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System Response III

- Nyquist Plot
- Bode Plot
- Stability – Routh Hurwitz

By

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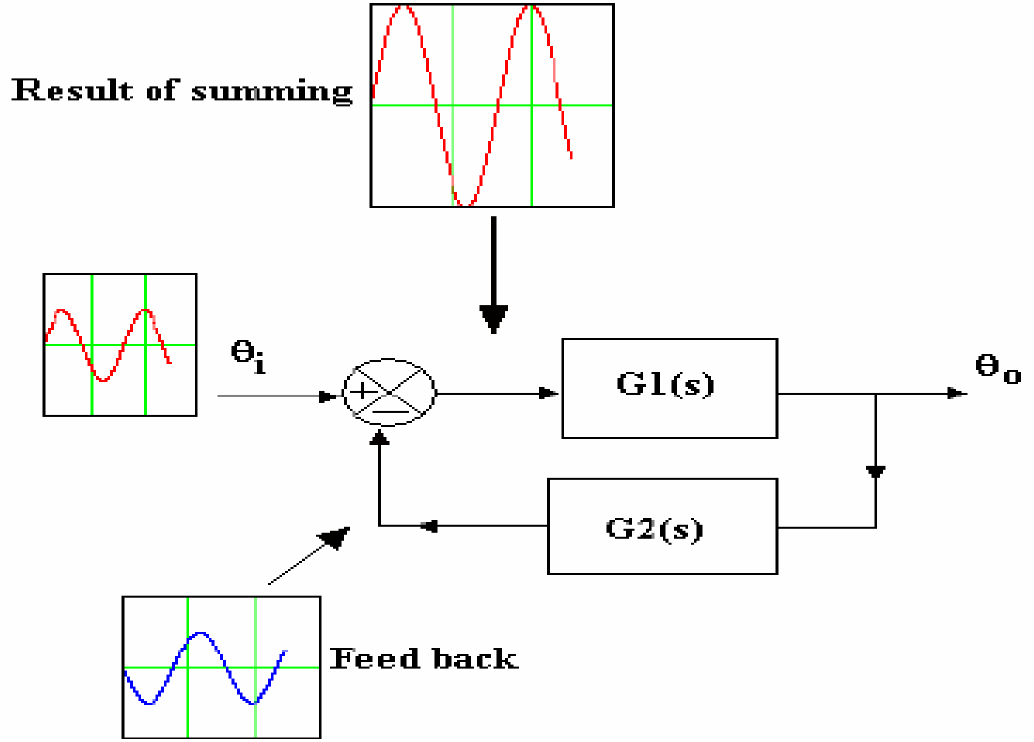
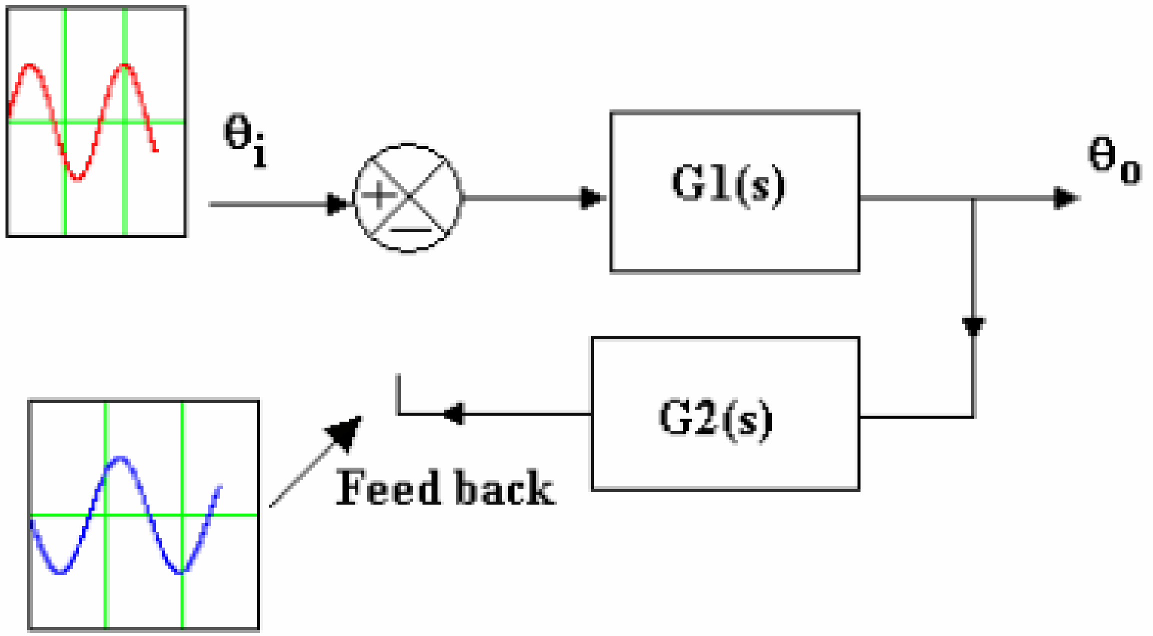
Nyquist Plot

A system with negative feed back becomes unstable if the signal arriving back at the summer is larger than the input signal and has shifted 180° relative to it. Consider the block diagram of the closed loop system. A sinusoidal signal is put in and the feed back is subtracted with the summer to produce the error. Due to time delay the feed back is 180° out of phase with the input. When they are summed the result is an error signal larger than the input signal. This will produce instability and the output will grow and grow.

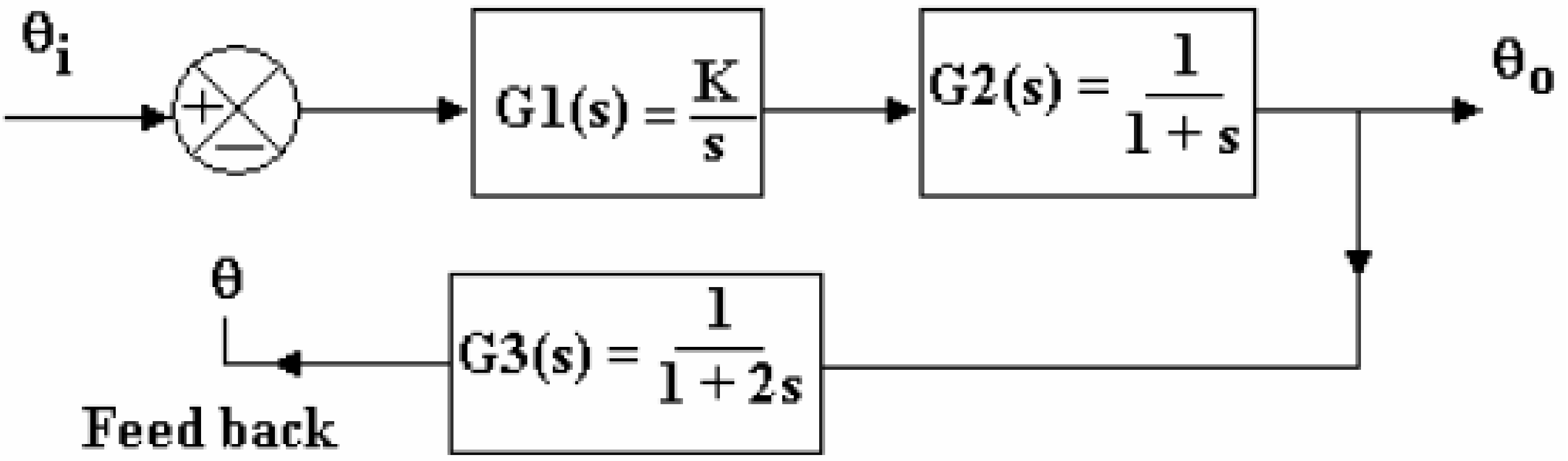
A method of checking if this is going to happen is to disconnect the feed back at the summer and measure the feed back over a wide range of frequencies.





Nyquist diagram is the locus of the open loop transfer function plotted on the complex plane. If a system is inherently unstable, the Nyquist diagram will enclose the point -1 (the point where the phase angle is 180° and unity gain).

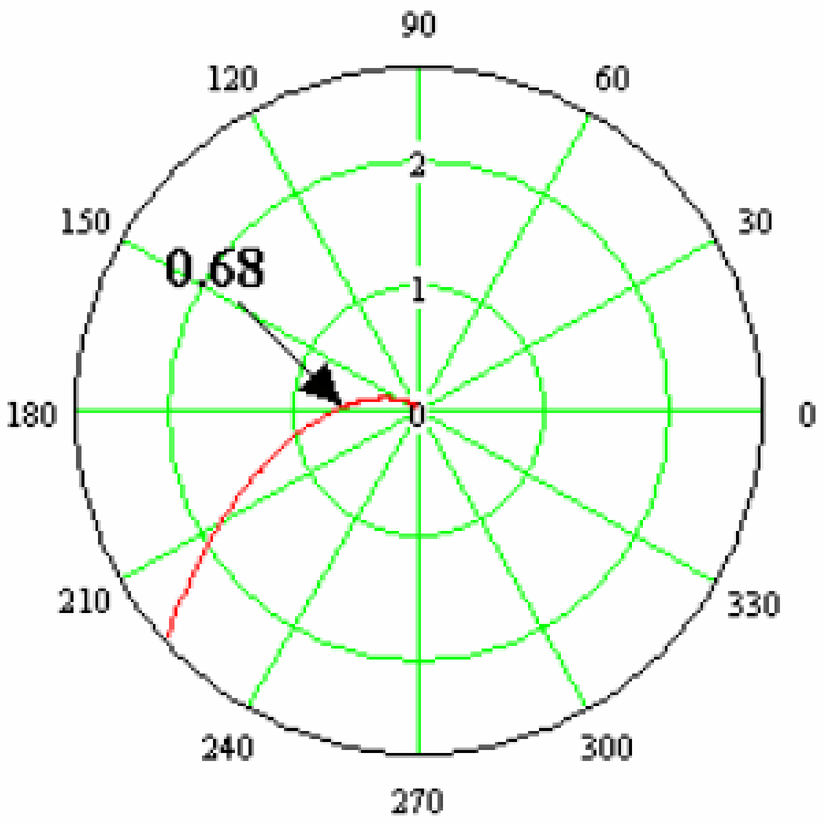
Consider the system.



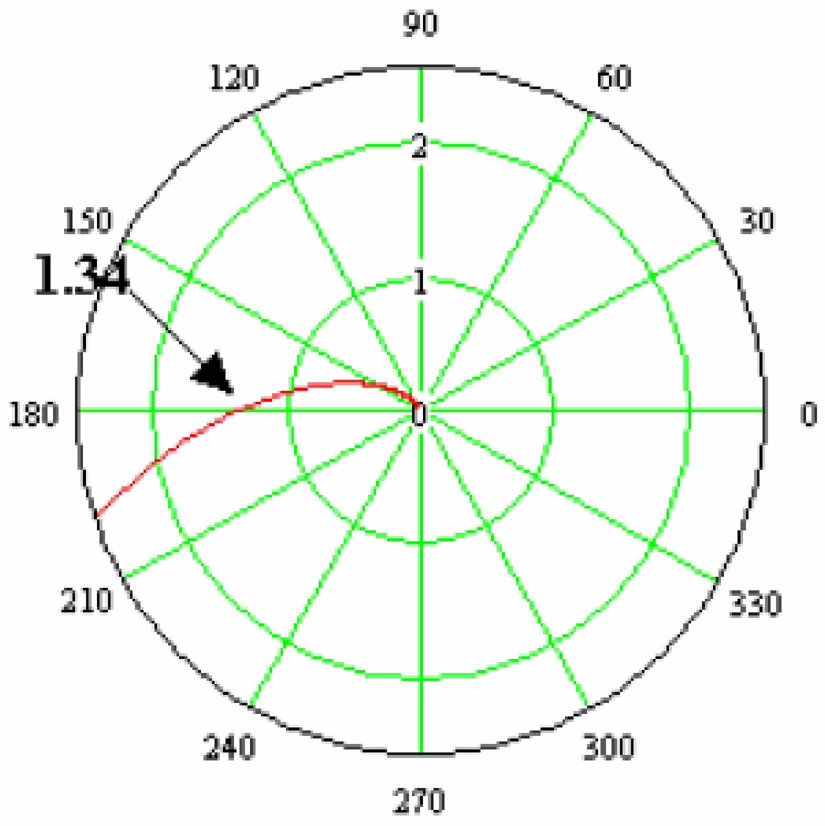
The transfer function relating θ_i and θ is G(s) G1 x G2 x G3 = K/{s(1+s)(1+2s)}. Converting this into a complex number (s = jω) we find

$$G(j\omega) = \frac{K\left\{-3\omega^2\right\}}{\left\{9\omega^4 + (\omega - 2\omega^3)\right\}} - j\frac{K(\omega - 2\omega^3)}{\left\{9\omega^4 + (\omega - 2\omega^3)\right\}}$$

The polar plot below (Nyquist Diagram) is shown for K = 1 and K = 0.4. We can see that at the 180° position the radius is less than 1 when K = 1 so the system will be stable. When K = 2 the radius is greater than 1 so the system is unstable. We conclude that turning up the gain makes the system become unstable.



K = 1



K = 2

The plot will cross the real axis when $\omega = 2\omega^3$ or $\omega = 0.707$ and this is true for all frequencies. The plot will enclose the -1 point if

$$\frac{K\left\{-3\omega^2\right\}}{\left\{9\omega^4 + (\omega - 2\omega^3)\right\}} \leq -1 \quad \text{so the limit is when } -K\,3\omega^2 = 9\omega^4 + (\omega - 2\omega^3)$$

Putting $\omega = 0.707$ the limiting value of K is 1.5

Method 2

There is another way to solve this and similar problems. The transfer function is broken down into separate components so in the above case we have:

$$G(s) = \frac{K}{s} \times \frac{1}{(1+s)} \times \frac{1}{(1+2s)}$$

Each is turned into polar co-ordinates

$\frac{K}{s}$ produces a radius of $\frac{K}{\omega}$ and an angle of -90° for all radii

$\frac{1}{1+s}$ produces a radius of $\frac{1}{\sqrt{1+\omega^2}}$ and angle $\tan^{-1}(\omega)$

$\frac{1}{1+2s}$ produces a radius of $\frac{1}{\sqrt{1+4\omega^2}}$ and angle $\tan^{-1}2\omega$

When we multiply polar coordinates remember that the resultant radius is the product of the individual radii and the resultant angle is the sum of the individual angles. The polar coordinates of the transfer function are then:

Radius is $\frac{K}{\omega} \times \frac{1}{\sqrt{1+\omega^2}} \times \frac{1}{\sqrt{1+4\omega^2}}$ Angle is $-90^\circ - \tan^{-1}(\omega) - \tan^{-1}2\omega$

Put $\omega = 0.707$ Radius = 1.414 x 0.8165 x 0.577 = 0.667 K Angle = -90 - 35.26 - 54.74 = -180°

If K = 1.5 the radius is 1 as stated previously.

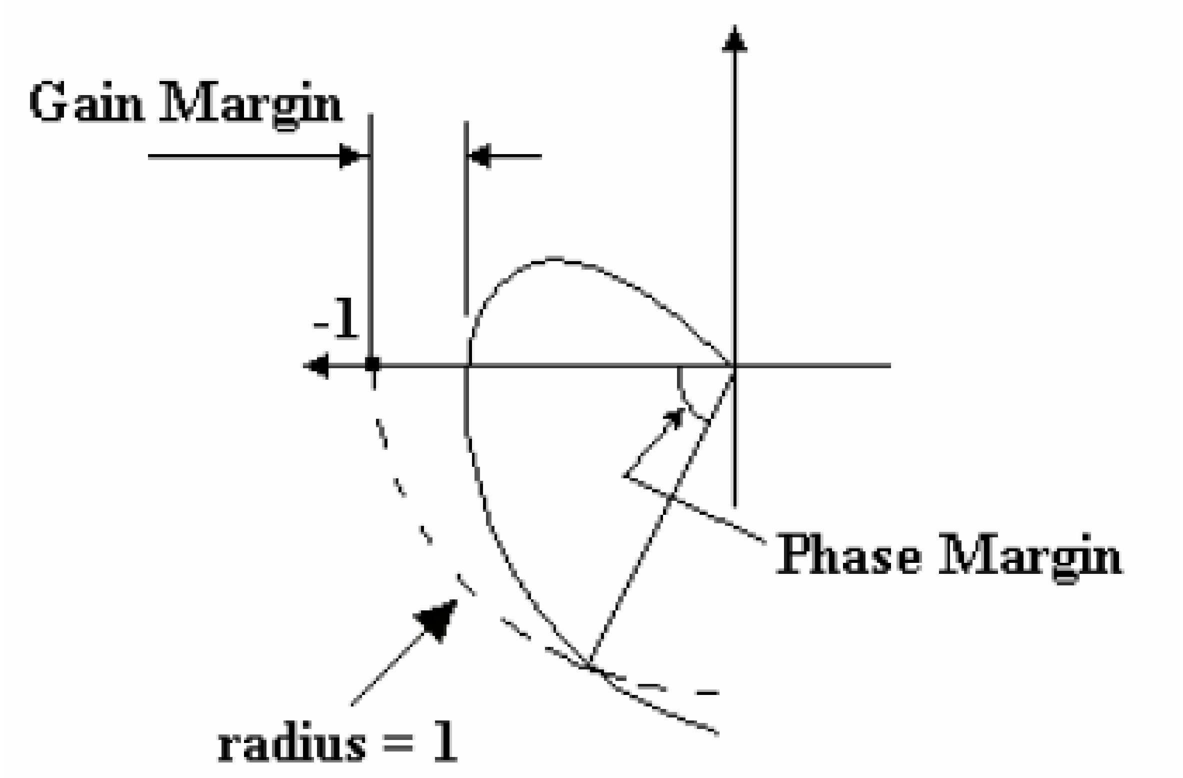
PHASE MARGIN and GAIN MARGIN

PHASE MARGIN

This is the additional phase lag which is needed to bring the system to the limit of stability. In other words it is the angle between the point -1 and the vector of magnitude 1.

GAIN MARGIN

This is the additional gain required to bring the system to the limit of stability.



Example

The open loop transfer function of a system is G(s) = 200/{(1+2S)(3+S)(5+S)}. Produce a polar plot for $\omega = 3$ to $\omega = 10$. Determine the phase and gain margin.

SOLUTION

Evaluate the polar coordinates for 200/(1 + 2s), then 1/(2+s) then 1/(5+s)

The radius is $$\frac{\frac{K}{n}}{\sqrt{1+\frac{T^2\cdot\omega^2}{n^2}}}$$

The angle is $$-\tan^{-1}\left(\frac{\omega\cdot T}{n}\right)$$

$\frac{200}{1+2s}$

T = 2
K = 200
n = 1

ω	R	φ
1	89.443	-63.435
2	48.507	-75.964
3	32.88	-80.538
4	24.807	-82.875
5	19.901	-84.289
6	16.609	-85.236
7	14.249	-85.914
8	12.476	-86.424
9	11.094	-86.82
10	9.988	-87.138

$\frac{1}{3+s}$

T = 1
K = 1
n = 3

ω	R	φ
1	0.316	-18.435
2	0.277	-33.69
3	0.236	-45
4	0.2	-53.13
5	0.171	-59.036
6	0.149	-63.435
7	0.131	-66.801
8	0.117	-69.444
9	0.105	-71.565
10	0.096	-73.301

$\frac{1}{5+s}$

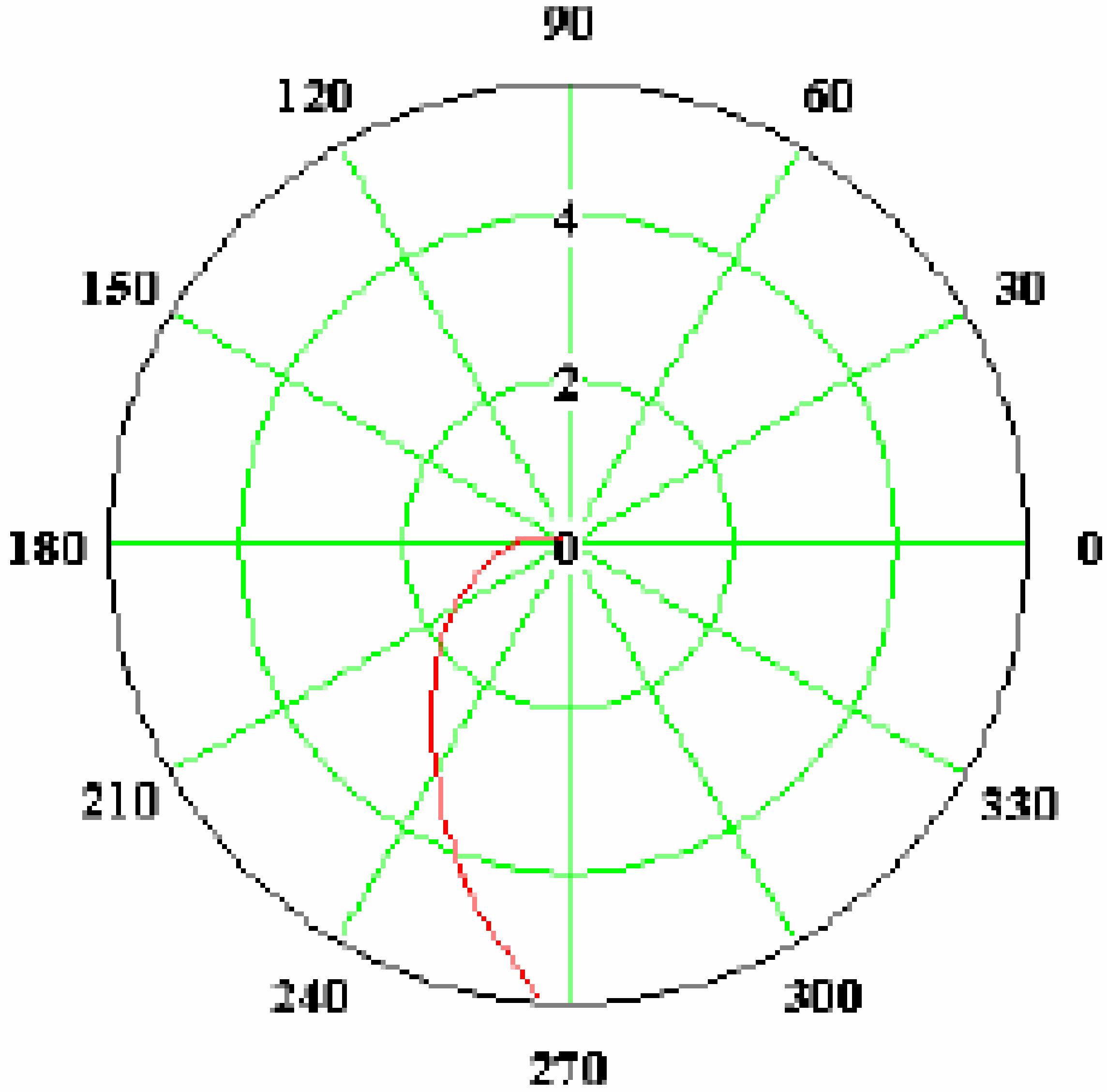
T = 1
K = 1
n = 5

ω	R	φ
1	0.196	-11.31
2	0.186	-21.801
3	0.171	-30.964
4	0.156	-38.66
5	0.141	-45
6	0.128	-50.194
7	0.116	-54.462
8	0.106	-57.995
9	0.097	-60.945
10	0.089	-63.435

Now add the three sets of angles and multiply the three sets of radii and plot the results.

Final Result

ω	R	ϕ
1	5.547	-93.18
2	2.498	-131.455
3	1.329	-156.501
4	0.775	-174.665
5	0.483	-188.326
6	0.317	-198.866
7	0.218	-207.178
8	0.155	-213.862
9	0.114	-219.331
10	0.086	-223.873



The region of interest is where the plot is -180° and the radius is 1. This would require a much more accurate plot around the region for ω = 3 to 5 as shown below.

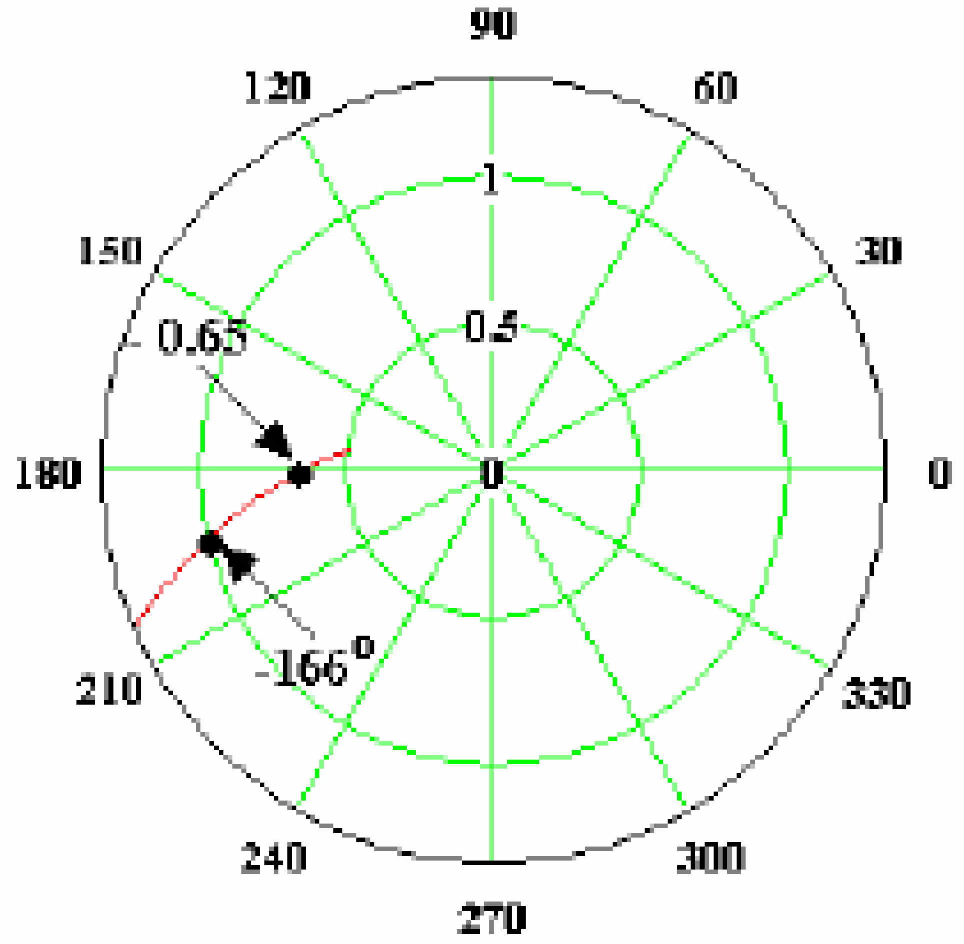


Figure 8

The phase margin is 180 – 166 = 14°   The gain margin is 1 – 0.65 = 0.35

Homework

1 Determine the steady state gain and primary time constant for $G(s) = 10/(s + 5)$. Determine the polar coordinates when $\omega = 1/T$
(Gain = 2 and Radius = 1.414 and angle = -45^{o})

2. Determine the steady state gain for $G(s) = 0.5/\{(s+2)(s+10)\}$. Determine the polar coordinates when $\omega = 0.5$
(Gain = 0.025 , $R_1 = 0.0243$ $\phi_1 = -16.9^{o}$)

3. The open loop transfer function of a system is $G(s) = 80/\{(s+1)(s+2)(s+4)\}$. Produce a polar plot for $\omega = 3$ to $\omega = 10$. Determine the phase and gain margin. (0.11 and 3.5^{o}).

Bode Plot

These are logarithmic plots of the magnitude (radius of the polar plot) and phase angle of the transfer function. First consider how to express the gain in decibels.

Strictly G is a power gain and G = Power out/Power In

If the power in and out were electric then we may say $G = \frac{V_{out}\, I_{out}}{V_{in}\, I_{in}}$

Using Ohms Law this with the same value of Resistance at input and output this becomes

$$G = \frac{V_o^2}{V_i^2} \text{ or } \frac{I_o^2}{I_i^2}$$

Expressing G in decibels $G(db) = 10 \log \frac{V_o^2}{V_i^2} = 20 \log \frac{V_o}{V_i} \text{ or } 20 \log \frac{I_o}{I_i}$

From this, it is usual to express the modulus of G as $|G| = 20 \log |(\theta_o/\theta_i)|$

Note that the gain in db is the 20 log R where R is the radius of the polar plot in previous examples.

Now consider the following transfer function

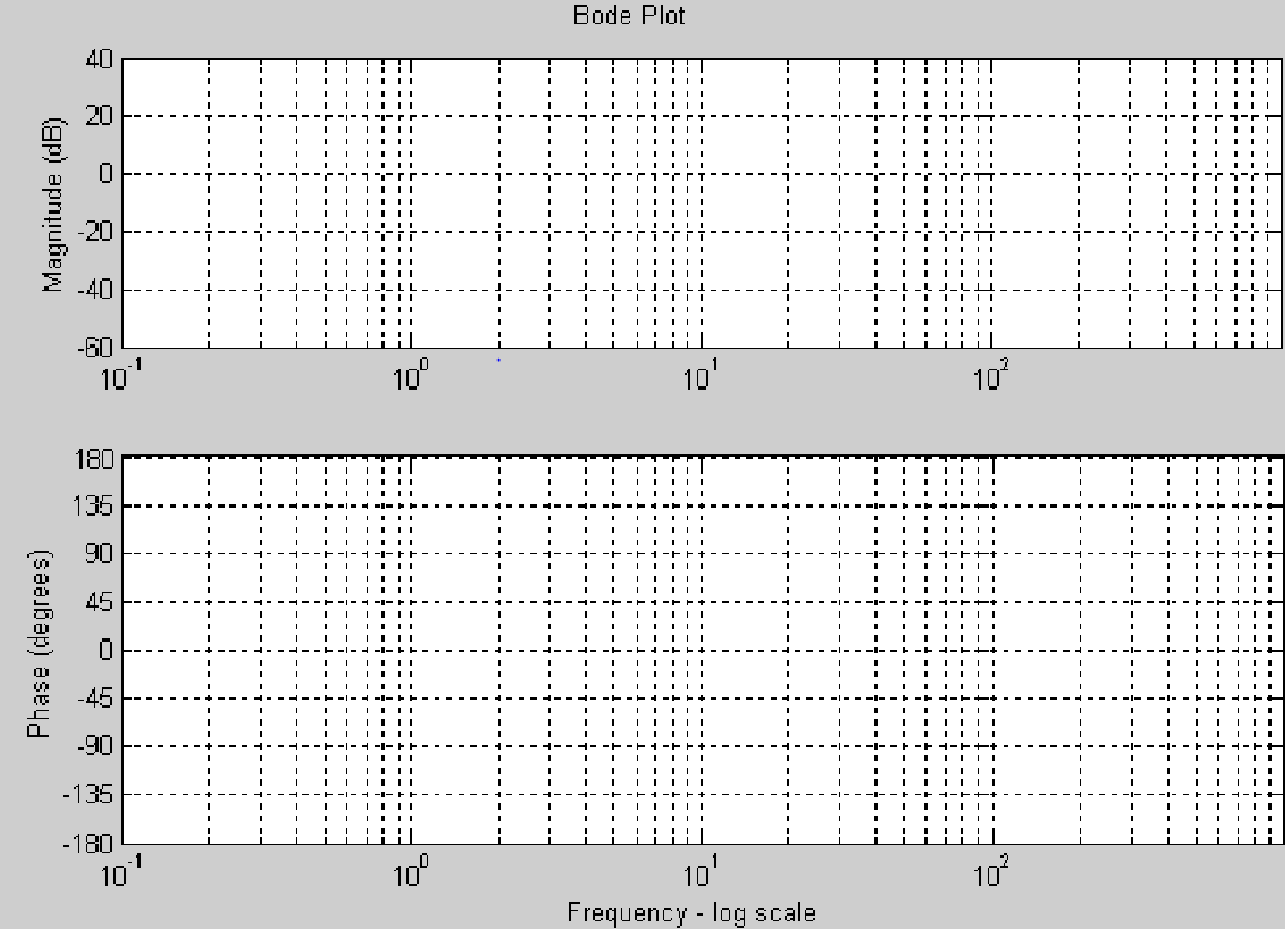
$$G(s) = \frac{1}{Ts + 1} \quad |G| = \left|\frac{1}{j\omega T + 1}\right| = \frac{1}{|j\omega T + 1|}$$

$G(j\omega) = \frac{1}{1 + \omega^2 T^2} - j\frac{\omega T}{1 + \omega^2 T^2}$ The radius of the polar coordinate is $\frac{1}{\sqrt{(T^2\omega^2 + 1)}}$ and this is the gain.

The gain in db is then

$$G(db) = 20\left[\log\left(\frac{1}{\sqrt{(T^2\omega^2 + 1)}}\right)\right] = -20\left[\frac{1}{2}\log\left(T^2\omega^2 + 1\right)\right] = -10\log\left(T^2\omega^2 + 1\right)$$

The phase angle is $-\tan^{-1}(\omega T)$



Bode Plot
Magnitude (dB)
40
20
0
-20
-40
-60
10^{-1}
10^{0}
10^{1}
10^{2}
Phase (degrees)
180
135
90
45
0
-45
-90
-135
-180
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency - log scale

- 2 plots – both have logarithm of frequency on x-axis
  - y-axis magnitude of transfer function, H(s), in dB
  - y-axis phase angle

The plot can be used to interpret how the input affects the output in both magnitude and phase over frequency.

Where do the Bode diagram lines comes from?

1) Determine the Transfer Function of the system:

$$H(s) = \frac{K(s+z_1)}{s(s+p_1)}$$

2) Rewrite it by factoring both the numerator and denominator into the standard form

$$H(s) = \frac{Kz_1(s/z_1 + 1)}{sp_1(s/p_1 + 1)}$$

where the z s are called zeros and the p s are called poles.

3) Replace *s* with *j?* . Then find the **Magnitude** of the Transfer Function.

$$H(jw) = \frac{Kz_1({}^{jw}\!/_{z_1} + 1)}{jwp_1({}^{jw}\!/_{p_1} + 1)}$$

If we take the $\log_{10}$ of this magnitude and multiply it by 20 it takes on the form of

$$20 \log_{10} (H(jw)) = 20\log_{10}\left(\frac{Kz_1({}^{jw}\!/_{z_1} + 1)}{jwp_1({}^{jw}\!/_{p_1} + 1)} \right) =$$

$$20\log_{10}|K| + 20\log_{10}|z_1| + 20\log_{10}\left|({}^{jw}\!/_{z_1} + 1)\right| - 20\log_{10}|p_1| - 20\log_{10}|jw| - 20\log_{10}\left|({}^{jw}\!/_{z_1} + 1)\right|$$

Each of these individual terms is very easy to show on a logarithmic plot. The entire Bode log magnitude pl the result of the superposition of all the straight line terms. This means with a little practice, we can quickly the effect of each term and quickly find the overall effect. To do this we have to understand the effect of the different types of terms.

These include: 1) Constant terms K

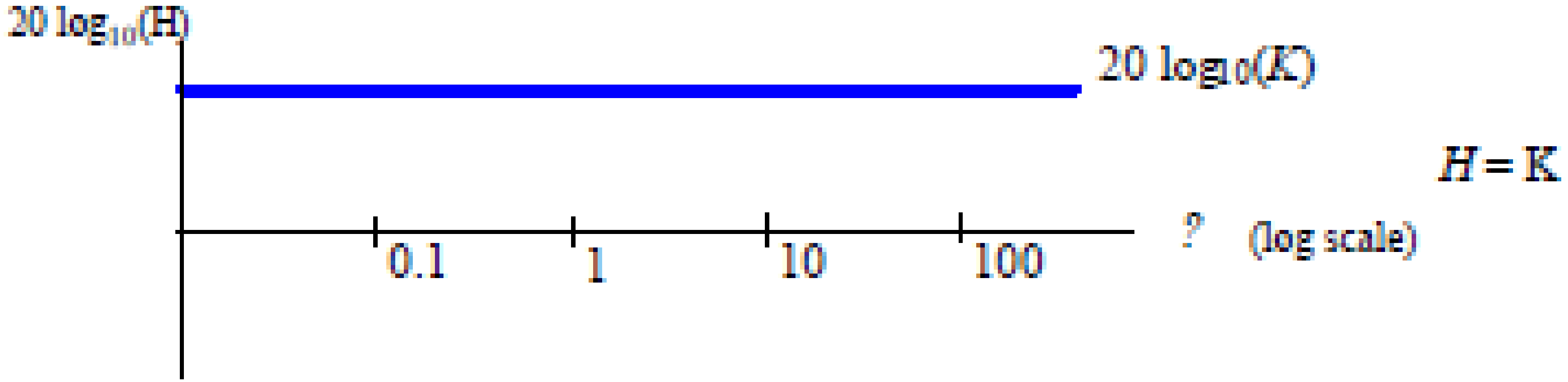
2) Poles and Zeros at the origin $|j?|$

3) Poles and Zeros not at the origin $\left|1 + \frac{j\omega}{p_1}\right|$ or $\left|1 + \frac{j\omega}{z_1}\right|$

4) Complex Poles and Zeros (addressed later)

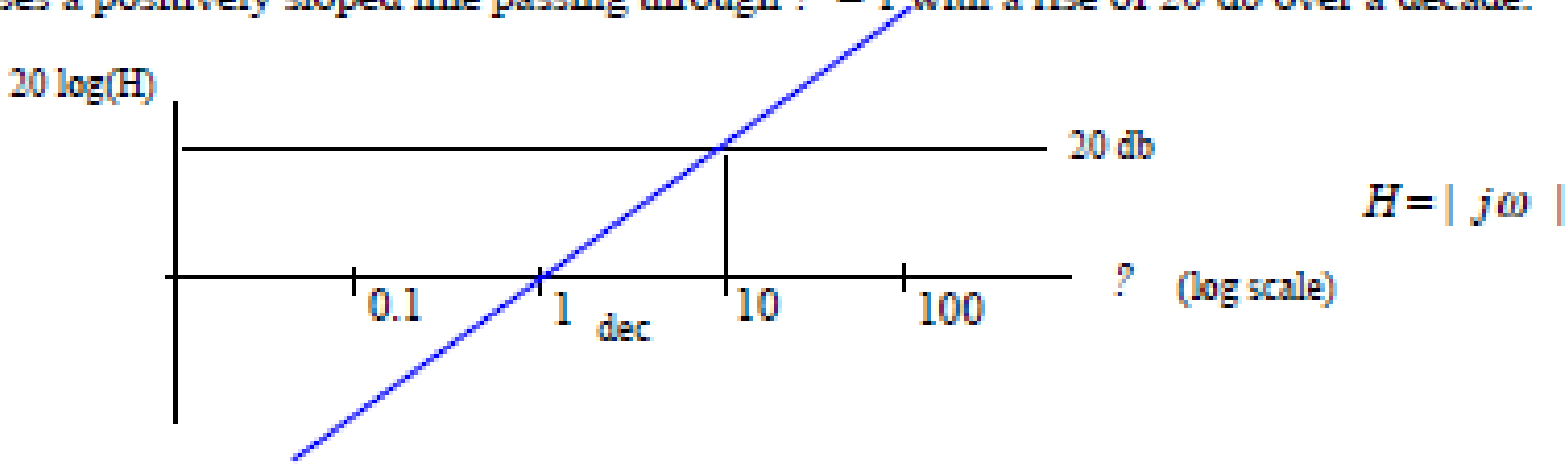
Effect of Constant Terms:

Constant terms such as K contribute a straight horizontal line of magnitude 20 $\log_{10}(K)$

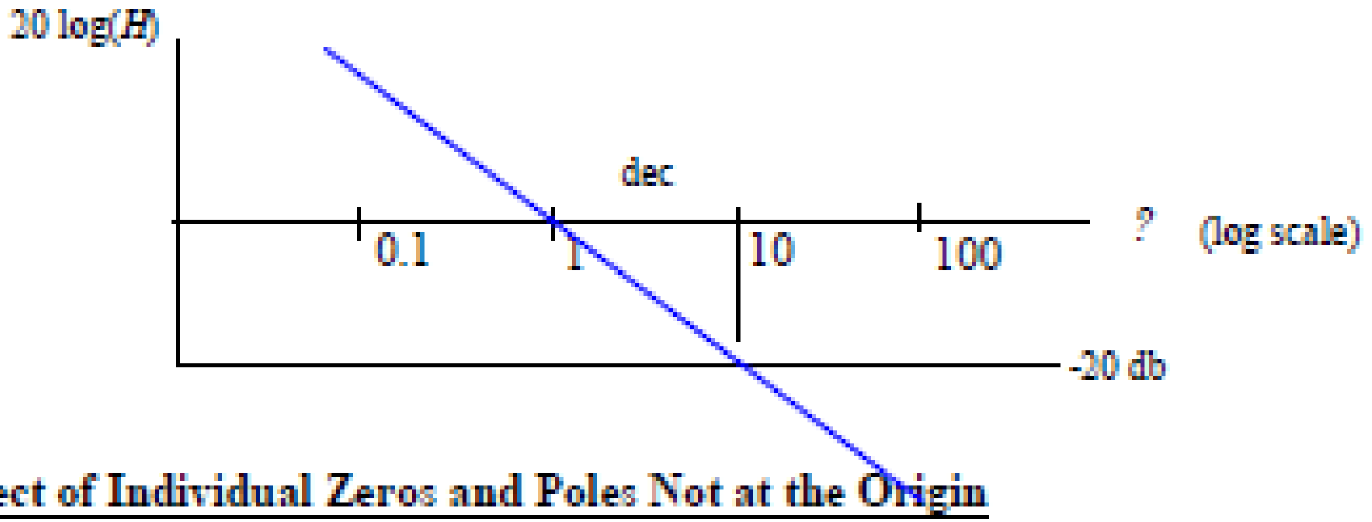


Effect of Individual Zeros and Poles at the origin:

A zero at the origin occurs when there is an s or $j?$ multiplying the numerator. Each occurrence of this causes a positively sloped line passing through ? = 1 with a rise of 20 db over a decade.



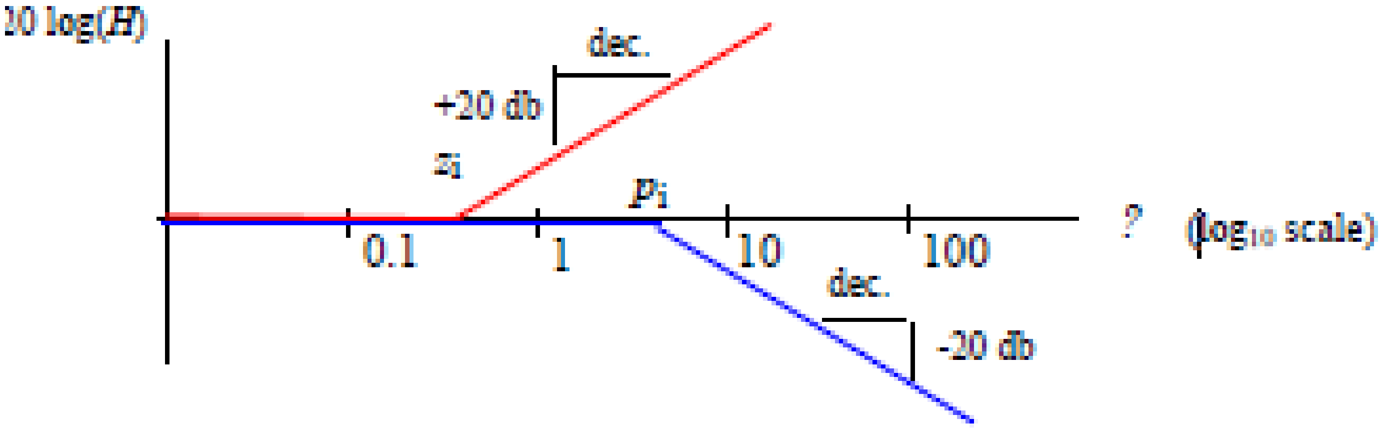
A pole at the origin occurs when there are *s* or *j?* multiplying the denominator. Each occurrence of this causes a negatively sloped line passing through ? = 1 with a drop of 20 db over a decade.



$$H = \frac{1}{j\omega}$$

Effect of Individual Zeros and Poles Not at the Origin

Zeros and Poles not at the origin are indicated by the (1+*j*? /z_i) and (1+*j*? /p_i). The values z_i and p_i in each of these expression is called a critical frequency (or break frequency). Below their critical frequency these terms do not contribute to the log magnitude of the overall plot. Above the critical frequency, they represent a ramp function of 20 db per decade. Zeros give a positive slope. Poles produce a negative slope.



$$H = \frac{\left|1 + \frac{j\omega}{z_i}\right|}{\left|1 + \frac{j\omega}{p_i}\right|}$$

- To complete the log magnitude vs. frequency plot of a Bode diagram, we superposition all the lines of the different terms on the same plot.

Example 1:

For the transfer function given, sketch the Bode log magnitude diagram which shows how the log magnitude of the system is affected by changing input frequency. (*TF*=*transfer function)*

$$TF = \frac{1}{2s+100}$$

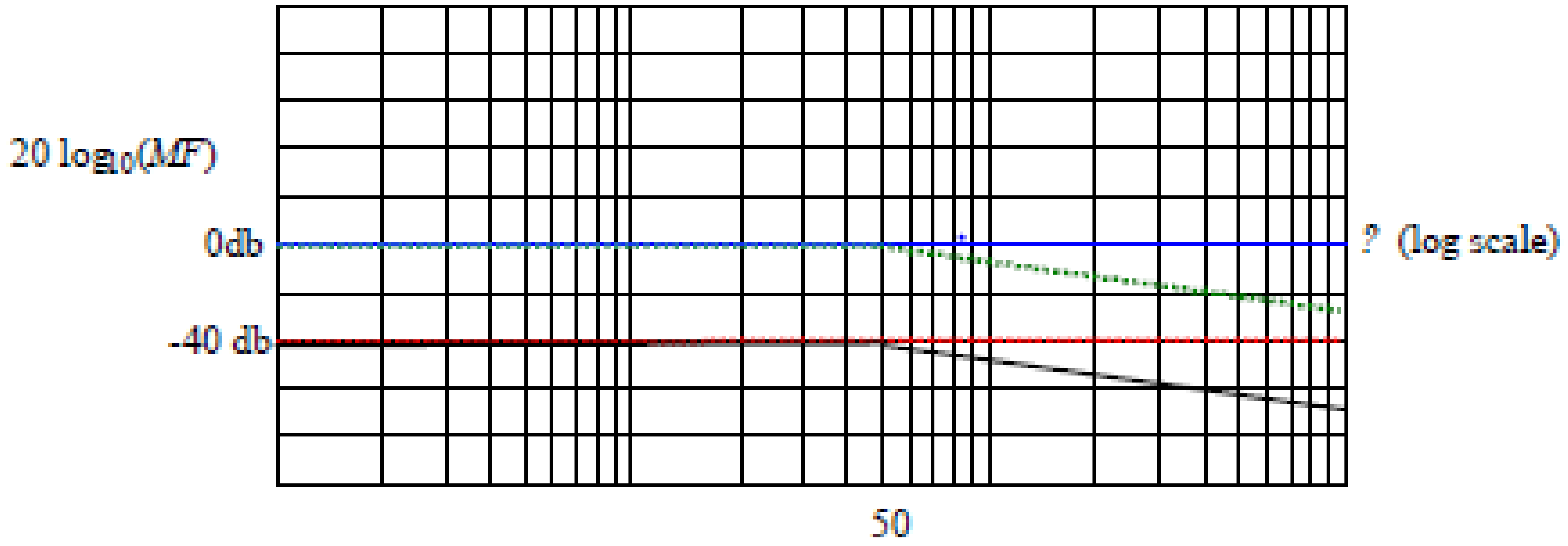
Step 1: Repose the equation in Bode plot form:

$$TF = \frac{\left(\frac{1}{100}\right)}{\frac{s}{50}+1} \qquad \text{recognized as} \qquad TF = \frac{K}{\frac{1}{p_1}s+1}$$

with $K = 0.01$ and $p_1 = 50$

For the constant, K: 20 $\log_{10}(0.01) = -40$

For the pole, with critical frequency, p_1: 



20 log10(MF)
0db
-40 db
? (log scale)
50

Example 2:

Find the Bode log magnitude plot for the transfer function,

$$TF = \frac{5x10^4 s}{s^2 + 505s + 2500}$$

Simplify transfer function form:

$$TF = \frac{5x10^4 \quad s}{(s+5)(s+500)} = \frac{\frac{5x10^4}{5*500} \quad s}{(\frac{s}{5}+1)(\frac{s}{500}+1)} = \frac{20 \quad s}{(\frac{s}{5}+1)(\frac{s}{500}+1)}$$

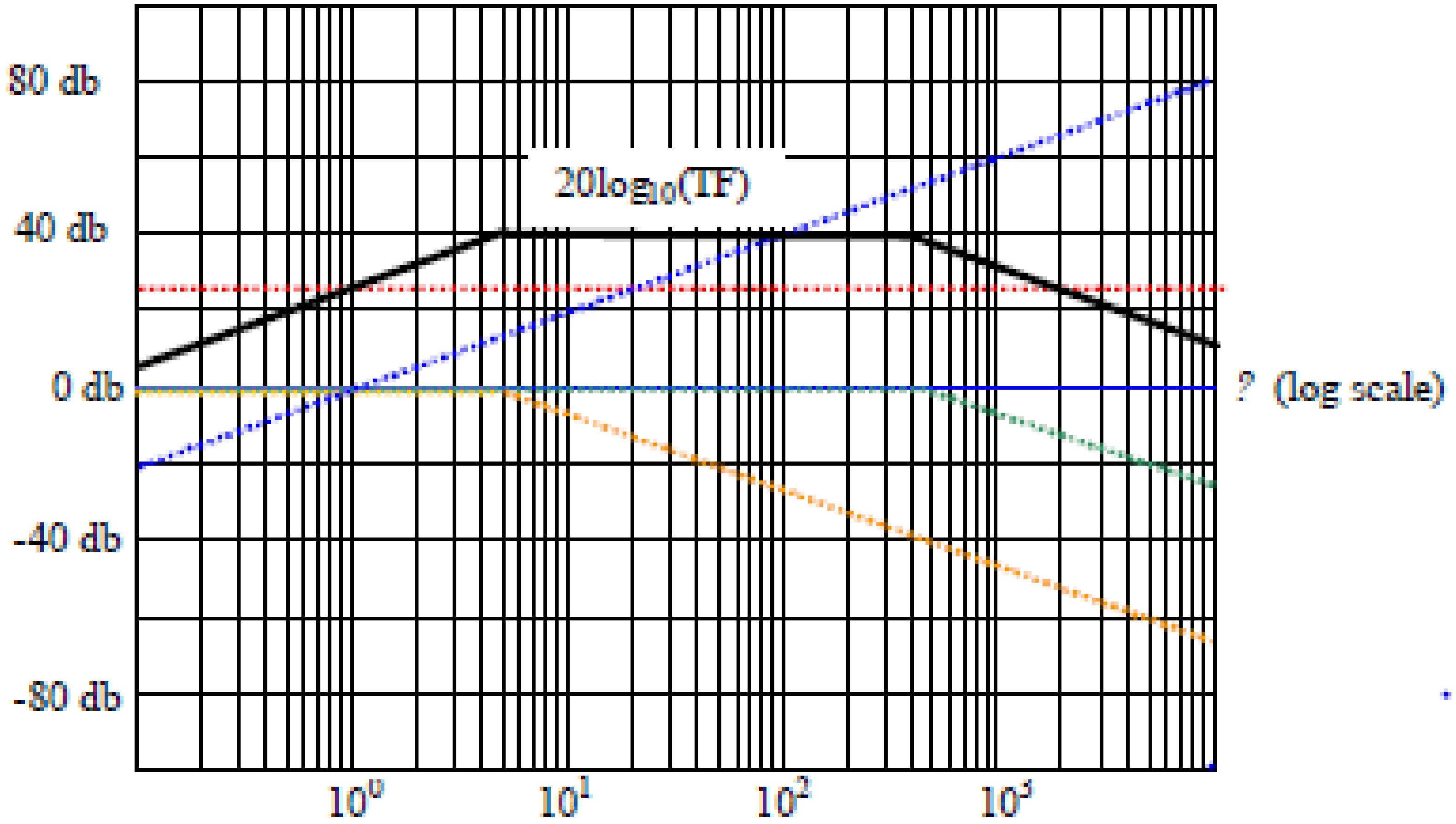
Recognize: $K = 20 \quad \rightarrow \quad 20\log10(20) = 26.02$

1 zero at the origin

2 poles: at $p_1 = 5$ and p_2=500

Technique to get started:

1) Draw the line of each individual term on the graph
2) Follow the combined pole-zero at the origin line back to the left side of the graph.
3) Add the constant offset, 20 $\log_{10}(K)$, to the value where the pole/zero at the origin line intersects the left side of the graph.
4) Apply the effect of the poles/zeros not at the origin. working from left (low values) to right (higher values) of the poles/zeros.



Example 3: Find the Bode log magnitude plot for the transfer function,

$$TF = \frac{200(s+20)}{s(2s+1)(s+40)}$$

Simplify transfer function form:

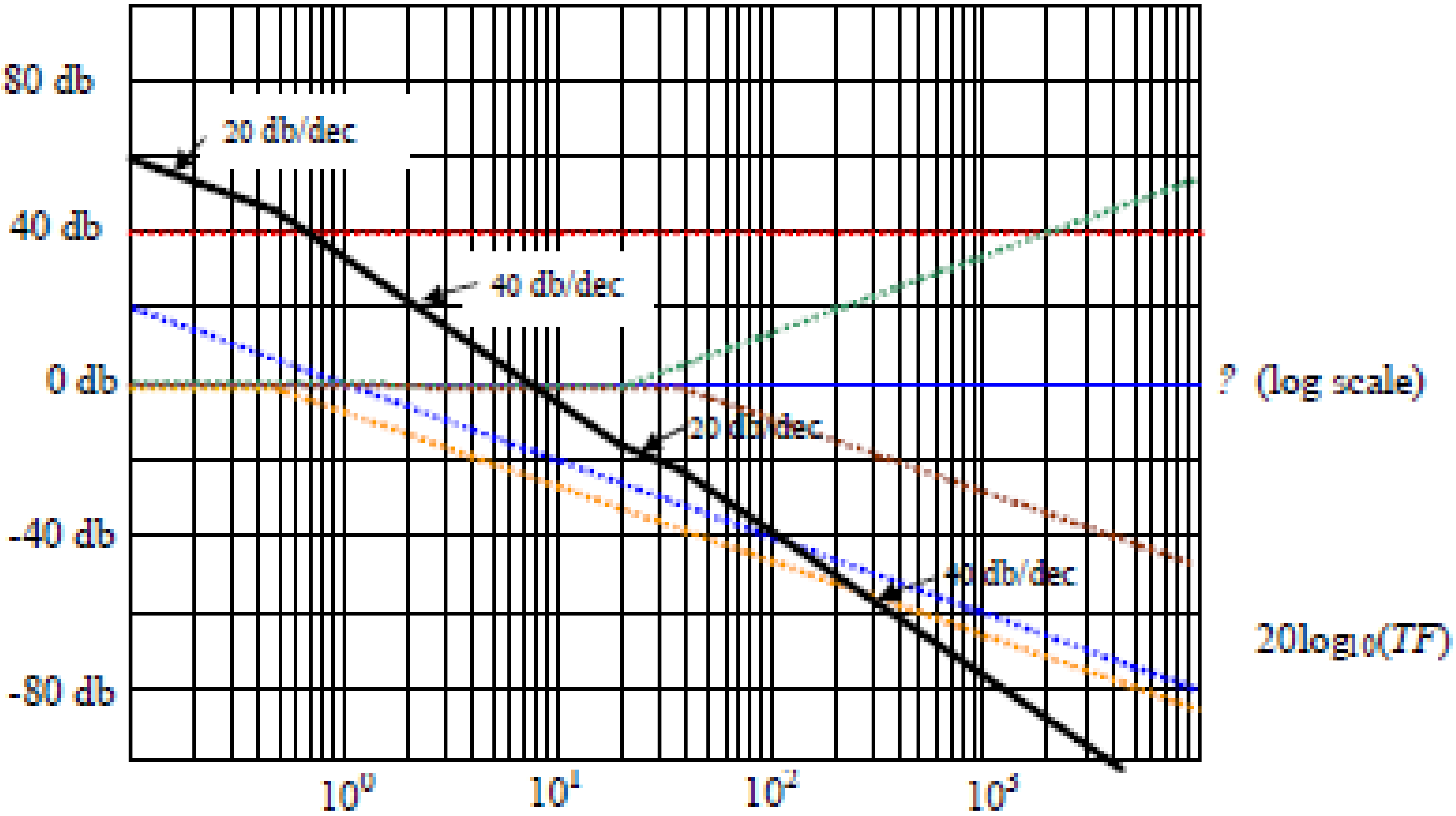
$$TF = \frac{200(s+20)}{s(2s+1)(s+40)} = \frac{\frac{200*20}{40}\quad(\frac{s}{20}+1)}{s(\frac{s}{0.5}+1)(\frac{s}{40}+1)} = \frac{100\quad(\frac{s}{20}+1)}{s(\frac{s}{0.5}+1)(\frac{s}{40}+1)}$$

Recognize: $K = 100$ → $20\log10(100) = 40$

1 pole at the origin

1 zero at $z_1 = 20$

2 poles: at $p_1 = 0.5$ and p_2=40



80 db
40 db
0 db
-40 db
-80 db
20 db/dec
40 db/dec
20 db/dec
40 db/dec
? (log scale)
20log10(TF)
10^0
10^1
10^2
10^3

The plot of the log magnitude vs. input frequency is only half of the story.
We also need to be able to plot the phase angle vs. input frequency on a log scale as well to complete the full Bode diagram..

For our original transfer function,

$$H(jw) = \frac{Kz_1 \left({}^{jw}\!/_{z_1} + 1 \right)}{jwp_1 \left({}^{jw}\!/_{p_1} + 1 \right)}$$

the cumulative phase angle associated with this function are given by

$$\angle H(jw) = \frac{\angle K \angle z_1 \angle \left({}^{jw}\!/_{z_1} + 1 \right)}{\angle jw \angle p_1 \angle \left({}^{jw}\!/_{p_1} + 1 \right)}$$

Then the cumulative phase angle as a function of the input frequency may be written as

$$\angle H(jw) = \angle \left[K + z_1 + \left({}^{jw}\!/_{z_1} + 1 \right) - (jw) - p_1 - \left({}^{jw}\!/_{p_1} + 1 \right) \right]$$

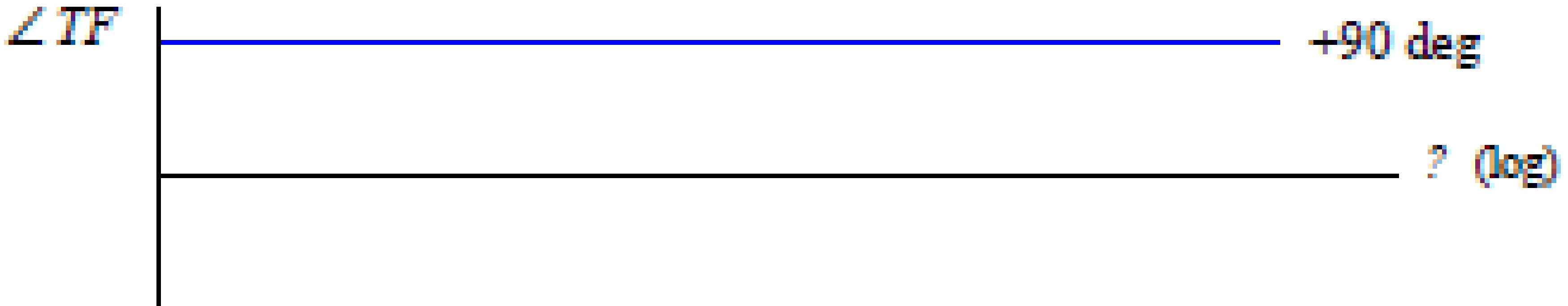
Once again, to show the phase plot of the Bode diagram, lines can be drawn for each of the different terms. Then the total effect may be found by superposition.

Effect of Constants on Phase:

A positive constant, $K>0$, has no effect on phase. A negative constant, $K<0$, will set up a phase shift of $\pm 180^\circ$. (Remember real vs imaginary plots – a negative real number is at $\pm 180^\circ$ relative to the origin)

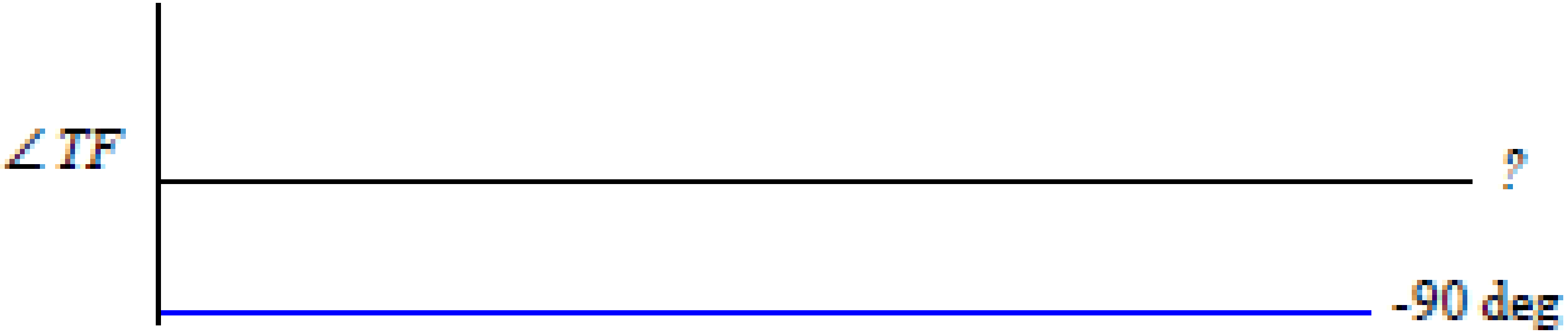
Effect of Zeros at the origin on Phase Angle:

Zeros at the origin, s, cause a constant +90 degree shift for each zero.



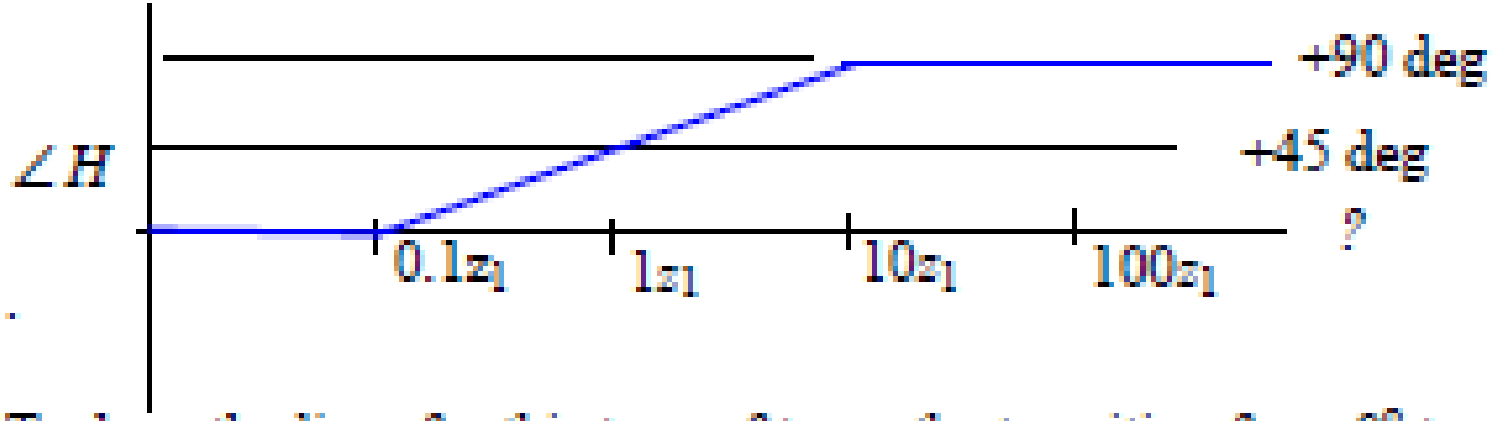
Effect of Poles at the origin on Phase Angle:

Poles at the origin, s^{-1}, cause a constant -90 degree shift for each pole.



Effect of Zeros not at the origin on Phase Angle:

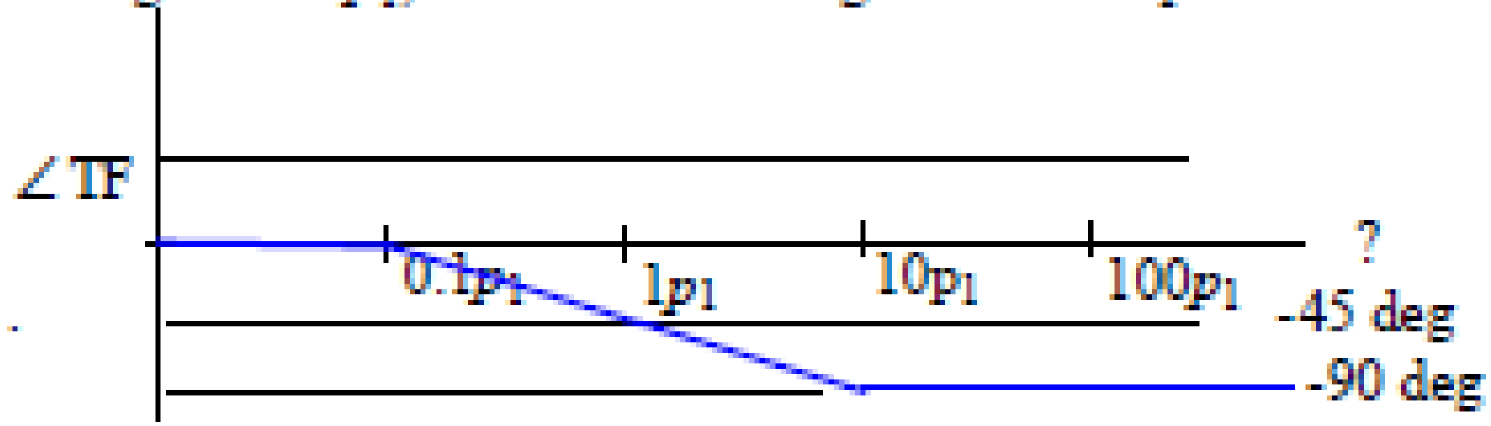
Zeros not at the origin, like $\left|1+\frac{j\omega}{z_1}\right|$, have no phase shift for frequencies much lower than z_1, have a + 45 deg shift at z_1, and have a +90 deg shift for frequencies much higher than z_1.



To draw the lines for this type of term, the transition from 0° to +90° is drawn over 2 decades, starting at $0.1z_1$ and ending at $10z_1$.

Effect of Poles not at the origin on Phase Angle:

Poles not at the origin, like $\dfrac{1}{\left|1 + \dfrac{j\omega}{p_1}\right|}$, have no phase shift for frequencies much lower than p_1, have a -45 deg shift at p_1, and have a -90 deg shift for frequencies much higher than p_1.



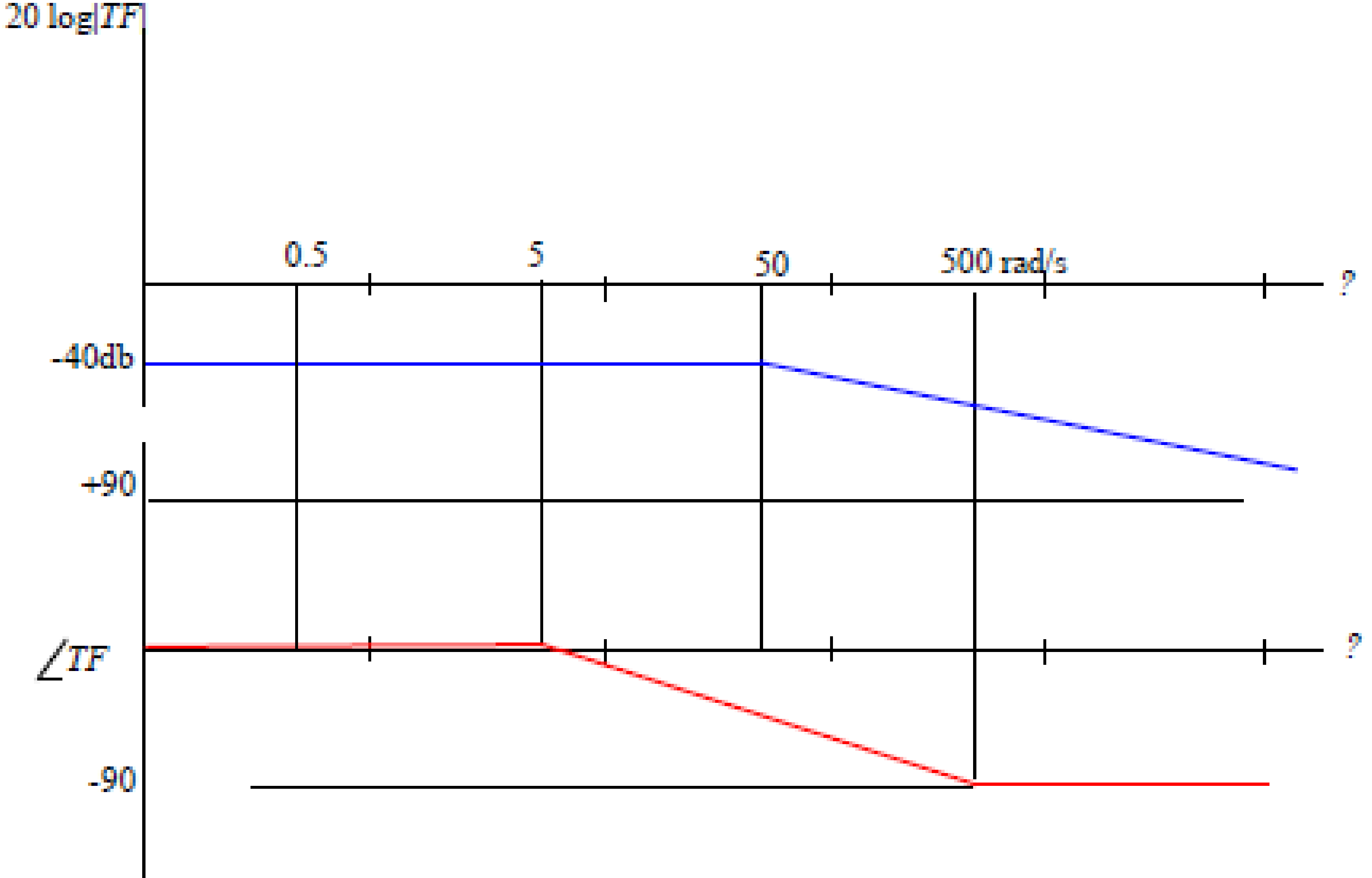
To draw the lines for this type of term, the transition from 0° to -90° is drawn over 2 decades, starting at $0.1p_1$ and ending at $10p_1$.

When drawing the phase angle shift for not-at-the-origin zeros and poles, first locate the critical frequency of the zero or pole. Then start the transition 1 decade before, following a slope of ±45° /decade. Continue the transition until reaching the frequency one decade past the critical frequency.

Example 1:

For the Transfer Function given, sketch the Bode diagram which shows how the phase of the system is affected by changing input frequency.

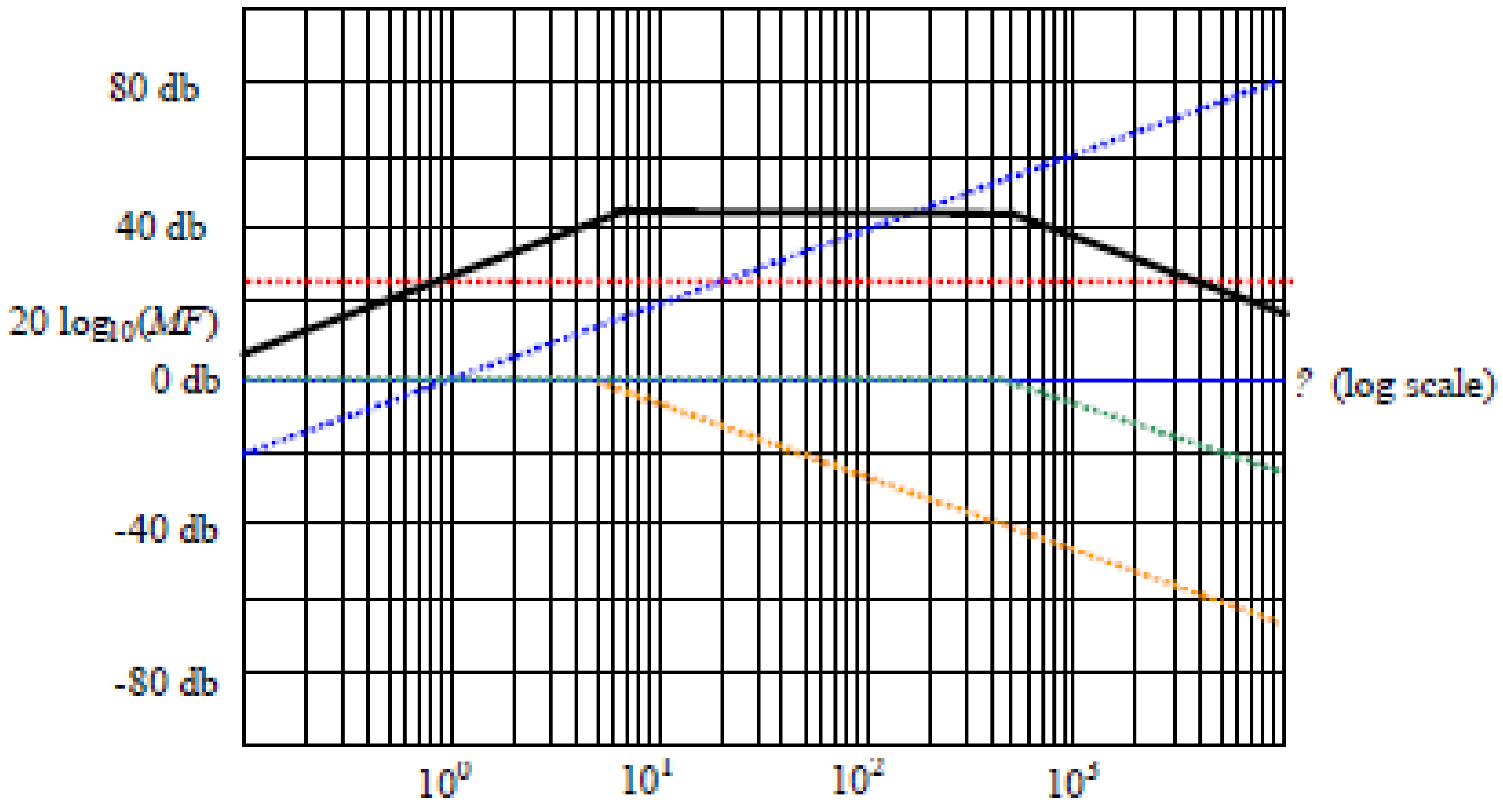
$$TF = \frac{1}{2s + 100} = \frac{(1/100)}{(\frac{s}{50} + 1)}$$

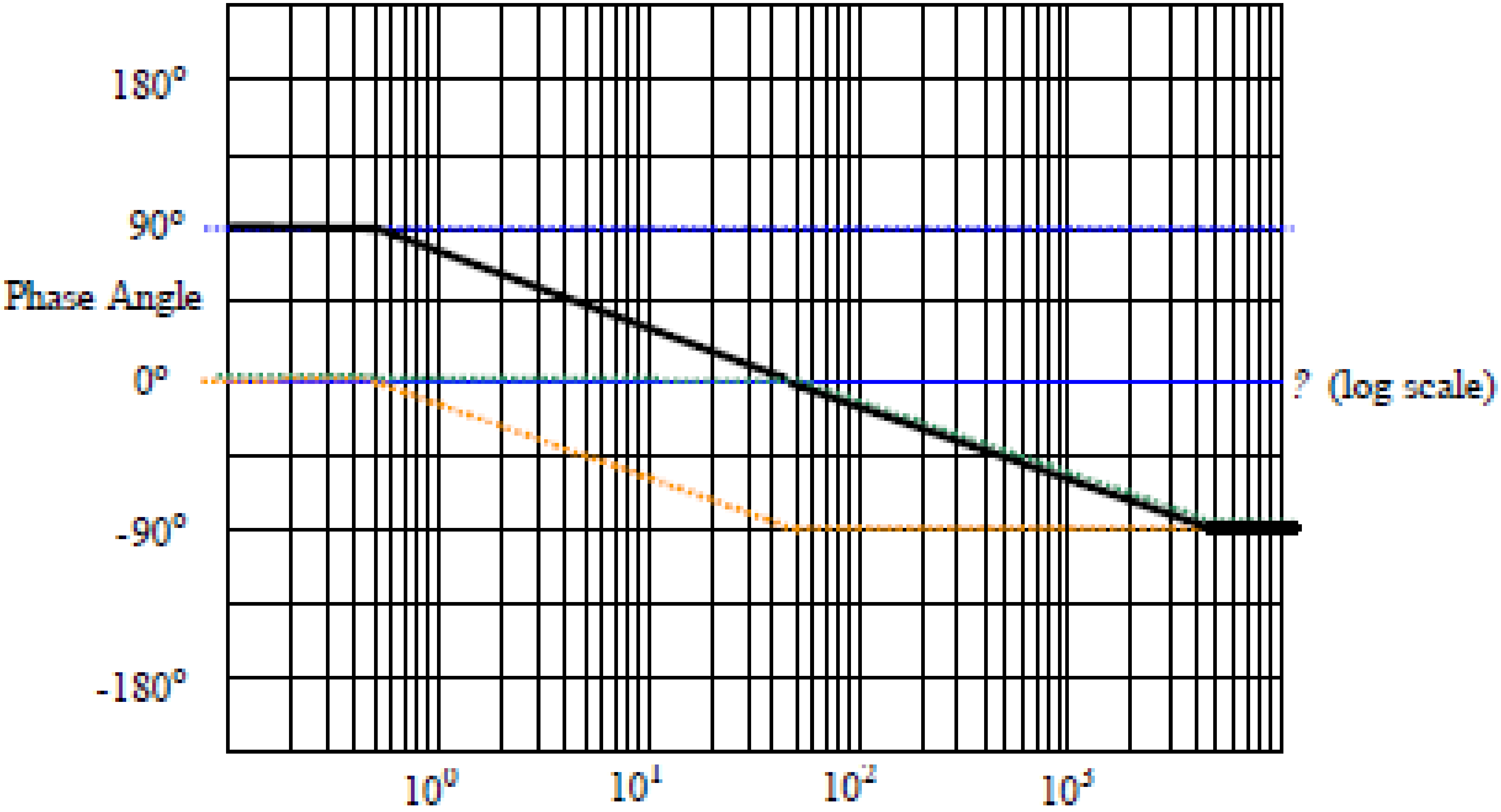


Example 2 Solution:
Repeat for the transfer function,

20log|*TF*|

$$TF = \frac{5x10^4 s}{s^2 + 505s + 2500} = \frac{20 \quad s}{(\frac{s}{5}+1)(\frac{s}{500}+1)}$$

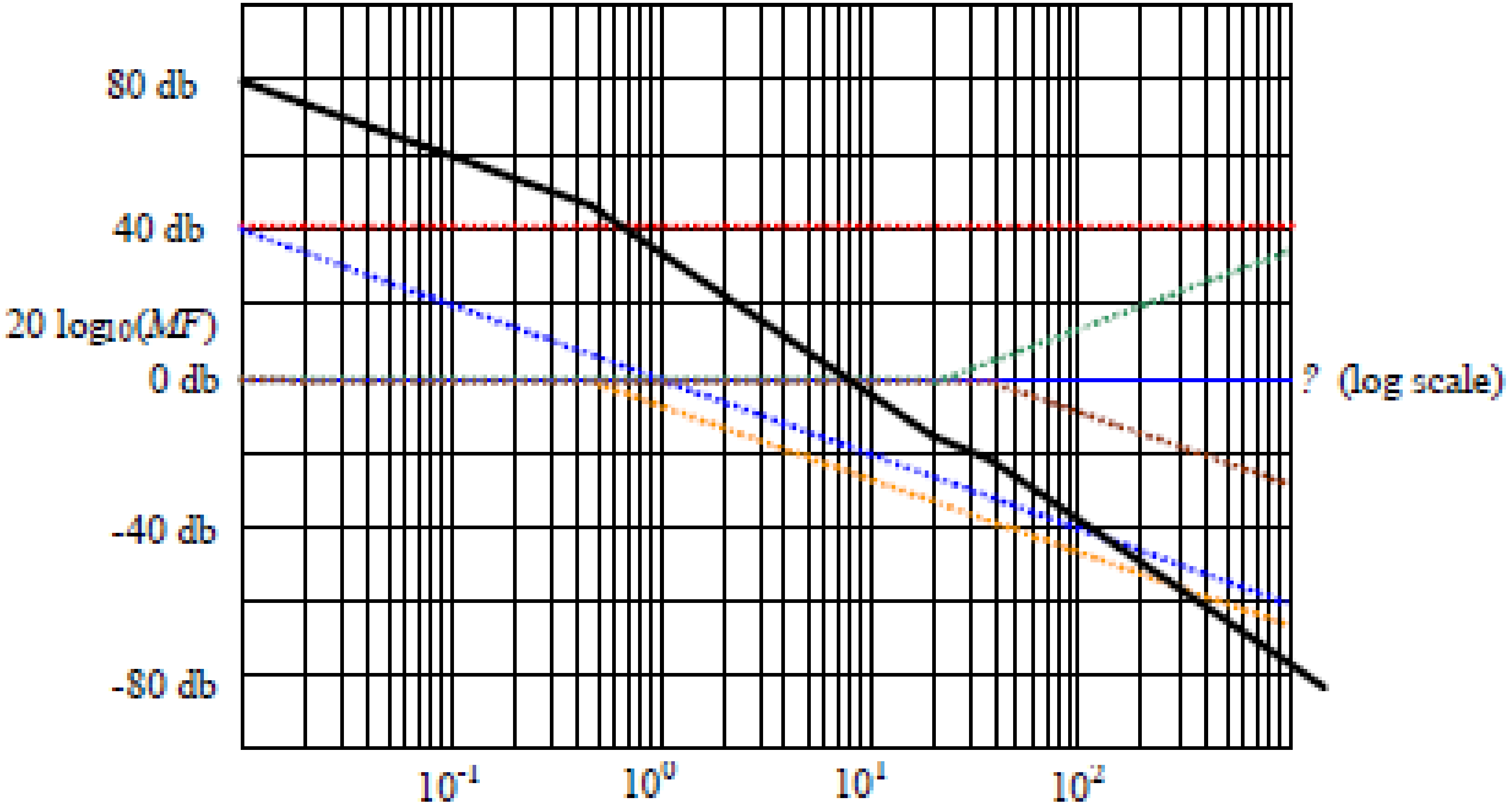


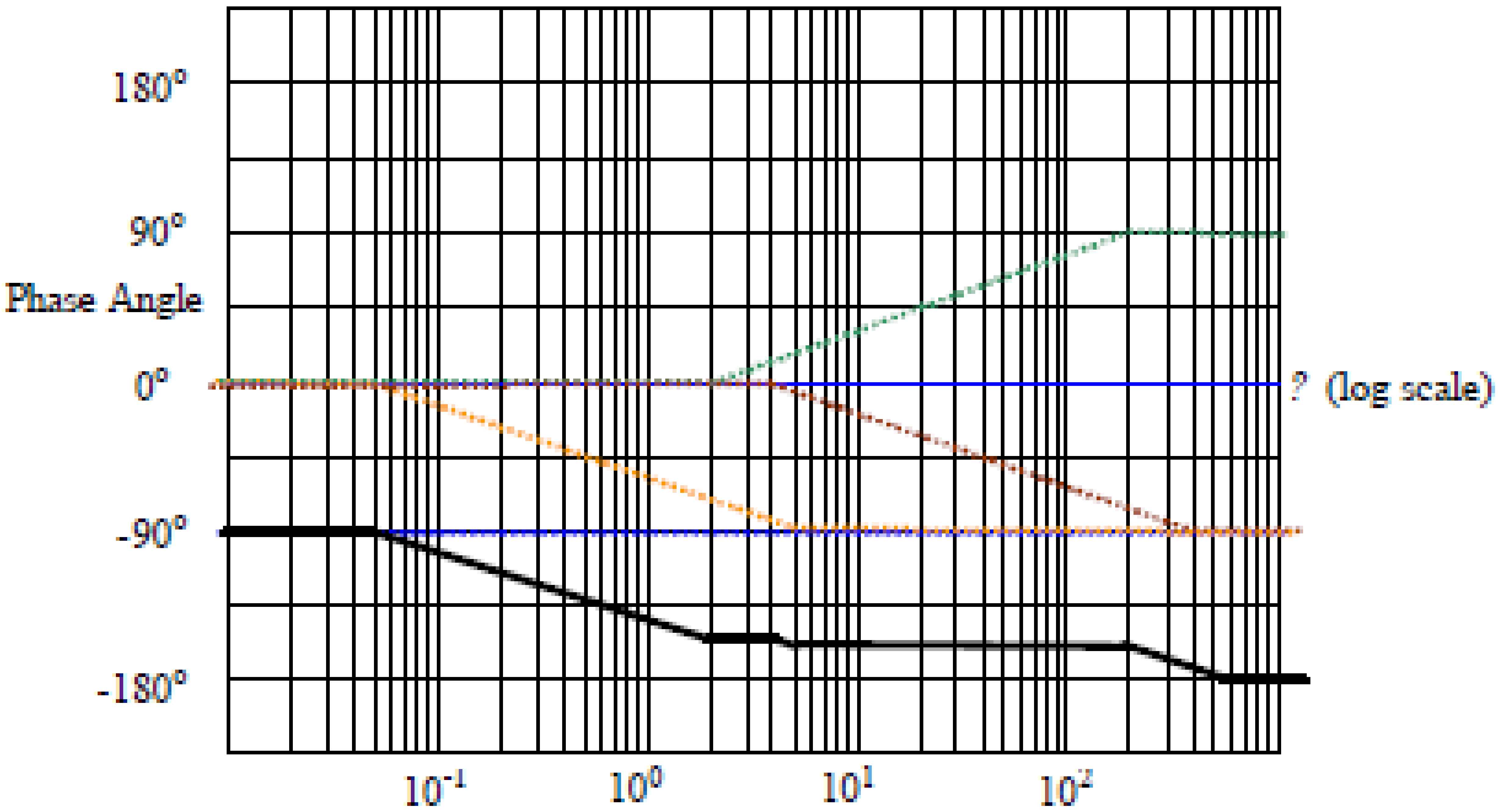


180°
90°
Phase Angle
0°
-90°
-180°
? (log scale)
10^0
10^1
10^2
10^3

Example 3: Find the Bode log magnitude and phase angle plot for the transfer function,

$$TF = \frac{200(s+20)}{s(2s+1)(s+40)} = \frac{100 \quad (\frac{s}{20}+1)}{s(\frac{s}{0.5}+1)(\frac{s}{40}+1)}$$




180°
90°
Phase Angle
0°
-90°
-180°
? (log scale)
10^{-1}
10^{0}
10^{1}
10^{2}