COURSE TITLE:

Introduction to Non-Linear Systems

COURSE CODE:

EEE 566

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Lecture 1

Course Outline

- Introduction to Nonlinearities and Nonlinear Systems
  - Non-linear differential equations, characteristics of nonlinear systems, common nonlinearities.
- Analysis of Nonlinear Systems
  - Linearization Approximations

    Piecewise linear approximation, the

    Describing Function Concept and derivation for

    common nonlinearities, the dual input describing function; stability analysis using the describing function. Limit cycle prediction.

Course Outline (Cont'd)

- Analysis of Nonlinear Systems
  - The Phase-Plane Method

    Construction of phase trajectories, transient analysis by the phase plane method.
  - Lyapunov's Indirect Method

    Stability Analysis of Non-linear Systems using Lyapunov's Method
- Introduction to Sampled-Data Systems

  The z-transforms; Pulse Transfer Functions; Stability Analysis in the z-plane.

Recommended Texts

- [1] Csaki, F. (1972), *Modern Control Theories: Nonlinear, Optimal and Adaptive Systems*, AkademiaiKiado, Budapest, Hungary.
- [2] Slotine, J.E., and Li, W. (1991), *Applied Nonlinear Control*, Prentice-Hall, Englewood Cliffs, New Jersey, United States of America.
- [3] Khalil, H. (1992), *Nonlinear Systems*, Macmillan Publishing Company, New York, United States of America.
- [4] Glad, T., and Ljung, L. (2000), *Control Theory: Multivariable and Nonlinear Methods*, Taylor and Francis, 11, New Fetter Lane, London EC4P 4EE, United Kingdom.
- [5] Vukic, Z., Kuljaca, L., Donlagic, D. and Tesnjak, S. (2003), *Nonlinear Control Systems*, Marcel-Dekker Inc., 270, Madison Avenue, New-York NY 10016, United States of America.
- [6] Marquez, H.J. (2003), *Nonlinear Control Systems*, John Wiley and Sons Inc., 111 River Street, Hoboken, New Jersey NJ 07030, United States of America.
- [7] Sastry, S. (1999), *Nonlinear Systems: Analysis, Stability and Control*, Springer-VerlagInc., 175 Fifth Avenue, New York NY 10010, United States of America.

INTRODUCTION

Representation of Systems

- In control terms, systems are commonly represented by:
  - Input-Output or Algebraic Equations

$$y(t) = f(u(t))$$

  - Differential Equations

$$f\left(y(t), \frac{dy(t)}{dt}, \dots, \frac{d^{n-1}y(t)}{dt^{n-1}}, \frac{d^{n}y(t)}{dt^{n}}\right) = g\left(u(t), \frac{du(t)}{dt}, \dots, \frac{d^{m-1}u(t)}{dt^{m-1}}, \frac{d^{m}u(t)}{dt^{m}}\right)$$

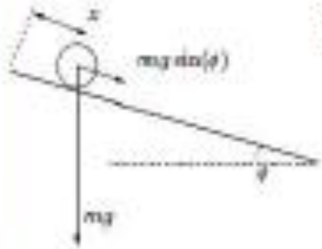
  - State-Space Equations

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$

$u(t), y(t)$ and $x(t)$ are the input, output and state functions respectively.

Representation of Systems (Cont'd)

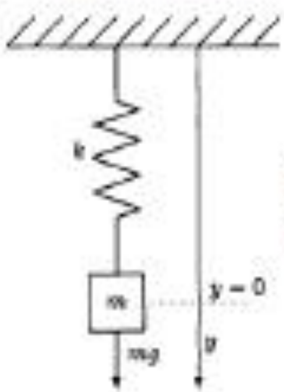
- Examples of:
  - Input-Output Equations



$a = g \sin \varphi$ (Ball-and-Beam Laboratory System)

a, g and φ are acceleration of ball on beam, acceleration due to gravity, and angle of inclination of beam to the horizontal respectively

- Necessarily inadequate in capturing the dynamics of a system
  - Differential Equations



Mass-Spring System with Hardening Spring (simple nonlinear mechanical system)

$$m\frac{d^2y}{dt^2} = \sum Forces = f(t) - f_k - f_\beta$$

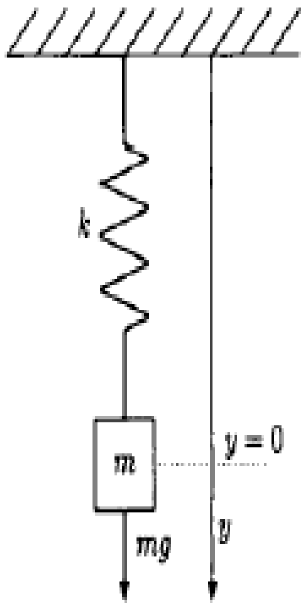
y is displacement from reference position, f_β is viscous frictional force, f_k is restoring force of spring, $f(t)$ is applied input force.

$$f_k = ky(1 + a^2y^2) \quad ==> \quad m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky + ka^2y^3 = f(t)$$

- More adequate than input-output equations in capturing the dynamics of a system

Representation of Systems (Cont'd)

- Examples of:
  - State-Space Equations



Mass-Spring System with Hardening Spring again

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky + ka^2y^3 = f(t)$$

Defining state variables $x_1 = y, x_2 = \frac{dy}{dt}$ and input $u = f(t)$

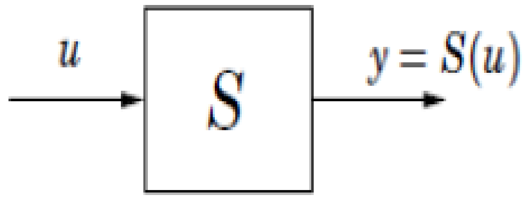
$$\dot{x_1} = x_2 = f_1(x_1, x_2, u)$$

$$\dot{x_2} = \frac{1}{m}(-\beta x_2 - kx_1 + ka^2{x_1}^3 + u) = f_2(x_1, x_2, u)$$

$$y = x_1 = g(x_1, x_2, u)$$

- First two equations above are state equations
- Last one is output equation
- State equations are the most commonly used means of describing dynamics of systems
- They always involve the use of differential equations

Overview of Linear Systems



Definitions: The system S is *linear* if

$$\begin{aligned} S(\alpha u) &= \alpha S(u), && \text{scaling} \\ S(u_1 + u_2) &= S(u_1) + S(u_2), && \text{superposition} \end{aligned}$$

Example (Input-Output/Algebraic Representation)

If $y = 17u$, then $S(u) = 17u$

(1)
$$\begin{aligned} S(10u) &= 17(10u) \\ &= 10(17u) = 10S(u) && \text{(Scaling)} \end{aligned}$$

(2)
$$\begin{aligned} S(u_1 + u_2) &= 17(u_1 + u_2) \\ &= 17u_1 + 17u_2 = S(u_1) + S(u_2) && \text{(Superposition)} \end{aligned}$$

Overview of Linear Systems (Cont'd)

Example (Linear Differential Equation)

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

Scaling:

$$b_m \frac{d^m [10u(t)]}{dt^m} + b_{m-1} \frac{d^{m-1} [10u(t)]}{dt^{m-1}} + \cdots + b_1 \frac{d[10u(t)]}{dt} + b_0 [10u(t)]$$

$$= 10 \left[a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) \right]$$

$$10\, f\left(y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{n-1} y(t)}{dt^{n-1}}, \frac{d^n y(t)}{dt^n} \right)$$
$$= g\left(10u(t), \frac{d[10u(t)]}{dt}, \ldots, \frac{d^{m-1}[10u(t)]}{dt^{m-1}}, \frac{d^m [10u(t)]}{dt^m} \right)$$

Overview of Linear Systems (Cont'd)

Superposition:

$$b_m \frac{d^m[u_1(t)+u_2(t)]}{dt^m} + b_{m-1}\frac{d^{m-1}[u_1(t)+u_2(t)]}{dt^{m-1}} + \cdots + b_1 \frac{d[u_1(t)+u_2(t)]}{dt} + b_0[u_1(t) + u_2(t)]$$

$$= b_m \frac{d^m[u_1(t)]}{dt^m} + b_{m-1}\frac{d^{m-1}[u_1(t)]}{dt^{m-1}} + \cdots + b_1 \frac{d[u_1(t)]}{dt} + b_0[u_1(t)]$$

$$+ b_m \frac{d^m[u_2(t)]}{dt^m} + b_{m-1}\frac{d^{m-1}[u_2(t)]}{dt^{m-1}} + \cdots + b_1 \frac{d[u_2(t)]}{dt} + b_0[u_2(t)]$$

$$= a_n \frac{d^n[y_1(t)]}{dt^n} + a_{n-1}\frac{d^{n-1}[y_1(t)]}{dt^{n-1}} + \cdots + a_1 \frac{d[y_1(t)]}{dt} + a_0[y_1(t)]$$

$$+ a_n \frac{d^n[y_2(t)]}{dt^n} + a_{n-1}\frac{d^{n-1}[y_2(t)]}{dt^{n-1}} + \cdots + a_1 \frac{d[y_2(t)]}{dt} + a_0[y_2(t)]$$

$$f\left([y_1+y_2], \frac{d[y_1+y_2]}{dt}, \dots, \frac{d^{n-1}[y_1+y_2]}{dt^{n-1}}, \frac{d^n[y_1+y_2]}{dt^n}\right)$$

$$= g\left([u_1+u_2], \frac{d[u_1+u_2]}{dt}, \dots, \frac{d^{m-1}[u_1+u_2]}{dt^{m-1}}, \frac{d^m[u_1+u_2]}{dt^m}\right)$$

$$= g\left([u_1], \frac{d[u_1]}{dt}, \dots, \frac{d^{m-1}[u_1]}{dt^{m-1}}, \frac{d^m[u_1]}{dt^m}\right)$$

$$+ g\left([u_2], \frac{d[u_2]}{dt}, \dots, \frac{d^{m-1}[u_2]}{dt^{m-1}}, \frac{d^m[u_2]}{dt^m}\right)$$

Overview of Linear Systems (Cont'd)

Example (Linear State Equations)

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Scaling:

If $u_n = Ku(t)$

$$K\dot{x}(t) = AKx(t) + BKu(t)$$

Thus

$$Ku(t) \text{ corresponds to } Kx(t)$$

Therefore, output equation becomes

$$C\big(Kx(t)\big) + D\big(Ku(t)\big) = K\big(Cx(t) + Du(t)\big) = Ky(t)$$

Therefore, for input $Ku(t)$

$$K\dot{x}(t) = f\big(Kx(t), K\,u(t)\big)$$
$$Ky(t) = g\big(Kx(t), Ku(t)\big)$$

Superposition

If u_A yields y_A by x_A, u_B yields y_B by x_B

$$\dot{x}_A(t) = Ax_A(t) + Bu_A(t)$$
$$y_A(t) = Cx_A(t) + Du_A(t)$$

$$\dot{x}_B(t) = Ax_B(t) + Bu_B(t)$$
$$y_B(t) = Cx_B(t) + Du_B(t)$$

$$(x_A \dot{+} x_B) = A(x_A + x_B) + B(u_A + u_B)$$
$$(y_A + y_B) = C(x_A + x_B) + D(u_A + u_B)$$

Time-Invariant Systems

A system is time-invariant (or autonomous) if the coefficients of the expressions in the representation of the system (input-output equations, differential equations or state equations) are all constants.

Mathematically,

- Algebraic input-output Expression
  - $\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{K}\boldsymbol{u}(\boldsymbol{t})$

    $\boldsymbol{K}$ is constant

  - $\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t})\big) = \boldsymbol{0}$

    Coefficients of $\boldsymbol{y}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t})$ or their products/powers are all constants.

Time-Invariant Systems (contd.)

- Differential-Equation Expression

➢ $a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$

$a_i, i = 1, 2, \ldots, n;\ b_j, j = 1, 2, \ldots, m$ are all constants

➢ $f\left(y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{n-1} y(t)}{dt^{n-1}}, \frac{d^n y(t)}{dt^n}\right) = g\left(u(t), \frac{du(t)}{dt}, \ldots, \frac{d^{m-1} u(t)}{dt^{m-1}}, \frac{d^m u(t)}{dt^m}\right)$

All coefficients of $\frac{d^i y(t)}{dt^i}, i = 0, 1, \ldots, n$; $\frac{d^j y(t)}{dt^j}, j = 0, 1, \ldots, m,$ and products/powers of these differentials are all constants (no functions of time in the expressions for the coefficients).

Time-Invariant Systems (contd.)

- State Equations

  - $$\dot{x}(t) = Ax(t) + Bu(t)$$
    $$y(t) = Cx(t) + Du(t)$$

    $A, B, C,$ and D are all constants

  - $$\dot{x}(t) = f\left(x(t), u(t)\right)$$
    $$y(t) = g\left(x(t), u(t)\right)$$

    All coefficients of $x(t), u(t)$, and their products/powers are all constants (no functions of time in the expressions for the coefficients).

Time-Invariant Systems (contd.)

- Alternatively, a system is time-invariant or autonomous if delaying the input results in a delayed output

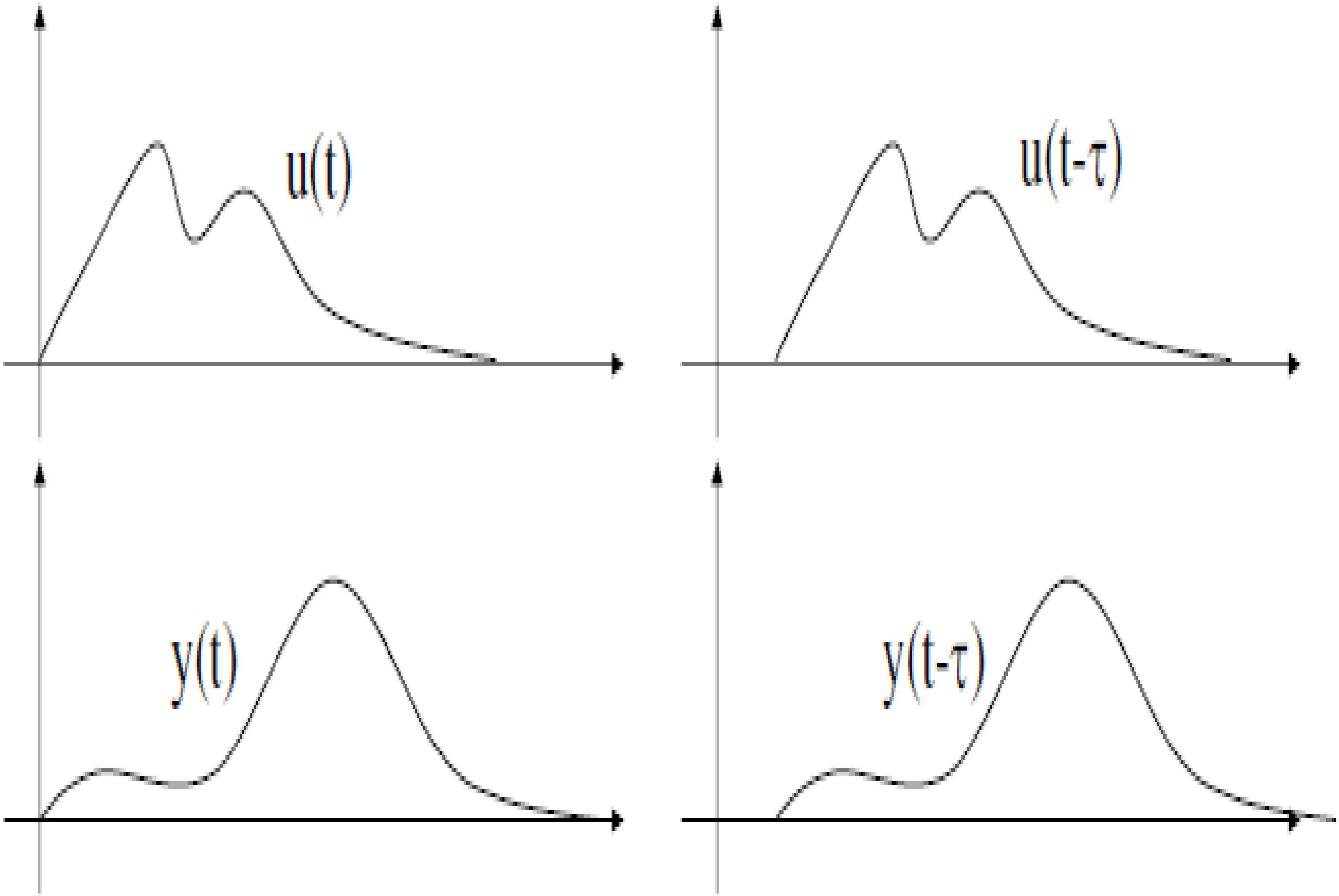
$$y(t-\tau) = g\big(u(t-\tau)\big)$$

- This can be shown for the input-output, differential and state equations above.

Simple Illustration:
For Algebraic Equation (Time-Invariant)

$$u(t) = \sin t\,, y(t) = \sin^4 t\,, y = u^4$$
$$y = g(u) = u^4$$
$$u(t-\tau) = \sin(t-\tau),$$
$$y(t-\tau) = \sin^4(t-\tau) = [\sin(t-\tau)]^4 = [u(t-\tau)]^4,$$
$$y(t-\tau) = g\big(u(t-\tau)\big)$$

Time-Invariant Systems (contd.)



Time-Varying Systems

A system is time-varying (or non-autonomous) if at least one of the coefficients of the expressions in the representation of the system (input-output equations, differential equations or state equations) is a function of time.

Mathematically,

- Algebraic input-output Expression
  - $\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{K}(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})$

    $\boldsymbol{K}$ is a function of time

  - $\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t})\big) = \boldsymbol{0}$

At least one of the coefficients of $\boldsymbol{y}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t})$ or their products/powers is a function of time.

Time-Varying Systems (Cont'd)

- Differential-Equation Expression

➢ $a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$

At least one of $a_i, i = 1, 2, \ldots, n;\ b_j, j = 1, 2, \ldots, m$ is a function of time.

➢ $f\left(y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{n-1} y(t)}{dt^{n-1}}, \frac{d^n y(t)}{dt^n}\right) = g\left(u(t), \frac{du(t)}{dt}, \ldots, \frac{d^{m-1} u(t)}{dt^{m-1}}, \frac{d^m u(t)}{dt^m}\right)$

At least one of the coefficients of $\frac{d^i y(t)}{dt^i}, i = 0, 1, \ldots, n;\ \frac{d^j y(t)}{dt^j}, j = 0, 1, \ldots, m$, or products/powers of these differentials is a function of time.

Time-Varying Systems (Cont'd)

- State Equations

➢
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

At least one of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, and $\boldsymbol{D}$ is a constant

➢
$$\dot{x}(t) = f\big(x(t), u(t)\big)$$
$$y(t) = g\big(x(t), u(t)\big)$$

At least one of the coefficients of $x(t), u(t)$, and their products/powers is a function of time.

Time-Varying Systems (Cont'd)

- Also, a system is time-varying or non-autonomous if delaying the input does not result in a delayed output

$$y(t-\tau) \neq g\big(u(t-\tau)\big)$$

- Again, this can be shown for the input-output, differential and state equations above.

Another Illustration:

For Algebraic Equation (Time-Varying)

$$u(t) = e^t, y(t) = t^2 e^{3t}, y = t^2 u^3$$

$$y = g(u,t) = t^2 u^3$$

$$u(t-\tau) = e^{t-\tau},$$

$$y(t-\tau) = (t-\tau)^2[u(t-\tau)]^3 = (t-\tau)^2[e^{t-\tau}]^3 = (t-\tau)^2 e^{3(t-\tau)},$$

$$g\big(u(t-\tau)\big) = t^2[u(t-\tau)]^3 = t^2[e^{t-\tau}]^3 = t^2 e^{3(t-\tau)}$$

$$y(t-\tau) \neq g\big(u(t-\tau)\big)$$

Classification of Systems (Linearity and Time-Variation Considerations)

- Linear, Time-Invariant (LTI) Systems
- Nonlinear, Time-Invariant Systems
- Linear, Time-Varying Systems
- Nonlinear, Time-Varying Systems

Classification Criteria for Nonlinear Systems

The following are some criteria for classification of nonlinearities:

- **Criterion 1: Level of Importance of Nonlinearity to Operation of System**
- **Criterion 2: Inherence or Otherwise of Nonlinearity within a System**
- **Criterion 3: Mathematical Properties (Continuity or Otherwise of Nonlinearity)**
- **Criterion 4: Mathematical Properties ("Single-Valuedness" or otherwise of Nonlinearity)**
- **Criterion 5: Dynamic Behaviour of System**

Assignment

1. **Write the equations representing the four system classifications based on linearity and time variations consideration**
2. **Compare 5 Linear and Non-Linear Systems properties**