32. The accounting firm in Exercise 31 raises its charge for an audit to \$2500. What number of audits and tax returns will bring in a maximum revenue?

In the simplex method, it may happen that in selecting the departing variable all the calculated ratios are negative. This indicates an *unbounded solution*. Demonstrate this in Exercises 33 and 34.

33. (Maximize)
Objective function:
$z = x_1 + 2x_2$
Constraints:
$$\begin{aligned} x_1 - 3x_2 &\leq 1 \\ -x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

34. (Maximize)
Objective function:
$z = x_1 + 3x_2$
Constraints:
$$\begin{aligned} -x_1 + x_2 &\leq 20 \\ -2x_1 + x_2 &\leq 50 \\ x_1, x_2 &\geq 0 \end{aligned}$$

If the simplex method terminates and one or more variables *not in the final basis* have bottom-row entries of zero, bringing these variables into the basis will determine other optimal solutions. Demonstrate this in Exercises 35 and 36.

35. (Maximize)
Objective function:
$z = 2.5x_1 + x_2$
Constraints:
$$\begin{aligned} 3x_1 + 5x_2 &\leq 15 \\ 5x_1 + 2x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

36. (Maximize)
Objective function:
$z = x_1 + \frac{1}{2}x_2$
Constraints:
$$\begin{aligned} 2x_1 + x_2 &\leq 20 \\ x_1 + 3x_2 &\leq 35 \\ x_1, x_2 &\geq 0 \end{aligned}$$

C **37.** Use a computer to maximize the objective function
$z = 2x_1 + 7x_2 + 6x_3 + 4x_4$
subject to the constraints
$$\begin{aligned} x_1 + x_2 + 0.83x_3 + 0.5x_4 &\leq 65 \\ 1.2x_1 + x_2 + x_3 + 1.2x_4 &\leq 96 \\ 0.5x_1 + 0.7x_2 + 1.2x_3 + 0.4x_4 &\leq 80 \end{aligned}$$
where $x_1, x_2, x_3, x_4 \geq 0$.

C **38.** Use a computer to maximize the objective function
$z = 1.2x_1 + x_2 + x_3 + x_4$
subject to the same set of constraints given in Exercise 37.

9.4 THE SIMPLEX METHOD: MINIMIZATION

In Section 9.3, we applied the simplex method only to linear programming problems in standard form where the objective function was to be *maximized*. In this section, we extend this procedure to linear programming problems in which the objective function is to be *minimized*.

A minimization problem is in **standard form** if the objective function $w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is to be minimized, subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. The basic procedure used to solve such a problem is to convert it to a *maximization problem* in standard form, and then apply the simplex method as discussed in Section 9.3.

In Example 5 in Section 9.2, we used geometric methods to solve the following minimization problem.

Minimization Problem: Find the minimum value of

$$w = 0.12x_1 + 0.15x_2 \qquad \text{Objective function}$$

subject to the following constraints

$$\left.\begin{aligned} 60x_1 + 60x_2 &\geq 300 \\ 12x_1 + 6x_2 &\geq 36 \\ 10x_1 + 30x_2 &\geq 90 \end{aligned}\right\} \qquad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$. The first step in converting this problem to a maximization problem is to form the augmented matrix for this system of inequalities. To this augmented matrix we add a last row that represents the coefficients of the objective function, as follows.

$$\begin{bmatrix} 60 & 60 & \vdots & 300 \\ 12 & 6 & \vdots & 36 \\ 10 & 30 & \vdots & 90 \\ \cdots & \cdots & \cdots & \cdots \\ 0.12 & 0.15 & \vdots & 0 \end{bmatrix}$$

Next, we form the **transpose** of this matrix by interchanging its rows and columns.

$$\begin{bmatrix} 60 & 12 & 10 & \vdots & 0.12 \\ 60 & 6 & 30 & \vdots & 0.15 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 300 & 36 & 90 & \vdots & 0 \end{bmatrix}$$

Note that the rows of this matrix are the columns of the first matrix, and vice versa. Finally, we interpret the new matrix as a *maximization* problem as follows. (To do this, we introduce new variables, y_1, y_2, and y_3.) We call this corresponding maximization problem the **dual** of the original minimization problem.

Dual Maximization Problem: Find the maximum value of

$$z = 300y_1 + 36y_2 + 90y_3 \qquad \text{Dual objective function}$$

subject to the constraints

$$\left.\begin{aligned} 60y_1 + 12y_2 + 10y_3 &\leq 0.12 \\ 60y_1 + 6y_2 + 30y_3 &\leq 0.15 \end{aligned}\right\} \qquad \text{Dual constraints}$$

where $y_1 \geq 0$, $y_2 \geq 0$, and $y_3 \geq 0$.

As it turns out, the solution of the original minimization problem can be found by applying the simplex method to the new dual problem, as follows.

y_1	y_2	y_3	s_1	s_2	b	Basic Variables	
(60)	12	10	1	0	0.12	s_1	← Departing
60	6	30	0	1	0.15	s_2	
−300	−36	−90	0	0	0		
↑ Entering							

y_1	y_2	y_3	s_1	s_2	b	Basic Variables	
1	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{60}$	0	$\frac{1}{500}$	y_1	
0	−6	(20)	−1	1	$\frac{3}{100}$	s_2	← Departing
0	24	−40	5	0	$\frac{3}{5}$		
		↑ Entering					

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
1	$\frac{1}{4}$	0	$\frac{1}{40}$	$-\frac{1}{120}$	$\frac{7}{4000}$	y_1
0	$-\frac{3}{10}$	1	$-\frac{1}{20}$	$\frac{1}{20}$	$\frac{3}{2000}$	y_3
0	12	0	3	2	$\frac{33}{50}$	
			↑ x_1	↑ x_2		

Thus, the solution of the dual maximization problem is $z = \frac{33}{50} = 0.66$. This is the same value we obtained in the minimization problem given in Example 5, in Section 9.2. The x-values corresponding to this optimal solution are obtained from the entries in the bottom row corresponding to slack variable columns. In other words, the optimal solution occurs when $x_1 = 3$ and $x_2 = 2$.

The fact that a dual maximization problem has the same solution as its original minimization problem is stated formally in a result called the **von Neumann Duality Principle,** after the American mathematician John von Neumann (1903–1957).

Theorem 9.2

The von Neumann Duality Principle

The objective value w of a minimization problem in standard form has a minimum value if and only if the objective value z of the dual maximization problem has a maximum value. Moreover, the minimum value of w is equal to the maximum value of z.

Solving a Minimization Problem

We summarize the steps used to solve a minimization problem as follows.

Solving a Minimization Problem

A minimization problem is in standard form if the objective function $w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is to be minimized, subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. To solve this problem we use the following steps.

1. Form the **augmented matrix** for the given system of inequalities, and add a bottom row consisting of the coefficients of the objective function.

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ & & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_1 & c_2 & \ldots & c_n & 0 \end{array}\right]$$

2. Form the **transpose** of this matrix.

$$\left[\begin{array}{cccc:c} a_{11} & a_{21} & \ldots & a_{m1} & c_1 \\ a_{12} & a_{22} & \ldots & a_{m2} & c_2 \\ & & & & \\ a_{1n} & a_{2n} & \ldots & a_{mn} & c_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_1 & b_2 & \ldots & b_m & 0 \end{array}\right]$$

3. Form the **dual maximization problem** corresponding to this transposed matrix. That is, find the maximum of the objective function given by $z = b_1y_1 + b_2y_2 + \cdots + b_my_m$ subject to the constraints

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &\leq c_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m &\leq c_2 \\ &\vdots \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m &\leq c_n \end{aligned}$$

where $y_1 \geq 0, y_2 \geq 0, \ldots,$ and $y_m \geq 0$.

4. Apply the **simplex method** to the dual maximization problem. The maximum value of z will be the minimum value of w. Moreover, the values of $x_1, x_2, \ldots,$ and x_n will occur in the bottom row of the final simplex tableau, in the columns corresponding to the slack variables.

We illustrate the steps used to solve a minimization problem in Examples 1 and 2.

EXAMPLE 1 *Solving a Minimization Problem*

Find the minimum value of

$$w = 3x_1 + 2x_2 \qquad \text{Objective function}$$

subject to the constraints

$$\left.\begin{aligned} 2x_1 + x_2 &\geq 6 \\ x_1 + x_2 &\geq 4 \end{aligned}\right\} \qquad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

Solution The augmented matrix corresponding to this minimization problem is

$$\begin{bmatrix} 2 & 1 & \vdots & 6 \\ 1 & 1 & \vdots & 4 \\ \dots & \dots & \vdots & \dots \\ 3 & 2 & \vdots & 0 \end{bmatrix}.$$

Thus, the matrix corresponding to the dual maximization problem is given by the following transpose.

$$\begin{bmatrix} 2 & 1 & \vdots & 3 \\ 1 & 1 & \vdots & 2 \\ \dots & \dots & \vdots & \dots \\ 6 & 4 & \vdots & 0 \end{bmatrix}.$$

This implies that the dual maximization problem is as follows.

Dual Maximization Problem: Find the maximum value of

$$z = 6y_1 + 4y_2 \qquad \text{Dual objective function}$$

subject to the constraints

$$\left.\begin{aligned} 2y_1 + y_2 &\leq 3 \\ y_1 + y_2 &\leq 2 \end{aligned}\right\} \qquad \text{Dual constraints}$$

where $y_1 \geq 0$ and $y_2 \geq 0$. We now apply the simplex method to the dual problem as follows.

y_1	y_2	s_1	s_2	b	Basic Variables	
(2)	1	1	0	3	s_1	← Departing
1	1	0	1	2	s_2	
−6	−4	0	0	0		
↑ Entering						

y_1	y_2	s_1	s_2	b	Basic Variables	
1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$	y_1	
0	$\frac{1}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	s_2	← Departing
0	−1	3	0	9		
	↑ Entering					

y_1	y_2	s_1	s_2	b	Basic Variables
1	0	1	−1	1	y_1
0	1	−1	2	1	y_2
0	0	2	2	10	
		↑ x_1	↑ x_2		

From this final simplex tableau, we see that the maximum value of z is 10. Therefore, the solution of the original minimization problem is

$$w = 10$$ **Minimum Value**

and this occurs when

$$x_1 = 2 \text{ and } x_2 = 2.$$

Both the minimization and the maximization linear programming problems in Example 1 could have been solved with a graphical method, as indicated in Figure 9.19. Note in Figure 9.19 (a) that the maximum value of $z = 6y_1 - 4y_2$ is the same as the minimum value of $w = 3x_1 + 2x_2$, as shown in Figure 9.19 (b). (See page 515.)

EXAMPLE 2 *Solving a Minimization Problem*

Find the minimum value of

$$w = 2x_1 + 10x_2 + 8x_3$$ **Objective function**

subject to the constraints

$$\left.\begin{aligned} x_1 + x_2 + x_3 &\geq 6 \\ x_2 + 2x_3 &\geq 8 \\ -x_1 + 2x_2 + 2x_3 &\geq 4 \end{aligned}\right\}$$ **Constraints**

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

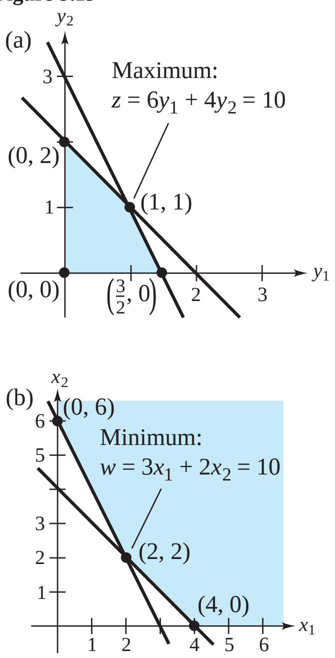
Solution The augmented matrix corresponding to this minimization problem is

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ -1 & 2 & 2 & \vdots & 4 \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 2 & 10 & 8 & \vdots & 0 \end{bmatrix}.$$

Thus, the matrix corresponding to the dual maximization problem is given by the following transpose.

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 2 \\ 1 & 1 & 2 & \vdots & 10 \\ 1 & 2 & 2 & \vdots & 8 \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 6 & 8 & 4 & \vdots & 0 \end{bmatrix}$$

Figure 9.19



This implies that the dual maximization problem is as follows.

Dual Maximization Problem: Find the maximum value of

$$z = 6y_1 + 8y_2 + 4y_3 \qquad \textbf{Dual objective function}$$

subject to the constraints

$$\left.\begin{aligned} y_1 \qquad - \ y_3 &\le 2 \\ y_1 + \ y_2 + 2y_3 &\le 10 \\ y_1 + 2y_2 + 2y_3 &\le 8 \end{aligned}\right\} \qquad \textbf{Dual constraints}$$

where $y_1 \ge 0$, $y_2 \ge 0$, and $y_3 \ge 0$. We now apply the simplex method to the dual problem as follows.

y_1	y_2	y_3	s_1	s_2	s_3	b	*Basic Variables*	
1	0	−1	1	0	0	2	s_1	
1	1	2	0	1	0	10	s_2	
1	2	2	0	0	1	8	s_3	← *Departing*
−6	−8	−4	0	0	0	0		
	↑ *Entering*							

y_1	y_2	y_3	s_1	s_2	s_3	b	*Basic Variables*	
1	0	−1	1	0	0	2	s_1	← *Departing*
$\frac{1}{2}$	0	1	0	1	$-\frac{1}{2}$	6	s_2	
$\frac{1}{2}$	1	1	0	0	$\frac{1}{2}$	4	y_2	
−2	0	4	0	0	4	32		
↑ *Entering*								

y_1	y_2	y_3	s_1	s_2	s_3	b	*Basic Variables*
1	0	-1	1	0	0	2	y_1
0	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	5	s_2
0	1	$\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	3	y_2
0	0	2	2	0	4	36	
			↑ x_1	↑ x_2	↑ x_3		

From this final simplex tableau, we see that the maximum value of z is 36. Therefore, the solution of the original minimization problem is

$$w = 36 \qquad \textbf{Minimum Value}$$

and this occurs when

$$x_1 = 2,\ x_2 = 0, \text{ and } x_3 = 4.$$

Applications

EXAMPLE 3 *A Business Application: Minimum Cost*

A small petroleum company owns two refineries. Refinery 1 costs \$20,000 per day to operate, and it can produce 400 barrels of high-grade oil, 300 barrels of medium-grade oil, and 200 barrels of low-grade oil each day. Refinery 2 is newer and more modern. It costs \$25,000 per day to operate, and it can produce 300 barrels of high-grade oil, 400 barrels of medium-grade oil, and 500 barrels of low-grade oil each day.

The company has orders totaling 25,000 barrels of high-grade oil, 27,000 barrels of medium-grade oil, and 30,000 barrels of low-grade oil. How many days should it run each refinery to minimize its costs and still refine enough oil to meet its orders?

Solution To begin, we let x_1 and x_2 represent the number of days the two refineries are operated. Then the total cost is given by

$$C = 20{,}000x_1 + 25{,}000x_2. \qquad \textbf{Objective function}$$

The constraints are given by

$$\left.\begin{aligned} &\text{(High-grade)} && 400x_1 + 300x_2 \ge 25{,}000 \\ &\text{(Medium-grade)} && 300x_1 + 400x_2 \ge 27{,}000 \\ &\text{(Low-grade)} && 200x_1 + 500x_2 \ge 30{,}000 \end{aligned}\right\} \quad \textbf{Constraints}$$

where $x_1 \ge 0$ and $x_2 \ge 0$. The augmented matrix corresponding to this minimization problem is

$$\begin{bmatrix} 400 & 300 & \vdots & 25{,}000 \\ 300 & 400 & \vdots & 27{,}000 \\ 200 & 500 & \vdots & 30{,}000 \\ \cdots & \cdots & \vdots & \cdots \\ 20{,}000 & 25{,}000 & \vdots & 0 \end{bmatrix}.$$

The matrix corresponding to the dual maximization problem is

$$\begin{bmatrix} 400 & 300 & 200 & \vdots & 20{,}000 \\ 300 & 400 & 500 & \vdots & 25{,}000 \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 25{,}000 & 27{,}000 & 30{,}000 & \vdots & 0 \end{bmatrix}.$$

We now apply the simplex method to the dual problem as follows.

y_1	y_2	y_3	s_1	s_2	b	Basic Variables	
400	300	200	1	0	20,000	s_1	
300	400	(500)	0	1	25,000	s_2	← Departing
−25,000	−27,000	−30,000	0	0	0		
		↑ Entering					

y_1	y_2	y_3	s_1	s_2	b	Basic Variables	
(280)	140	0	1	$-\frac{2}{5}$	10,000	s_1	← Departing
$\frac{3}{5}$	$\frac{4}{5}$	1	0	$\frac{1}{500}$	50	y_3	
−7,000	−3,000	0	0	60	1,500,000		
↑ Entering							

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{280}$	$-\frac{1}{700}$	$\frac{250}{7}$	y_1
0	$\frac{1}{2}$	1	$-\frac{3}{1400}$	$\frac{1}{350}$	$\frac{200}{7}$	y_3
0	500	0	25	50	1,750,000	
			↑ x_1	↑ x_2		

From the third simplex tableau, we see that the solution to the original minimization problem is

$$C = \$1{,}750{,}000 \qquad \textbf{Minimum cost}$$

and this occurs when $x_1 = 25$ and $x_2 = 50$. Thus, the two refineries should be operated for the following number of days.

Refinery 1: 25 days

Refinery 2: 50 days

Note that by operating the two refineries for this number of days, the company will have produced the following amounts of oil.

$$\begin{aligned} \text{High-grade oil:} &\quad 25(400) + 50(300) = 25{,}000 \text{ barrels} \\ \text{Medium-grade oil:} &\quad 25(300) + 50(400) = 27{,}500 \text{ barrels} \\ \text{Low-grade oil:} &\quad 25(200) + 50(500) = 30{,}000 \text{ barrels} \end{aligned}$$

Thus, the original production level has been met (with a surplus of 500 barrels of medium-grade oil).

SECTION 9.4 ❑ EXERCISES

In Exercises 1–6, determine the dual of the given minimization problem.

1. Objective function:

$w = 3x_1 + 3x_2$

Constraints:

$$\begin{aligned} 2x_1 + x_2 &\geq 4 \\ x_1 + 2x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

2. Objective function:

$w = 2x_1 + x_2$

Constraints:

$$\begin{aligned} 5x_1 + x_2 &\geq 9 \\ 2x_1 + 2x_2 &\geq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3. Objective function:

$w = 4x_1 + x_2 + x_3$

Constraints:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &\geq 23 \\ x_1 + x_3 &\geq 10 \\ 8x_1 + x_2 + 2x_3 &\geq 40 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

4. Objective function:

$w = 9x_1 + 6x_2$

Constraints:

$$\begin{aligned} x_1 + 2x_2 &\geq 5 \\ 2x_1 + 2x_2 &\geq 8 \\ 2x_1 + x_2 &\geq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

5. Objective function:

$w = 14x_1 + 20x_2 + 24x_3$

Constraints:

$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 7 \\ x_1 + 2x_2 + x_3 &\geq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

6. Objective function:

$w = 9x_1 + 4x_2 + 10x_3$

Constraints:

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &\geq 6 \\ 6x_1 + x_2 + x_3 &\geq 9 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

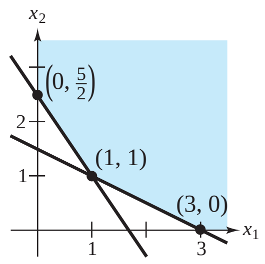
In Exercises 7–12, (a) solve the given minimization problem by the graphical method, (b) formulate the dual problem, and (c) solve the dual problem by the graphical method.

7. Objective function:

$w = 2x_1 + 2x_2$

Constraints:

$$\begin{aligned} x_1 + 2x_2 &\geq 3 \\ 3x_1 + 2x_2 &\geq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

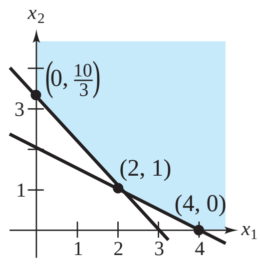


8. Objective function:

$w = 14x_1 + 20x_2$

Constraints:

$$\begin{aligned} x_1 + 2x_2 &\geq 4 \\ 7x_1 + 6x_2 &\geq 20 \\ x_1, x_2 &\geq 0 \end{aligned}$$

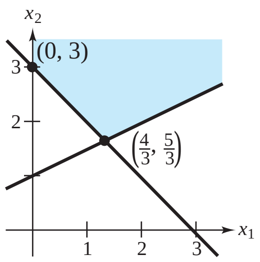


9. Objective function:

$w = x_1 + 4x_2$

Constraints:

$$\begin{aligned} x_1 + x_2 &\geq 3 \\ -x_1 + 2x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

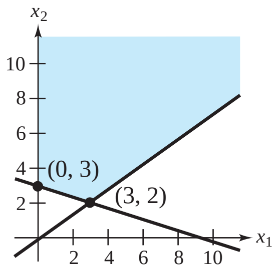


10. Objective function:

$w = 2x_1 + 6x_2$

Constraints:

$$\begin{aligned} -2x_1 + 3x_2 &\geq 0 \\ x_1 + 3x_2 &\geq 9 \\ x_1, x_2 &\geq 0 \end{aligned}$$

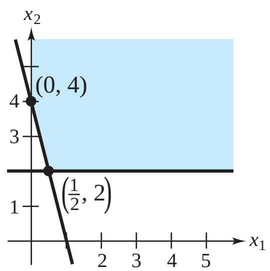


11. Objective function:

$w = 6x_1 + 3x_2$

Constraints:

$$\begin{aligned} 4x_1 + x_2 &\geq 4 \\ x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

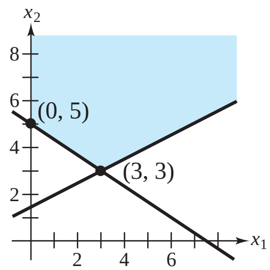


12. Objective function:

$w = x_1 + 6x_2$

Constraints:

$$\begin{aligned} 2x_1 + 3x_2 &\geq 15 \\ -x_1 + 2x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$



In Exercises 13–20, solve the given minimization problem by solving the dual maximization problem with the simplex method.

13. Objective function:

$w = x_2$

Constraints:

$$\begin{aligned} x_1 + 5x_2 &\geq 10 \\ -6x_1 + 5x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

14. Objective function:

$w = 3x_1 + 8x_2$

Constraints:

$$\begin{aligned} 2x_1 + 7x_2 &\geq 9 \\ x_1 + 2x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

15. Objective function:

$w = 2x_1 + x_2$

Constraints:

$$\begin{aligned} 5x_1 + x_2 &\geq 9 \\ 2x_1 + 2x_2 &\geq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

16. Objective function:

$w = 2x_1 + 2x_2$

Constraints:

$$\begin{aligned} 3x_1 + x_2 &\geq 6 \\ -4x_1 + 2x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

17. Objective function:

$w = 8x_1 + 4x_2 + 6x_3$

Constraints:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &\geq 6 \\ 4x_1 + x_2 + 3x_3 &\geq 7 \\ 2x_1 + x_2 + 4x_3 &\geq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

18. Objective function:

$w = 8x_1 + 16x_2 + 18x_3$

Constraints:

$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 &\geq 4 \\ -4x_1 + 3x_2 - x_3 &\geq 1 \\ x_1 - x_2 + 3x_3 &\geq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

19. Objective function:

$w = 6x_1 + 2x_2 + 3x_3$

Constraints:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &\geq 28 \\ 6x_1 + x_3 &\geq 24 \\ 3x_1 + x_2 + 2x_3 &\geq 40 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

20. Objective function:

$w = 42x_1 + 5x_2 + 17x_3$

Constraints:

$$\begin{aligned} 3x_1 - x_2 + 7x_3 &\geq 5 \\ -3x_1 - x_2 + 3x_3 &\geq 8 \\ 6x_1 + x_2 + x_3 &\geq 16 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

In Exercises 21–24, two dietary drinks are used to supply protein and carbohydrates. The first drink provides 1 unit of protein and 3 units of carbohydrates in each liter. The second drink supplies 2 units of protein and 2 units of carbohydrates in each liter. An athlete requires 3 units of protein and 5 units of carbohydrates. Find the amount of each drink the athlete should consume to minimize the cost and still meet the minimum dietary requirements.

21. The first drink costs \$2 per liter and the second costs \$3 per liter.

22. The first drink costs \$4 per liter and the second costs \$2 per liter.

23. The first drink costs \$1 per liter and the second costs \$3 per liter.

24. The first drink costs \$1 per liter and the second costs \$2 per liter.

In Exercises 25–28, an athlete uses two dietary drinks that provide the nutritional elements listed in the following table.

Drink	*Protein*	*Carbohydrates*	*Vitamin D*
I	4	2	1
II	1	5	1

Find the combination of drinks of minimum cost that will meet the minimum requirements of 4 units of protein, 10 units of carbohydrates, and 3 units of vitamin D.

25. Drink I costs \$5 per liter and drink II costs \$8 per liter.

26. Drink I costs \$7 per liter and drink II costs \$4 per liter.

27. Drink I costs \$1 per liter and drink II costs \$5 per liter.

28. Drink I costs \$8 per liter and drink II costs \$1 per liter.

29. A company has three production plants, each of which produces three different models of a particular product. The daily capacities (in thousands of units) of the three plants are as follows.

	Model 1	*Model 2*	*Model 3*
Plant 1	8	4	8
Plant 2	6	6	3
Plant 3	12	4	8

The total demand for Model 1 is 300,000 units, for Model 2 is 172,000 units, and for Model 3 is 249,500 units. Moreover, the daily operating cost for Plant 1 is \$55,000, for Plant 2 is \$60,000, and for Plant 3 is \$60,000. How many days should each plant be operated in order to fill the total demand, and keep the operating cost at a minimum?

30. The company in Exercise 29 has lowered the daily operating cost for Plant 3 to \$50,000. How many days should each plant be operated in order to fill the total demand, and keep the operating cost at a minimum?

31. A small petroleum company owns two refineries. Refinery 1 costs \$25,000 per day to operate, and it can produce 300 barrels of high-grade oil, 200 barrels of medium-grade oil, and 150 barrels of low-grade oil each day. Refinery 2 is newer and more modern. It costs \$30,000 per day to operate, and it can produce 300 barrels of high-grade oil, 250 barrels of medium-grade oil, and 400 barrels of low-grade oil each day. The company has orders totaling 35,000 barrels of high-grade oil, 30,000 barrels of medium-grade oil, and 40,000 barrels of low-grade oil. How many days should the company run each refinery to minimize its costs and still meet its orders?

32. A steel company has two mills. Mill 1 costs \$70,000 per day to operate, and it can produce 400 tons of high-grade steel, 500 tons of medium-grade steel, and 450 tons of low-grade steel each day. Mill 2 costs \$60,000 per day to operate, and it can produce 350 tons of high-grade steel, 600 tons of medium-grade steel, and 400 tons of low-grade steel each day. The company has orders totaling 100,000 tons of high-grade steel, 150,000 tons of medium-grade steel, and 124,500 tons of low-grade steel. How many days should the company run each mill to minimize its costs and still fill the orders?

33. Use a computer to minimize the objective function

$$w = x_1 + 0.5x_2 + 2.5x_3 + 3x_4$$

subject to the constraints

$$\begin{aligned} 1.5x_1 + x_2 + 2x_4 &\geq 35 \\ 2x_2 + 6x_3 + 4x_4 &\geq 120 \\ x_1 + x_2 + x_3 + x_4 &\geq 50 \\ 0.5x_1 + 2.5x_3 + 1.5x_4 &\geq 75 \end{aligned}$$

where $x_1, x_2, x_3, x_4 \geq 0$.

34. Use a computer to minimize the objective function

$$w = 1.5x_1 + x_2 + 0.5x_3 + 2x_4$$

subject to the same set of constraints given in Exercise 33.