = - A_1 \qquad (50) \quad \text{and}$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = - A_2 \qquad (51)$$

where A_1 and A_2 are constants. This implies

$$A_1 + A_2 = k^2 \qquad (52)$$

Eqns (50) and (51) each involves one independent variable.

A solution of eqn (50) is

$$Y = c_1 \sin \sqrt{A_1}\, y \qquad (53)$$

Another solution is

$$Y = c_2 \cos \sqrt{A_1}\, y \qquad (54)$$

If eqns (53) and (54) are each a solution of Y, then the sum of both will also be a solution, i. e.

$$Y = c_1 \sin \sqrt{A_1}\, y + c_2 \cos \sqrt{A_1}\, y \qquad (55)$$

Similarly, a solution of eqn (51) for Z is given by

$$Z = c_3 \sin\sqrt{A_2}\, z + c_4 \cos\sqrt{A_2}\, z \tag{56}$$

Thus, eqn (47), $H_x = YZ$ becomes,

$$H_x = c_1 c_3 \sin\sqrt{A_1}\, y \sin\sqrt{A_2}\, z + c_2 c_3 \cos\sqrt{A_1}\, y \sin\sqrt{A_2}\, z$$

$$c_1 c_4 \sin\sqrt{A_1}\, y \cos\sqrt{A_2}\, z + c_2 c_4 \cos\sqrt{A_1}\, y \cos\sqrt{A_2}\, z \tag{57}$$

Substituting eqn (57) into eqns (41) and (42) and introducing the boundary conditions,

$E_y = 0$ at $z = 0,\ z = z_1,$ (58) and

$E_z = 0$ at $y = 0,\ y = y_1$ (59) (See Fig. 6.1)

It can be shown that only the last term of eqn (57) satisfies the boundary conditions provided,

$\sqrt{A_1} = \frac{n\pi}{y_1}$ and $\sqrt{A_2} = \frac{m\pi}{z_1}$ (60),

where m, n, are integers (0, 1, 2, 3, …), which may be of same integers or of different integers.

The solution for H_x may now be written as

$$H_x(y, z) = H_0 \cos\frac{n\pi y}{y_1} \cos\frac{n\pi z}{z_1} \tag{61}$$

where $H_0 = c_2 c_4 =$ constant. (62)

If this constant is multiplied by the constant term exp(jwt – γx), the solution will still be preserved.

Thus, the complete solution for H_x is

$$\mathbf{H_x\ (y, z, x, t)} = \boldsymbol{H_0} \ \mathbf{cos}\frac{\boldsymbol{n\pi y}}{\boldsymbol{y_1}} \ \mathbf{cos}\frac{\boldsymbol{n\pi z}}{\boldsymbol{z_1}} \ \boldsymbol{e}^{(\mathbf{jwt - \gamma x})} \qquad \mathbf{(63)}$$

Step 7 is now completed.

To perform step 8, H_x will be substituted into the equations of step 5 to determine the other four field components, H_y, H_z, E_y, and E_z, using eqns (39) to (42),respectively, resulting in the following expressions:

$$\mathbf{H_y} = \frac{\gamma}{k^2}\boldsymbol{H_0}\frac{\boldsymbol{n\pi}}{\boldsymbol{y_1}}\ \mathbf{sin}\frac{\boldsymbol{n\pi y}}{\boldsymbol{y_1}}\ \mathbf{cos}\frac{\boldsymbol{m\pi z}}{\boldsymbol{z_1}} \ \boldsymbol{e}^{(-\gamma \mathbf{x})} \qquad \mathbf{(64)}$$

$$\mathbf{H_z} = \frac{\gamma}{k^2}\boldsymbol{H_0}\frac{\boldsymbol{m\pi}}{\boldsymbol{z_1}}\ \mathbf{cos}\frac{\boldsymbol{n\pi y}}{\boldsymbol{y_1}}\ \mathbf{sin}\frac{\boldsymbol{m\pi z}}{\boldsymbol{z_1}} \ \boldsymbol{e}^{(-\gamma \mathbf{x})} \qquad \mathbf{(65)}$$

$$\mathbf{E_y} = \frac{\gamma}{k^2}\boldsymbol{Z_{yz}}\boldsymbol{H_0}\frac{\boldsymbol{m\pi}}{\boldsymbol{z_1}}\ \mathbf{cos}\frac{\boldsymbol{n\pi y}}{\boldsymbol{y_1}}\ \mathbf{sin}\frac{\boldsymbol{m\pi z}}{\boldsymbol{z_1}} \ \boldsymbol{e}^{(-\gamma \mathbf{x})} \qquad \mathbf{(66)}$$

$$\mathbf{E_z} = -\frac{\gamma}{k^2}\boldsymbol{Z_{yz}}\boldsymbol{H_0}\frac{\boldsymbol{n\pi}}{\boldsymbol{y_1}}\ \mathbf{sin}\frac{\boldsymbol{n\pi y}}{\boldsymbol{y_1}}\ \mathbf{cos}\frac{\boldsymbol{m\pi z}}{\boldsymbol{z_1}} \ \boldsymbol{e}^{(-\gamma \mathbf{x})} \qquad \mathbf{(67)}$$

Eqns (63) to (67), to which may be added the TE mode condition of E_x = 0 are the complete equations for the six scalar field components in the hollow rectangular waveguide, having width z_1 and height y_1.

[Note that subscript (y, z, x, t) has been dropped in eqns (64) to (67) for simplicity, and e^{jwt} is also omitted in accordance with phasor notation]

11.2 TRANSMISSION MODES $\mathbf{TE_{MN}}$ AND $\mathbf{TM_{MN}}$

Integers m, n appearing in all the field components constitute the various transmission modes within the waveguide.

11.2.1 $\mathbf{TE_{10}}$ MODE

For this mode, m = 1, n = 0. It is apparent that the components $E_z = 0$, $H_y = 0$ and the three components, E_y, H_x , and H_z are not zero, resulting in the following expressions:

$$\mathrm{E_y} = \frac{\gamma}{k^2} Z_{yz} H_0 \frac{\pi}{z_1} \sin \frac{\pi z}{z_1} e^{(-\gamma x)} \tag{1}$$

$$\mathrm{H_x} = H_0 \cos \frac{\pi z}{z_1} e^{(-\gamma x)} \tag{2}$$

$$\mathrm{H_z} = \frac{\gamma}{k^2} H_0 \frac{\pi}{z_1} \sin \frac{\pi z}{z_1} \; e^{(-\gamma x)} \tag{3}$$

The variations of these three components as a function of z for modes TE_{10} , TE_{20} are as shown in Figs 11.2 (a) and (b).

(see Fig 13 – 11, p. 551 J. D. Kraus 3rd Ed.)

Fig 11.2 Half-cycle field variations in the waveguide

For m = 1, there is a half-cycle variation with respect to z, for each field component. The variation for E_y, for instance, shows a maximum value at the centre of the guide and zero at the walls. For m = 2, the variation consists of two half-cycles.

When n = 1, there is a half-cycle variation of each field component with respect to y.

In general, the value of m or n indicates the number of half-cycle variations of each field component with respect to z and y, respectively. Each combination of m and n values represents a different field configuration or mode in the guide. The notation TE_{mn} (or TM_{mn}) is adopted to indicate mode m, n in Transverse Electric (or Transverse Magnetic) transmission in the guide. The z-dimension is generally regarded as the larger dimension of the guide.

Figs 11.3 (a) and (b) illustrate the TE_{10}, TE_{20} electric and magnetic cross-sectional field configurations of the guide.

Fig. 11.3 Waveguide cross-sectional field configuration for TE_{10}, TE_{20} modes

11.3 PROPAGATION CONSTANT, CUT-OFF FREQUENCY AND CUT-OFF WAVELENGTH IN A LOSSLESS HOLLOW RECTANGULAR WAVEGUIDE.

Going back to eqns (20), (21), (38), (45), (52) and (60), the expression for the propagation constant can be obtained from the expression,

$$\left(\frac{n\pi}{y_1}\right)^2 + \left(\frac{m\pi}{z_1}\right)^2 = k^2 \tag{1}$$

where $$k^2 = \gamma^2 - j\omega\mu\,(\sigma + j\omega\,\epsilon\,) \tag{2}$$

For a lossless medium inside the waveguide, $\sigma = 0$, and so,

$$\gamma = \sqrt{\left(\frac{n\pi}{y_1}\right)^2 + \left(\frac{m\pi}{z_1}\right)^2 - \omega^2\mu\,\epsilon} \tag{3}$$

At sufficiently low frequencies, the last term of eqn (3) is smaller than the sum of the first two terms under the square root sign. Under this condition, γ is real and so the wave (or mode) is not propagated but attenuated.

Conversely, at sufficiently high frequencies, the last term becomes larger than the sum of the two terms, and γ becomes imaginary, resulting in unattenuated wave propagation within the guide.

At some intermediate frequency where the last term is equal to the sum of the two terms, γ becomes zero. The frequency is referred to as the **cut-off frequency** for the mode m, n under consideration.

At frequencies higher than the cut-off frequency, the mode propagates without attenuation, while at frequencies lower than the cut-off frequency, the mode is attenuated and not propagated.

The cut-off frequency, f_c, is expressed by

$$f_c = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{n}{y_1}\right)^2 + \left(\frac{m}{z_1}\right)^2} \tag{5}$$

And the cut-off wavelength,

$$\lambda_{oc} = \frac{2}{\sqrt{\left(\frac{n}{y_1}\right)^2 + \left(\frac{m}{z_1}\right)^2}} \tag{6}$$

The cut -off frequency for the TE_{10} mode in a rectangular waveguide is from eqn (5)

$$f_{c(1,0)} = \frac{1}{2\sqrt{\mu\epsilon}}\frac{1}{z_1} \tag{7}$$

This gives the lowest frequency of any of the TE modes, and is referred to as the **dominant mode**.

The **cut-off wavelength** for the TE_{10} mode is from eqn (6)

$$\lambda_{0c(1,0)} = 2z_1 \tag{8}$$

11.4 PHASE VELOCITY IN THE GUIDE

The phase velocity in an unbounded medium of the same dielectric material as in the guide is

$$v_0 = \frac{1}{\sqrt{\mu\epsilon}} \tag{1}$$

and the phase constant is

$$\beta_0 = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda_0} \tag{2}$$

where λ_0 is the wavelength in unbounded medium.

Eqn (11.3.3) becomes

$$= \sqrt{k^2 - \beta_0{}^2} \tag{3}$$

At frequencies higher than cut-off, $\beta_0 > k$,

Therefore, $= j\beta$, where

$$\beta = \frac{2\pi}{\lambda} = \sqrt{\beta_0^2 - k^2} = \sqrt{\omega^2\mu\epsilon - \left(\frac{n\pi}{y_1}\right)^2 - \left(\frac{m\pi}{z_1}\right)^2} \tag{4}$$

is the phase constant, and λ is the wavelength in the guide.

The phase velocity in the guide is, therefore,

$$v_p = \frac{\omega}{\beta} = \frac{v_0}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_{oc}}\right)^2}} \tag{5}$$

$$= \frac{v_0}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \tag{6}$$

Eqn (6) implies that the phase velocity in the guide is real and greater than the velocity in the unbounded medium when the frequency of propagation is greater than the cut-off frequency. When the frequency is equal to or less than cut-off, there's no wave propagation in the guide, but attenuation, as explained in section (6.3) above.

APPENDICES

APPENDIX 1

$$I = \int_{-\infty}^{\infty} \frac{xdr}{x^2 + r^2}$$

Let $r = x\ tan\ \theta$

$$dr = xSec^2\theta d\theta$$

$$x^2 + r^2 = (x^2 + x^2 tan^2\theta)$$

$$= x^2(1 + tan^2\theta)$$

$$= x^2(sec^2\theta)$$

When $= \pm\infty$, $\theta = \pm\infty$, Therefore,

$$I = \int_{-\infty}^{\infty} \frac{x(xSec^2\theta)d\theta}{x^2 Sec^2\theta}$$

$$I = \int_{-\infty}^{\infty} d\theta = [\theta]\ {}_{-\infty}^{\infty}$$

$$tan\theta = \frac{r}{x}$$

$$\theta = \tan^{-1}\left(\frac{r}{x}\right)$$

Therefore,

$$I = \left[\tan^{-1}\left(\frac{r}{x}\right)\right]{}_{-\infty}^{\infty} = \pi.$$