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System Response I

- Impulse input
- Step input
- Ramp input

By

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System Models

Systems can be modelled mathematically, by being represented with equations. The standard equations are as follows:

- First order equations

$$\theta_i = \tau \frac{d\theta_0}{dt} + \theta_0 \qquad \textit{(i)}$$

- Second order equations

$$\theta_i = \tau^2 \frac{d^2\theta_0}{dt^2} + 2\delta\tau \frac{d\theta_0}{dt} + \theta_0 \qquad \textit{(ii)}$$

where:

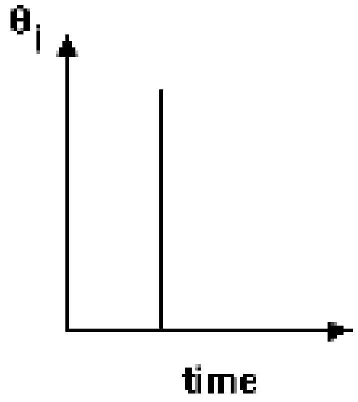
τ is the time constant, δ is the damping ratio

θ_0 is the output function, θ_i is the input function

In order to solve for how the output change with time, the input change with time must be decided

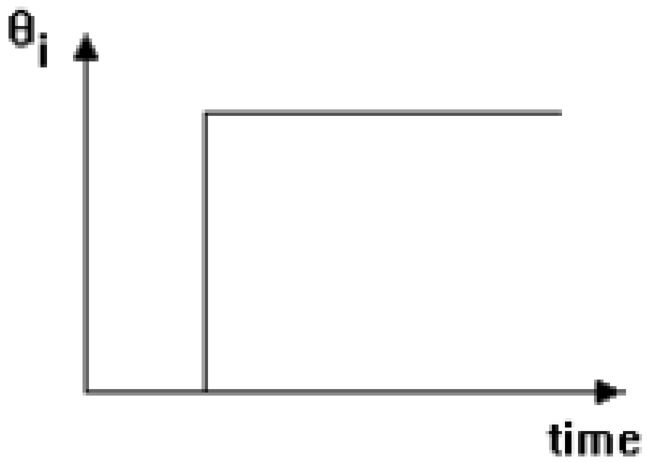
Standard Inputs

i. AN IMPULSE



This is an instantaneous change in θ_i lasting for zero length of time and returning to the initial value. This is mostly applied to digital systems where instantaneous values are sampled by digital to analogue converters. It is also widely used as a standard input to a system to compare the responses of different systems.

ii. A STEP CHANGE

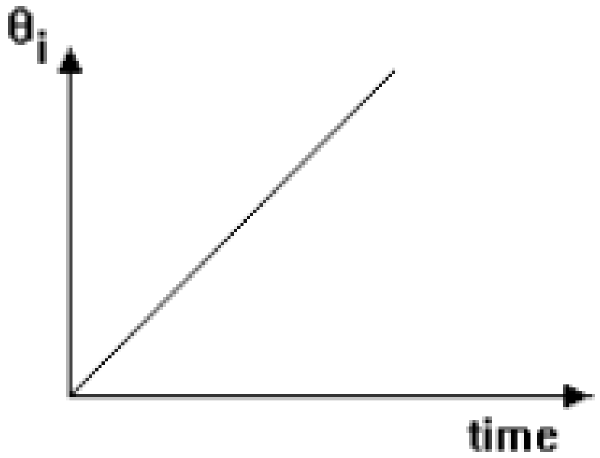


This is an instantaneous change in the input which then remains at the new value.

$\theta_i = H$ at all values of time after $t = 0$.

H is the change or height of the step.

iii. A RAMP OR VELOCITY CHANGE

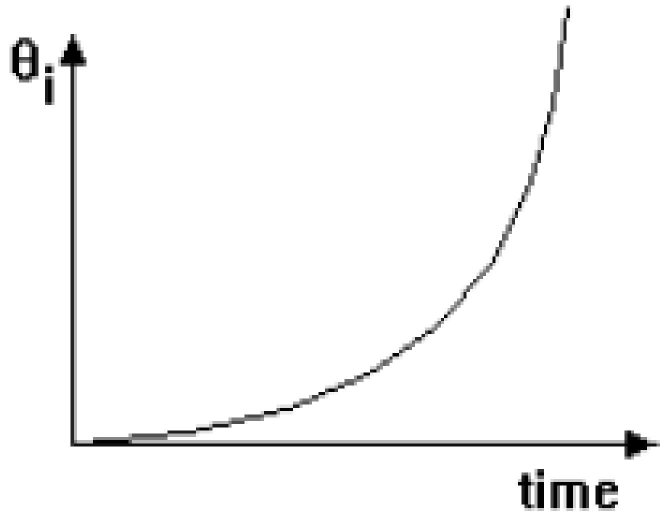


This is when the input changes at a constant rate. It is also called a velocity input.

$$d\theta_i/dt = c \quad \text{or} \quad \theta_i = ct$$

c is the rate of change (velocity).

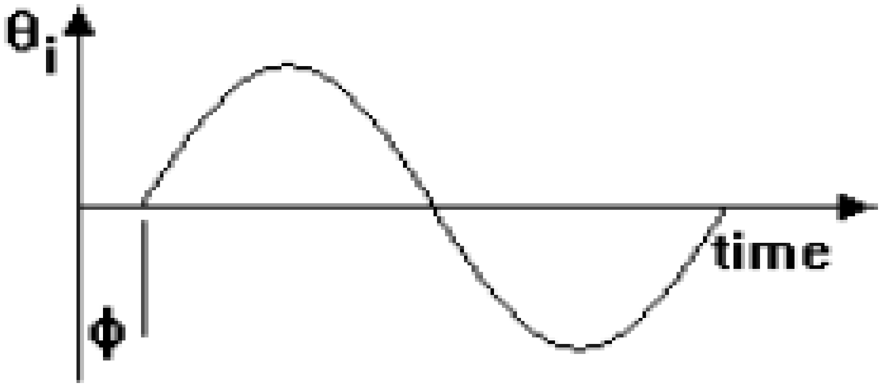
iv. A PARABOLIC or ACCELERATION CHANGE



This is when $d^2\theta_i/dt^2 = a$ or $\theta_i = at^2/2$

This is also known an acceleration since the rate of rate of change is a constant a.

v. A SINUSOIDAL CHANGE

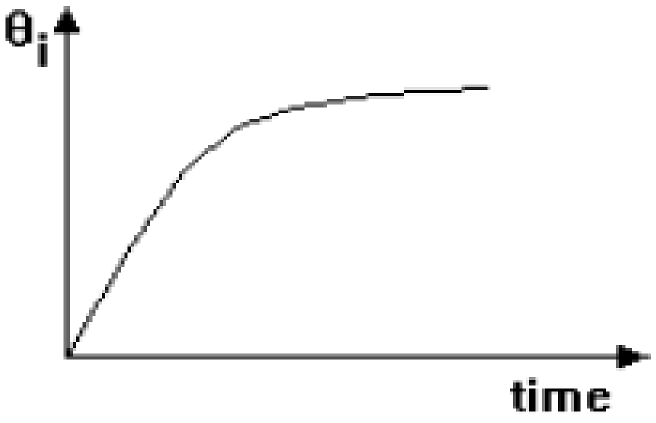


In this case the input changes with time sinusoidally.

$$\theta_i = A \sin(\omega t + \phi)$$

A is the amplitude and ϕ is the phase angle.

vi. AN EXPONENTIAL CHANGE



This is when the input changes exponentially with time.

$$\theta_i = A(1 - e^{-at})$$

There are several exponential forms e.g. representing growth and decay.

Laplace transform is therefore used to solve for the ratio output-time response with respect to the input-time change

Laplace Transforms (Revised)

	Time domain f(t)	Frequency domain f(s)	Description
1	e^{-at} f(t) dt	f(s + a)	
2	δt	1	Unit Impulse
3	H	$\frac{H}{s}$	Step H
4	ct	$\frac{c}{s^2}$	Ramp
5	H(t - T)	$H\frac{e^{-sT}}{s}$	Delayed Step
6		$\frac{1 - e^{-sT}}{s}$	Rectangular pulse
7	k e^{-at}	$\frac{k}{s + a}$	Exponential
8	kt e^{-at}	$\frac{k}{(s + a)^2}$	
9	K(e^{-at} - e^{-bt})	$\frac{k(b - a)}{(s + a)(s + b)}$	
	k sin (ωt)	$\frac{k\omega}{s^2 + \omega^2}$	Sinusoidal

	Time domain f(t)	Frequency domain f(s)	Description
10	$k\,\omega\, t \sin(\omega t)$	$\dfrac{2k\omega^2 s}{(s^2 + \omega^2)^2}$	
11	$k \cos(\omega t)$	$\dfrac{ks}{s^2 + \omega^2}$	Co sinusoidal
12	$k\, e^{-at} \sin(\omega t)$	$\dfrac{k\omega}{(s + a)^2 + \omega^2}$	Damped sinusoidal
13	$k\, e^{-at} \cos(\omega t)$	$\dfrac{s + a}{(s + a)^2 + \omega^2}$	Damped co sinusoidal
14	$k\left\{1 - e^{-\frac{t}{T}}\right\}$	$\dfrac{ka}{s(s + a)}$ $a = 1/T$	Exponential growth
15	$k\left\{t - \left(1 - e^{-\frac{t}{T}}\right)\right\}$	$\dfrac{ka}{s^2(s + a)}$	
16	$k(1-\cos\omega t)$	$\dfrac{k\omega^2}{s(s^2 + \omega^2)}$	
17	$k \sin(\omega t + \phi)$	$\dfrac{k\{\omega \cos\varphi + s \sin\varphi\}}{s^2 + \omega^2}$	
18	$1 - e^{-at}(\cos bt + \frac{a}{b}\sin bt)$	$\dfrac{a^2 + b^2}{s\left[(s+a)^2 + b^2\right]}$	Standard second order system step response

Therefore, the Laplace transforms of equation (i) and (ii) respectively give equation (iii) and (iv):

$$\theta_i = \tau s\theta_0 + \theta_0 \qquad (iii)$$

$$\theta_i = \theta_0(\tau s + 1)$$

$$\frac{\theta_0}{\theta_i} = \frac{1}{\tau s+1}$$

$$\theta_i = \tau^2 s^2\theta_0 + 2\delta\tau\theta_0 + \theta_0 \qquad (iv)$$

$$\theta_i = \theta_0(\tau^2 s^2 + 2\delta\tau + 1)$$

$$\frac{\theta_0}{\theta_i} = \frac{1}{\tau^2 s^2+2\delta\tau+1}$$

Gains of First Order Systems

DC GAIN :- this is an electrical term and is the gain when there is no variation of the input in the time domain. When an input is constant all the derivative values of the transfer function are zero so the D.C. is the magnitude of G(s) when s = 0.

Hence the gain of the standard 1st order closed loop transfer function $\frac{\theta_o}{\theta_i}(s) = \frac{1}{Ts + 1}$ is unity.

When the transfer function is of the form $\frac{\theta_o}{\theta_i}(s) = \frac{K}{Ts + 1}$, the D.C. gain is K.

The D.C. gain is not always obvious at first glance e derivation. Consider the case $\frac{\theta_o}{\theta_i}(s) = \frac{A}{Ts + B}$

Put s = 0 and the D.C. gain is A/B. We could rearrange the transfer function to $\frac{\theta_o}{\theta_i}(s) = \frac{A/B}{(T/B)s + 1}$

This is now in the form $\frac{\theta_o}{\theta_i}(s) = \frac{K}{T_2 s + 1}$

T_2 is a new time constant resulting from the change. In the steady state s = 0 so K is the steady state gain of the system.

D.C. GAIN = A/B $\quad T_2 = T/B \quad$ where T is the time constant

System Response to Inputs

Example 1

Compare the time response of a system with the transfer function $\frac{\theta_o}{\theta_i} = \frac{1}{Ts+1}$ to a unit impulse and a unit step input.

where T is the time constant

Solution

UNIT IMPULSE

$\theta_o(s) = \dfrac{1\theta_i}{Ts+1}$ substitute $\theta_i(s) = 1$ for an impulse input.

$\theta_o(t) =$ inverse Laplace transform of $\dfrac{1(1)}{Ts+1}$

rearrange into a recognisable transform and $\dfrac{1/T}{s+1/T}$ $\quad \theta_o(t) = (1/T)e^{-t/T}$

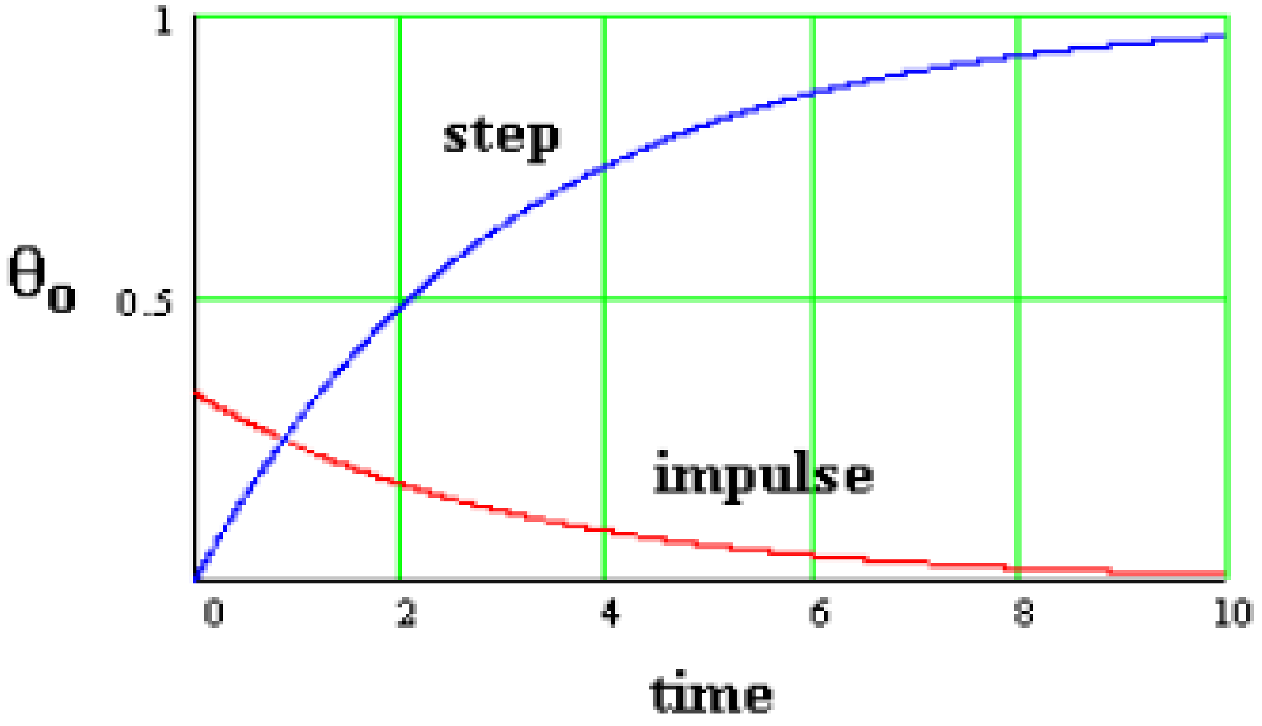
UNIT STEP

$\theta_o(s) = \dfrac{1\theta_i}{Ts+1}$ substitute $\theta_i(s) = 1/s$ for an step input.

$\theta_o(t) = \text{inverse Laplace transform of } \dfrac{1}{s(Ts+1)}$

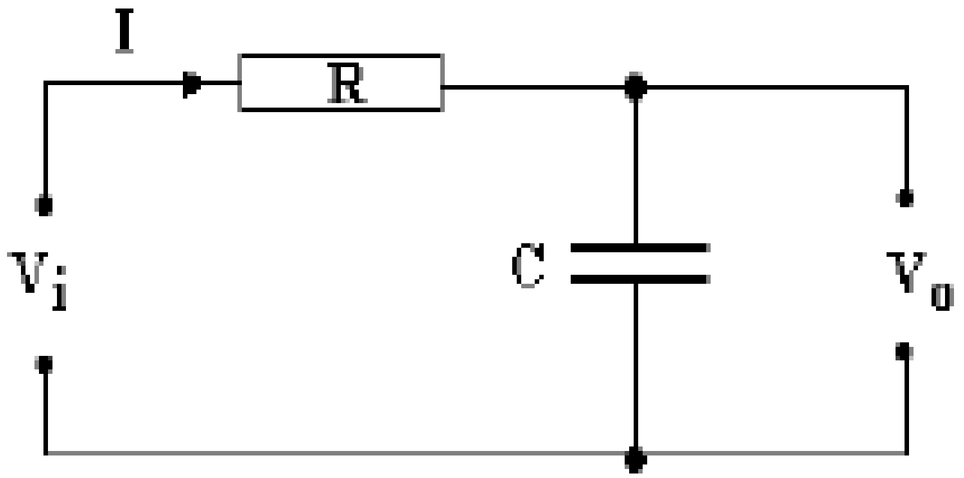
rearrange into a recognisable transform and $\dfrac{1/T}{s(s+1/T)}$ $\qquad \theta_o(t) = 1 - e^{-t/T}$

The diagram shows the two responses.



Example 2

A R- C circuit is shown in which R = 200 Ω and C = 15 μF. The voltage V_i is suddenly changed from 0 to 10 Volts. Determine the time constant and how long it takes V_o to reach 9.99V.



where T is the time constant

Solution

The model for the R – C circuit shown was derived in tutorial 1 and shown to be $V_i = TdV_o/dt + V_o$

First replace dV_o/dt by $s\,V_o$ $\quad V_i = T\,s\,V_o + V_o$

Next rearrange into a transfer function $\quad G(s) = V_o/V_i = 1/(Ts + 1)$

This may be represented diagrammatically as shown.

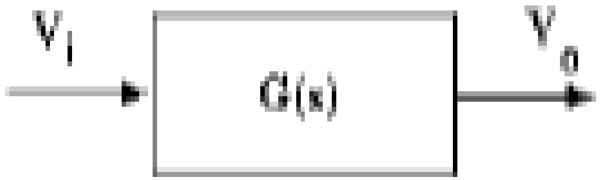


Figure 14

Next change the input from a function of time into a function of s by making a Laplace transformation. For a step input $V_i(t) = H$ the transform is $V_i(s) = H/s$. In this case H = 10 Volts.

$$V_o = \frac{V_i}{Ts + 1} = \frac{H}{s(Ts + 1)}$$

Next manipulate the equation into a recognisable Laplace transform as follows.

$$V_o = \frac{H/T}{s(s + 1/T)}$$

Looking in the table of transforms we see $Ka/s(s+a)$ is the transform of $K(1 - e^{-at})$. This means that if we put $a = 1/T$ and $K = H$ the solution for V_o is $\quad V_o = \mathbf{H(1 - e^{-t/T})}$

The time constant $T = RC = 200 \times 15 \times 10^{-6} = 0.003$ seconds

Put $V_o = 9.99$ V and $H = 10$ V

$$9.99 = 10\left(1 - e^{-\frac{t}{0.003}}\right) \qquad 0.999 = \left(1 - e^{-\frac{t}{0.003}}\right)$$

$$e^{-\frac{t}{0.003}} = 0.001 \qquad \ln(0.001) = \frac{-t}{0.003} = -6.9077 \qquad t = 0.0207\,\text{s}$$

Example 3

A position control system has a transfer function $G(s) = 1/(0.2\ s + 1)$. The input is changed at a constant rate of 5 degrees/s from the zero position. Calculate the error after 0.4 seconds and the steady state error.

where T is the time constant

Solution

A ramp or velocity input occurs when $\theta_i(t) = ct$ and $\theta_i(s) = c/s^2$

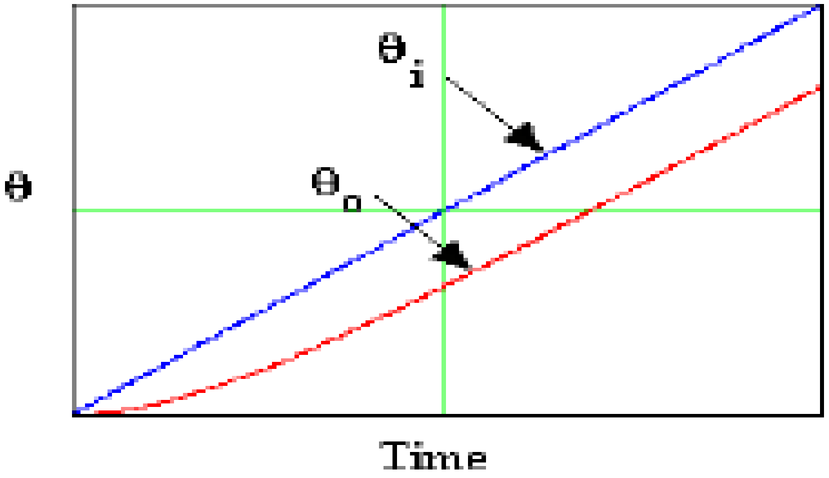


Figure 16

$$\frac{\theta_o}{\theta_i}(s) = \frac{1}{(Ts+1)}$$

$$\theta_o(s) = \frac{\theta_i(s)}{(Ts+1)} = \frac{c}{s^2(Ts+1)} = \frac{c(1/T)}{s^2(s+1/T)}$$

If we replace 'c' with 'k' and '1/T' with 'a' we have

$$\theta_o(s) = \frac{ca}{s^2(s+a)}$$ and we can find this in the table of transforms. The inverse transform gives the solution

$$\theta_o(t) = c\{t - T(1 - e^{-t/T})\}$$

Plotting the output θ_o against time produces the result shown. The equation may be written as:

$$\theta_o(t) = c\,t - cT(1 - e^{-t/T})\}$$

At large values of time t the term ($e^{-t/T}$) becomes negligibly small and the output becomes:

$$\theta_o(t) = c\{t - T(1)\} = c(t - T)$$

The error becomes $\theta_e(t) = \theta_i - \theta_o = ct - ct + cT$ $\qquad \theta_e(t) = cT$

In other words, a constant error cT result after an initial transient stage and this is the steady state error.

Comparing parameters it is apparent that $T = 0.2$ and $c = 5$ deg/s.

After 0.4 seconds $\theta_i = 5 \times 0.4 = 2$ degree.

$\theta_o = c\,t - cT(1 - e^{-t/T})\} = 5 \times 0.4 - 5 \times 0.2 \times (1 - e^{-0.4/0.2}) = 2 - 0.835 = 1.135$ degrees.
$\theta_e = 2 - 1.135 = 0.865$ degrees

The steady state error is cT where $T = 0.2$ and $c = 5$ degrees/s. $\theta_e = 5 \times 0.2 = 1$ degree.

Note:

The D.C. gain also affects the magnitude of a dynamic response

The response to a step change H will hence settle down at KH. The response to a ramp input will not settle down at a fixed value but $\theta_o(t) = Kc\{t - T(1 - e^{-t/T})\}$. The affect of introducing gain is shown on the response diagram.

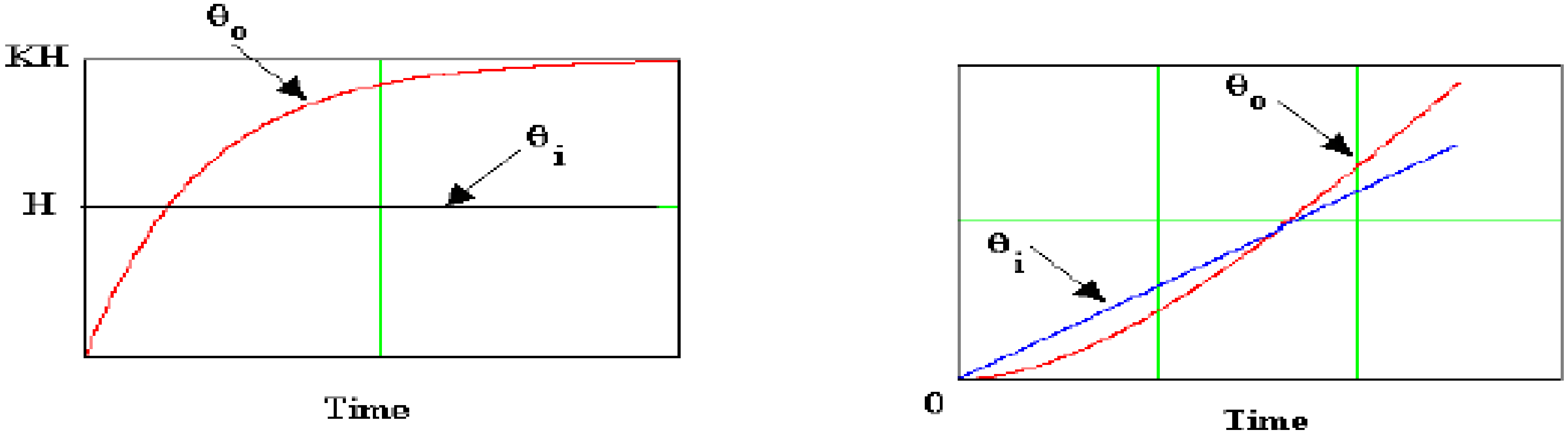


Figure 19

In the case of the step input $\theta_i = H$, the output θ_o reaches a value of K H.

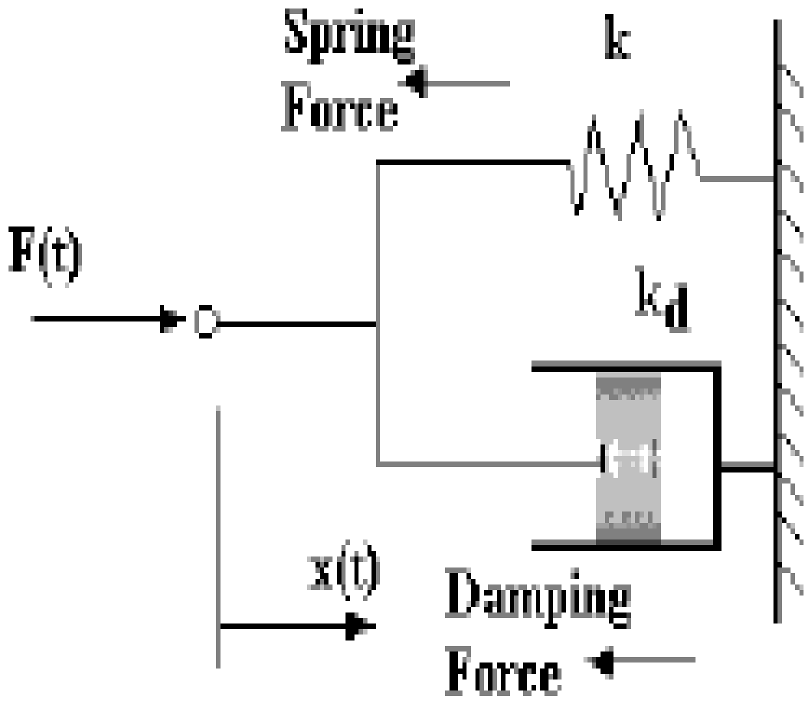
In the case of the ramp input $\theta_i = ct$, the output passes and exceeds the input more and more as time progresses.

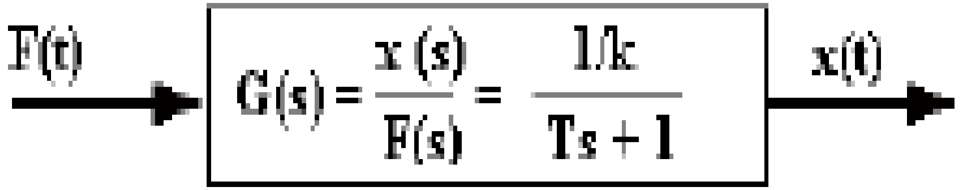
Future Note:

- 1^{st} order system response to sinusoidal input
- 2^{nd} order system response to ALL inputs

Homework

1. Show the derivation of the transfer function for spring and damper system shown. Given that the damping coefficient k_d is 0.03 and the spring stiffness k is 4 kN/m, determine the time constant for the system. (Answer 7.5 μs)
   If a force of F = 100 N is suddenly applied, calculate the value of x after T seconds. (Answer 16 mm)





2. A block of metal has a mass of 0.5 kg, specific heat capacity 346 J/kg K and temperature of $\theta_1 = 20^{o}C$. It is dropped into a large tank of oil at $\theta_2 = 120^{o}C$ and it is found that the temperature of the block takes 6 minutes to reach 119 °C.

Assume that the temperature of the block is changes by the law $\frac{\theta_1}{\theta_2}(s) = \frac{1}{(Ts + 1)}$

Show that the temperature of the block changes with time by the law $\theta = \theta_1 + (\theta_2 - \theta_1)(1-e^{-t/T})$

Determine the time constant T and hence the thermal resistance between the block and the oil.

(Answer R=0.452 K/W)

3. A hydraulic motor has a nominal displacement of k_1 m^3/radian. The speed ω is controlled by a simple valve such that the pressure to the motor is k_2x where x is the input position of the valve.

The motor has a moment of inertia J kg m^2 and a damping coefficient of k_3 Nm s/radian.

Given that the torque developed by the motor is k_1p, show that the open loop transfer function relating output ω to input x is given by $\dfrac{\omega}{k_m x} = \dfrac{1}{Ts + 1}$

$$T = \frac{J}{k_3} \quad \text{and} \quad k_m = \frac{k_1 k_2}{k_3}$$

The input is given a step change. Sketch the response of the output. Determine the % change in the output at t = T and t = 4T. (63.2% and 99.9%)

Show on the sketch the affect of increasing the moment of inertia.

4. A position control system has a transfer function G(s) = 1/(3 s + 1). The input is changed at a constant rate of 4 mm/s from the zero position. Calculate the error after 2 seconds and the steady state error. (2.161 mm and 12 mm)

5 Find the D.C. gain and time constant for the following transfer functions.

i. $G(s) = \dfrac{2}{0.2s + 0.5}$ (4 and 0.4)

ii. $G(s) = \dfrac{0.2}{0.05s + 0.1}$ (2 and 0.5)

iii. $G(s) = \dfrac{2}{3s+1}$ (2 and 3)

iv. $G(s) = \dfrac{16}{8s+4}$ (4 and 2)

6 The output speed of a motor (ω rad/s) is related to the angle of the input sensor (θ radian) by the transfer function $\dfrac{\omega}{\theta}(s) = \dfrac{k_m}{T_m s + 2}$

Where $k_m = 15\ s^{-1}$ and $T_m = 4$ s

Determine the D.C. gain and time constant of the system. (7.5 s^{-1} and 2 seconds)