COURSE TITLE:

Introduction to Non-Linear Systems

COURSE CODE:

EEE 566

LECTURE 3

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Modelling of Simple Nonlinear Systems

Some Background Information

- In Control Engineering, there are three major tasks that are performed in the process of trying to improve upon the operations of systems:
  - Modelling
  - Analysis
  - Design of Controllers

- Taking the first letters of the tasks above, I like to say that control involves the MAD combination!

Crude Definition of Modelling

- Modelling is the investigation of the behaviour of a system as indicated by how certain variables of interest (outputs) change with time under the influence of changes in manipulated variables (inputs) and external disturbances.

- The processes of modelling and analysis lead to the subsequent design of appropriate corrective measures (controllers) for the system.

Importance of Modelling

- Modelling is important for many reasons. Some of them are:
  - **Operator Training**: People can learn the proper ways to respond to different system conditions before having to experience them on the real system.
  - **Design of Equipment**: The information provided by a model can help in design of equipment for desired rates of production or performance.
  - **Design of Safety Systems**: Models can provide information for taking safety steps in the event of failure of some vital components.
  - **Control of Systems and Processes**:

❖ Control system design is based on models.

❖ Before implementation, models are tested by simulation.

Approaches to Modelling

- Two main approaches:
  - **Experimental Approach**:
    - Physical equipment of the system available to the Control Engineer.
    - Values of various inputs (manipulated variables and disturbances) are changed;
    - Through use of appropriate measuring devices, outputs of the system are observed and recorded.
    - This procedure is tedious and time-consuming;
    - However, this procedure is applicable for systems where mathematical modelling may not be possible.
  - **Theoretical Approach**
    - This is usually a set of mathematical equations (differential or algebraic) whose solution yields the dynamic or static behaviour of the system under consideration
    - All tools we have been introduced to in earlier control courses are in aid of mathematical modelling.

Examples of Physical and Material-Balance Laws for Modelling

- Physical Laws:
  - Newtonian Motion Laws (First, Second, and Third)
  - Quantum Mechanical Laws
  - Relativistic Mechanical Laws
  - Hooke's Law (Spring Systems)
  - Kirchhoff's, Ohm's Laws (Electrical Systems)

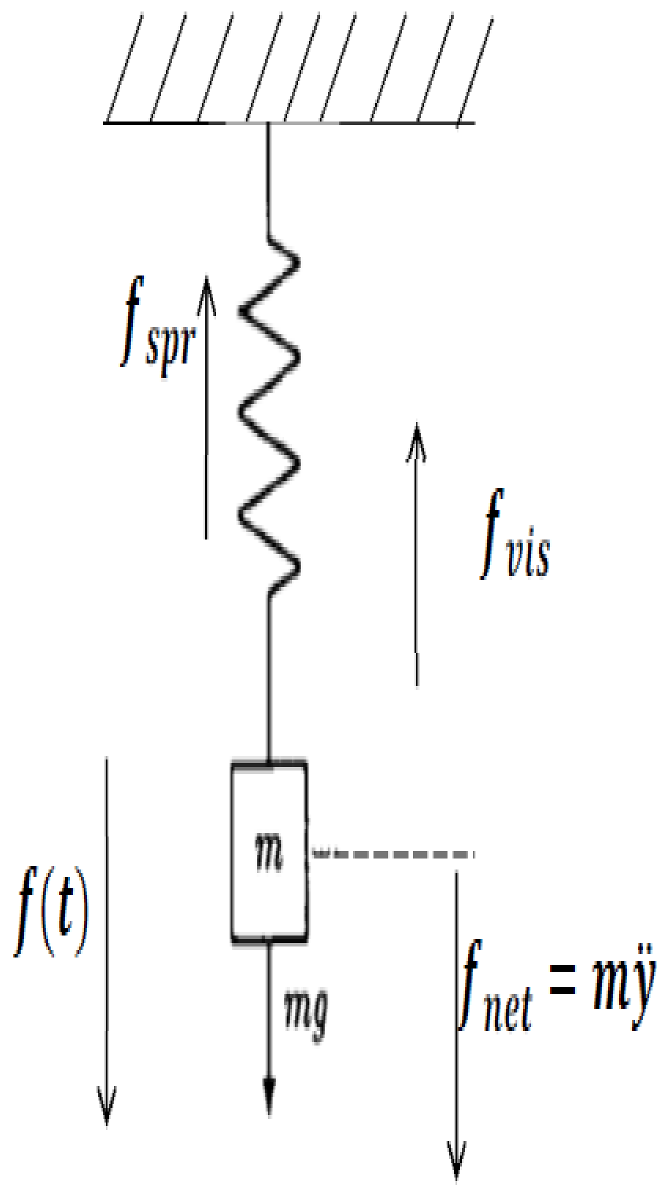
- Material-Balance Laws
  - Bernoulli and Torricelli's Law (Chemical Process Systems)
  - Mass and Energy-Balance Equations
  - Transport Rate Equations (Transport Phenomena)
  - Kinetic Rate Equations (Chemical Kinetics)
  - Reactions and Phase-Equilibria Equations (Thermodynamics)

Linear Constitutive Relations for Common Systems

System	Constitutive Relation for		
	Energy-Storage Elements		Energy Dissipating Elements
Type	*A*-Type (across) Element	*T*-Type (through) Element	*D*-Type (dissipative) Element
Translatory-mechanical v = velocity f = force	Mass $m\frac{dv}{dt} = f$ (Newton's second law) m = mass	Spring $\frac{df}{dt} = kv$ (Hooke's law) k = stiffness	Viscous damper $f = bv$ b = damping constant
Electrical v = voltage i = current	Capacitor $C\frac{dv}{dt} = i$ C = capacitance	Inductor $L\frac{di}{dt} = v$ L = inductance	Resistor $Ri = v$ R = resistance
Thermal T = temperature difference Q = heat transfer rate	Thermal capacitor $C_t\frac{dT}{dt} = Q$ C_t = thermal capacitance	None	Thermal resistor $R_t Q = T$ R_t = thermal resistance
Fluid P = pressure difference Q = volume flow rate	Fluid capacitor $C_f\frac{dP}{dt} = Q$ C_f = fluid capacitance	Fluid inertor $I_f\frac{dQ}{dt} = P$ I_f = inertance	Fluid resistor $R_f Q = P$ R_f = fluid resistance

Example 1: Modeling of a Mechanical System

A Mechanical System



System: Mass-Spring-Damper System with a Hardening Spring

Preliminaries

- This system is commonly used to model many mechanical systems where the application of force is being used to effect changes in displacement and associated variables.
- Here, a displacement takes place when a force is applied on a mass suspended by a spring
- The applied force $f(t)$ and the gravitational force mg act to pull the mass down, while forces f_{spr} and f_{vis} due respectively to the spring and viscous friction act to oppose these forces.
- Unlike springs with linear force-displacement relationships, this particular spring has a **nonlinear** force-displacement relationship

$$f_{spr} = ky(1 + a^2y^2)$$

A Mechanical System (Cont'd)

- The use of Newton's Second Law of Motion, Viscous Friction Relationship and Spring Laws give rise to the relevant force-balance differential equation

$$m\ddot{y} = f(t) + mg - \beta\dot{y} - ky(1 + a^2y^2)$$

- This can be rearranged to give

$$m\ddot{y} + \beta\dot{y} + ky(1 + a^2y^2) = f(t) + mg$$

- Defining state variables

$$x_1 = y; x_2 = \dot{y}$$

gives the following state equations

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -\frac{k}{m}x_1 - \frac{k}{m}a^2{x_1}^3 - \frac{\beta}{m}x_2 + \frac{f(t)}{m} + g$$

- Since $f(t)$ is the force applied on the mass, it is the input and therefore can be replaced with the symbol u
- We can then re-write the state equations above as

$$\dot{x_1} = f_1(x_1, x_2, u)$$

$$\dot{x_2} = f_2(x_1, x_2, u)$$

and the output equation as

$$y = g(x_1, x_2, u)$$

where

$$f_1(x_1, x_2, u) = x_2$$

$$f_2(x_1, x_2, u) =$$

$$-\frac{k}{m}x_1 - \frac{k}{m}a^2{x_1}^3 - \frac{\beta}{m}x_2 + \frac{u}{m} + g$$

$$g(x_1, x_2, u) = x_1$$

A Mechanical System (Cont'd)

Determination of Equilibrium Points

- For equilibrium, the states must be steady i.e. the states must be unchanging with time i.e.

$$\dot{x}_1 = 0, \dot{x}_2 = 0$$

- This leads to

$$\dot{x}_1 = x_2 = 0$$

and

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{k}{m}a^2{x_1}^3 - \frac{\beta}{m}x_2 + \frac{u}{m} + g = 0$$

- Putting $x_2 = 0$ into the expression for $\dot{x}_2$ yields

$$-\frac{k}{m}x_1 - \frac{k}{m}a^2{x_1}^3 + \frac{u}{m} + g = 0$$

A Mechanical System (Cont'd)

- Because we have a single equation and 2 unknowns, this cannot be solved as it is.

- We therefore assign a value to the input for which we desire to determine steady-state information. We often call this input value the "nominal input value" or "nominal manipulated variable value"

- For a chosen nominal input value U_{nom}, and for determined parameter values k, m, and a, the equation

$$-\frac{k}{m}x_1 - \frac{k}{m}a^2 {x_1}^3 + \frac{U_{nom}}{m} + g = 0$$

  can then be solved to yield three values of x_1 i.e. X_{11}, X_{12} and X_{13}

- Then, we can say that the points of equilibrium of the system, for the nominal input value of U_{nom}, and parameter values of k, m, and a, are located where

$$x_1 = y = X_{11}\,;\, x_2 = \dot{y} = 0$$
$$x_1 = y = X_{12};\, x_2 = \dot{y} = 0$$
$$x_1 = y = X_{13};\, x_2 = \dot{y} = 0$$

Example 2: Modeling of PMSMs

An Electromechanical System

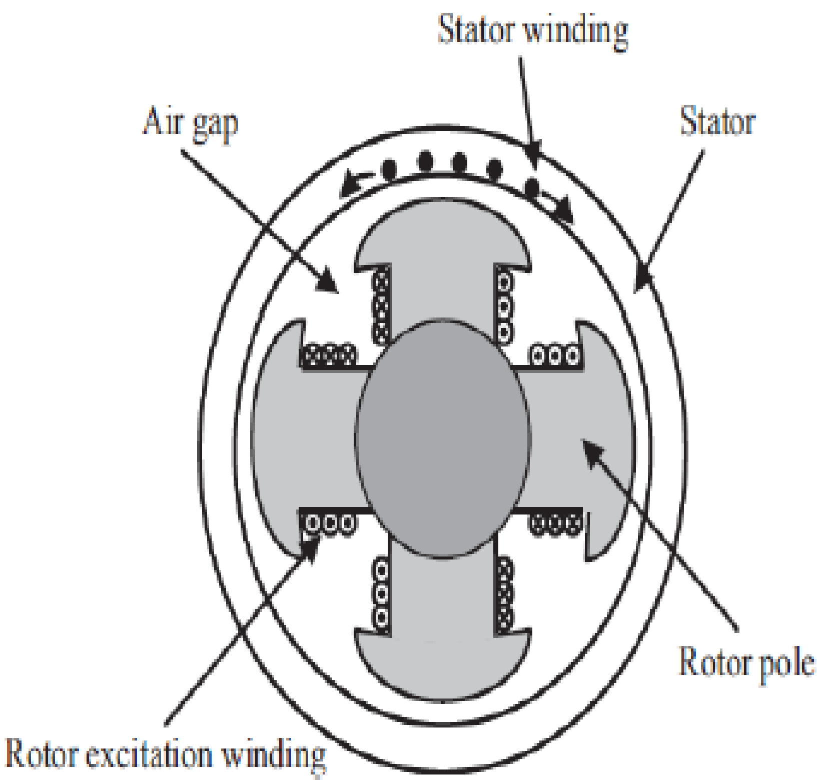
System : Permanent-Magnet Synchronous Machines

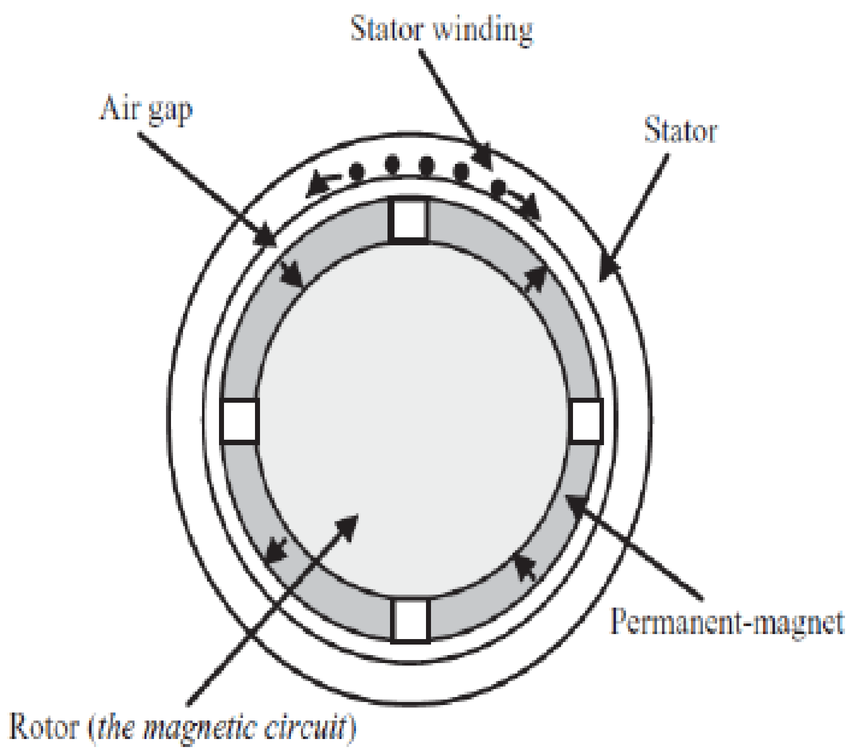
Introductory Remarks

- Synchronous machines are machines that have the rotor speed and the speed of the rotating stator-generated magnetic field synchronized, hence the name.
- Synchronous machines are well known in applications requiring speed reversions and wide-range power variations.
- The stator is composed of three identical winding distributed in space such that any two successive windings has a space of $\mathbf{120^0}$ between them.
- When the stator windings are current-fed by a balanced three-phase AC supply, a turning field is generated along the air gap between the stator and the rotor.
- The turning field generated by the stator does not make the rotor to rotate.
- The rotor therefore needs to be excited separately to begin its own rotation.

An Electromechanical System (cont'd)

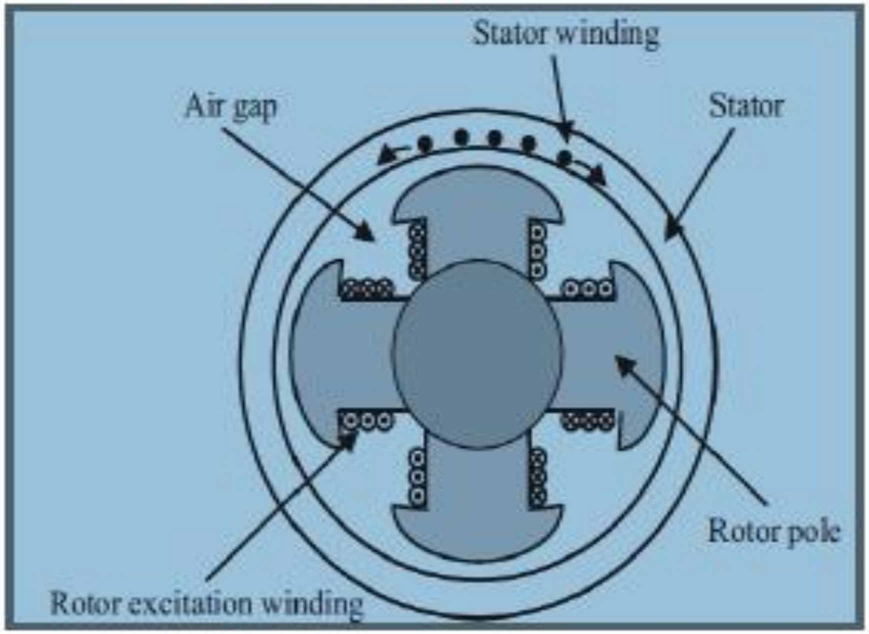
- Based on the source of this excitation, and hence the elements attached to, or associated with the rotor, synchronous machines exist in two variants i.e. **wound-rotor synchronous machines (WRSMs)** and **permanent-magnet synchronous machines (PMSMs)**.



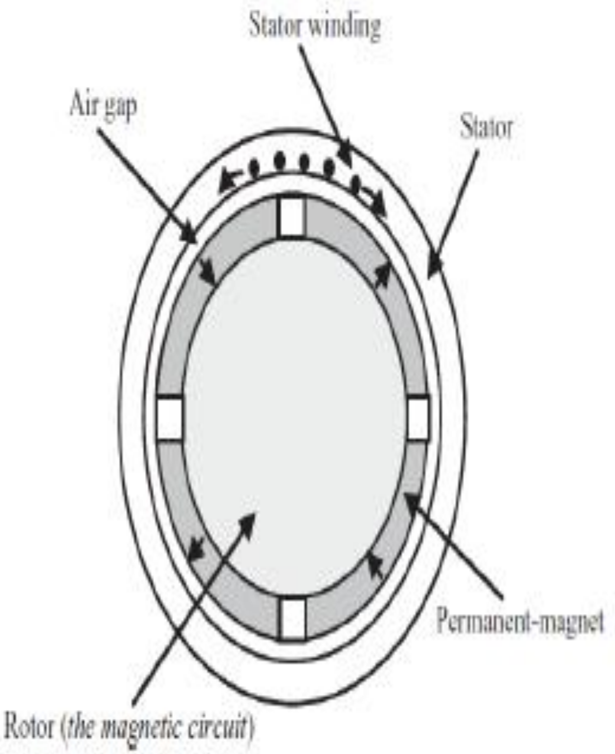


An Electromechanical System (cont’d)

- In **WRSMs**, the rotor magnetic field is generated by windings fixed on the rotor.
- These windings are fed by a dc generator to create a magnetomotive force (MMF) along the air gap between the stator and the rotor.
- The interaction between the turning field created by the stator and the magnetomotive force created by the windings on the rotor generates an electromagnetic torque that gets applied to the rotor and generates a rotation.

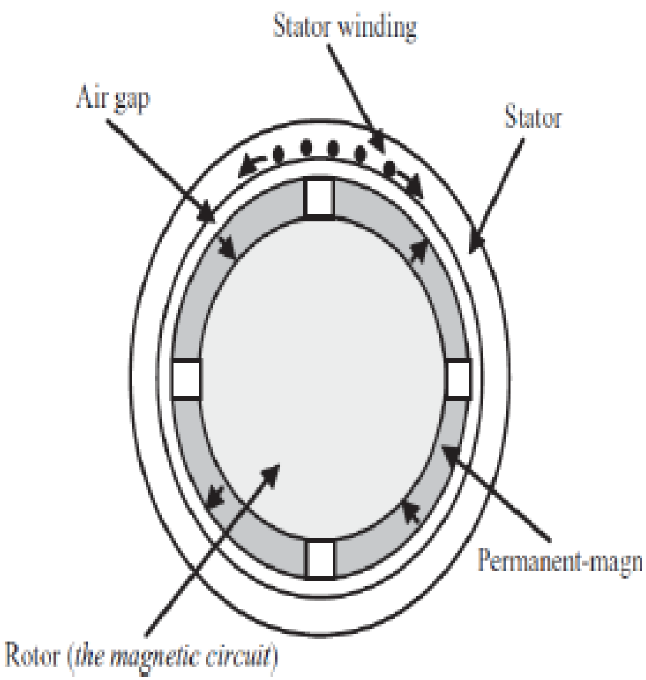


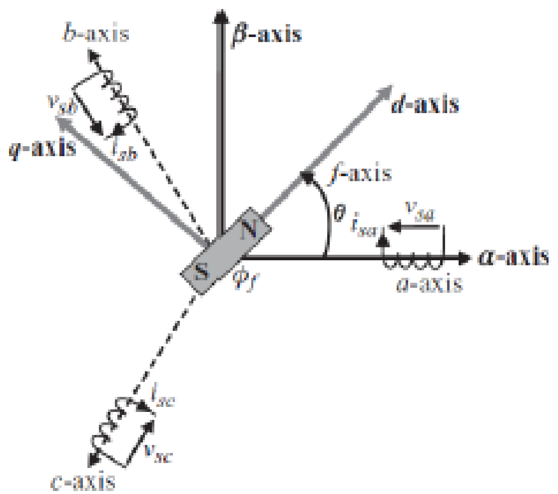
An Electromechanical System (cont'd)



- In **PMSMs**, the rotor magnetic field is generated by permanent magnets fixed on the rotor.
- These magnets need no external excitation and generate a magnetomotive force (MMF) along the air gap between the stator and the rotor.
- Again, the interaction between the turning field created by the stator and the magnetomotive force created by the permanent magnets generates an electromagnetic torque that gets applied to the rotor and generates a rotation.
- The motions of the turning stator-generated magnetic field and the rotor reach steady-state when the rotor speed becomes equal to the speed of the turning field generated by the stator.

An Electromechanical System (cont'd)





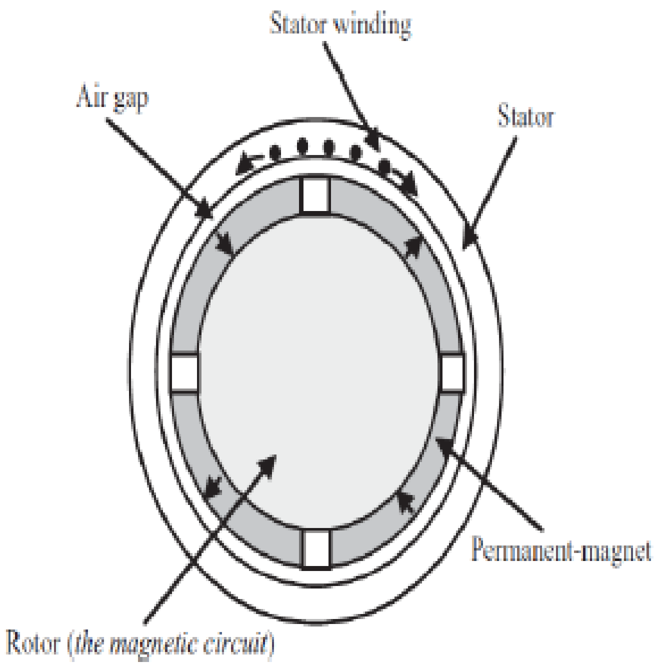
Mathematical Modelling

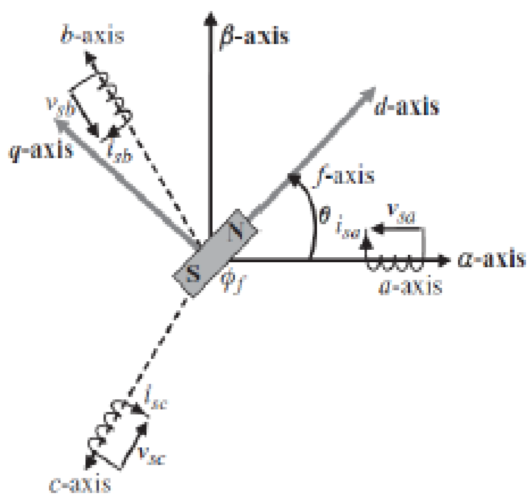
- Since we are dealing with three-phase systems, the balanced three-phase positive (or *abc*) phase sequence yields the following triplet of equations

$$x_a = A\cos(\omega t + \varphi)$$
$$x_b = A\cos\left(\omega t + \varphi - \frac{2\pi}{3}\right)$$
$$x_c = A\cos\left(\omega t + \varphi + \frac{2\pi}{3}\right)$$

  where $\boldsymbol{x}$ could represent, in this case, voltages or currents or magnetic fluxes.
- The three-coordinate frame above is usually stationary or stator-related.
- This frame is difficult to deal with when control-related applications are being considered. This is because the voltage expressions that take the derivatives of the respective fluxes in the system comprise expressions of self and mutual inductances, and the tri-dimensionality of the equations makes the equations unduly cumbersome.
- Also, there is a dependence of the fluxes on both time and rotor position.

An Electromechanical System (cont'd)





- Because of these issues, a coordinate transformation system was developed by Park and Concordia to take the stator-related, position-dependent three-phase frame to an equivalent, lower-size, position-independent rotating direct-axis-quadrature axis (or $\boldsymbol{d}-\boldsymbol{q}$) frame.
- This frame has constant inductance terms and all signals are steady-state sinusoidal along the d- and q- axes.

- Going back to the abc frame, the application of Faraday's and Ohm's laws yields the following three-phase stator voltage equations:

$$\begin{bmatrix} v_{sa} \\ v_{sb} \\ v_{sc} \end{bmatrix} = \begin{bmatrix} R_s & 0 & 0 \\ 0 & R_s & 0 \\ 0 & 0 & R_s \end{bmatrix} \begin{bmatrix} i_{sa} \\ i_{sb} \\ i_{sc} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \emptyset_{sa} \\ \emptyset_{sb} \\ \emptyset_{sc} \end{bmatrix}$$

where

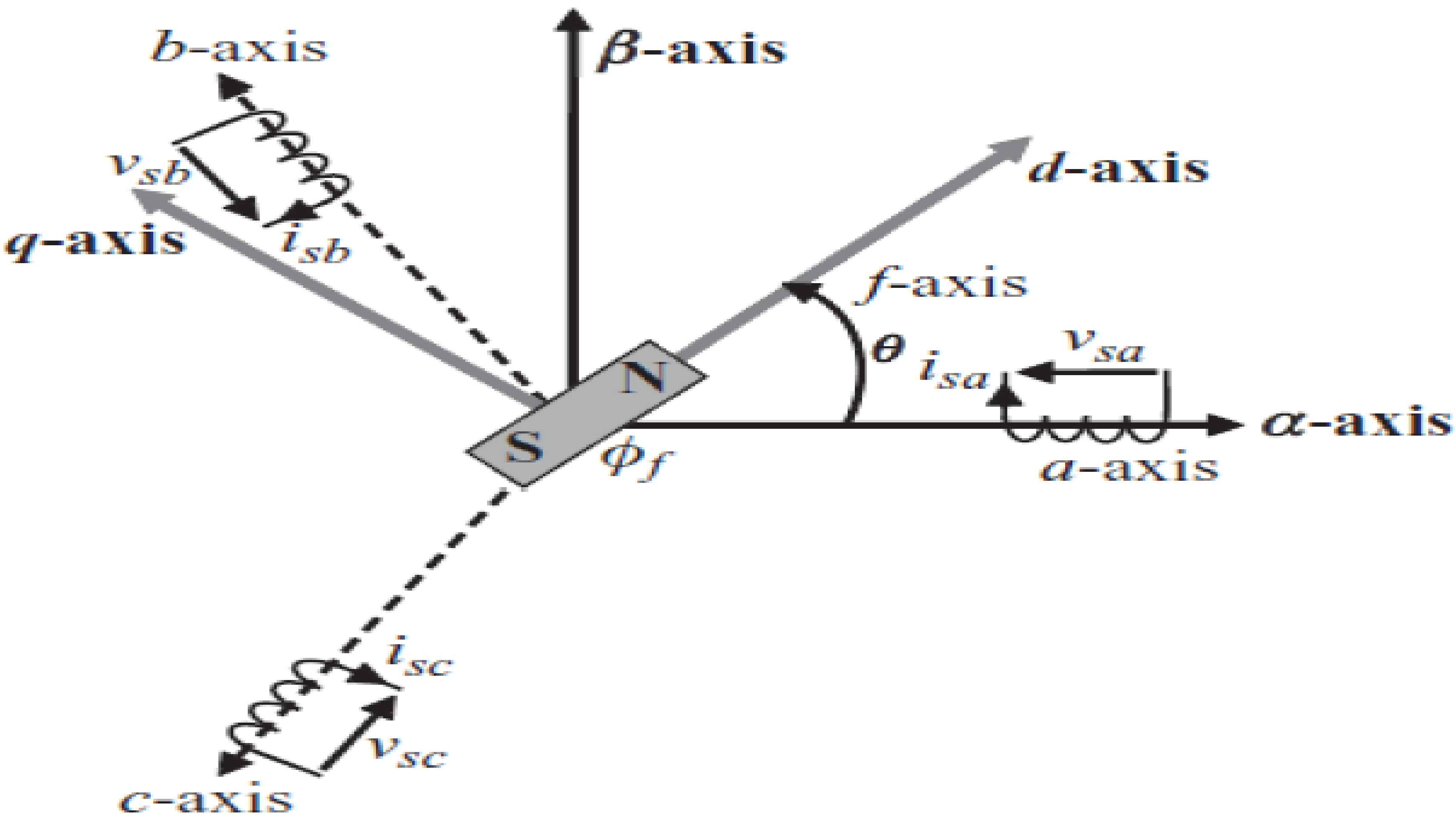
$v_{si}(i = a, b, c)$ is the stator voltage for phase i;

$i_{si}(i = a, b, c)$ is the stator current for phase i;

$\emptyset_{si}(i = a, b, c)$ is the induced flux in the stator windings for phase i

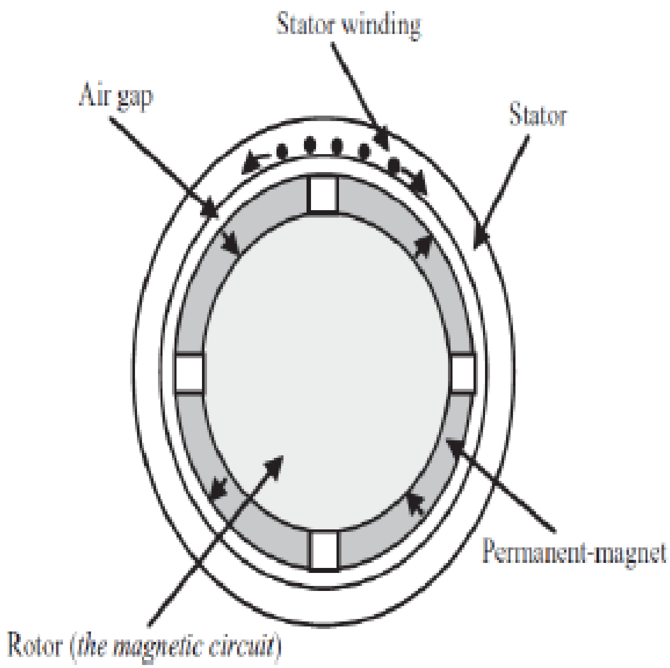
R_s is the stator winding resistance.

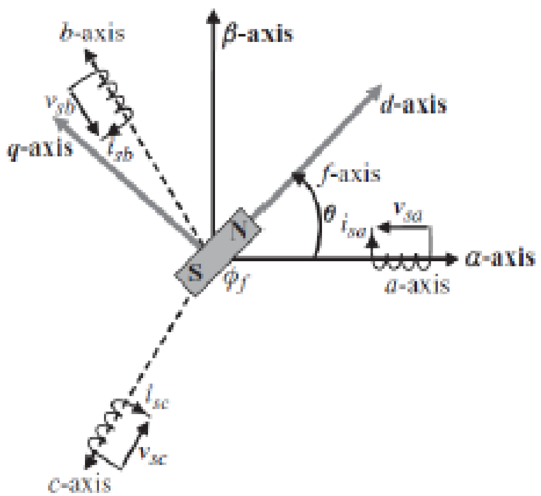
An Electromechanical System (cont'd)



The Three-Phase abc-Coordinate Frame, The $\alpha\beta$ Stationary Two-Phase Coordinate Frame, and the dq Rotating Two-Phase Coordinate Frame for the PMSM

An Electromechanical System (cont'd)





- We can write the above equation in shorthand form as

$$[v_{sabc}] = [R_s][i_{sabc}] + \frac{d}{dt}[\emptyset_{sabc}]$$

- In the rotor, a constant flux is created by the permanent magnets and a set of mutual fluxes is generated between the magnetic field of the rotor's permananet magnets and the rotating magnetic field generated by the stator.
- These fluxes can be written in the *abc*-frame as:

$$\emptyset_a = \emptyset_r \cos(p\theta)$$

$$\emptyset_b = \emptyset_r \cos\left(p\theta - \frac{2\pi}{3}\right)$$

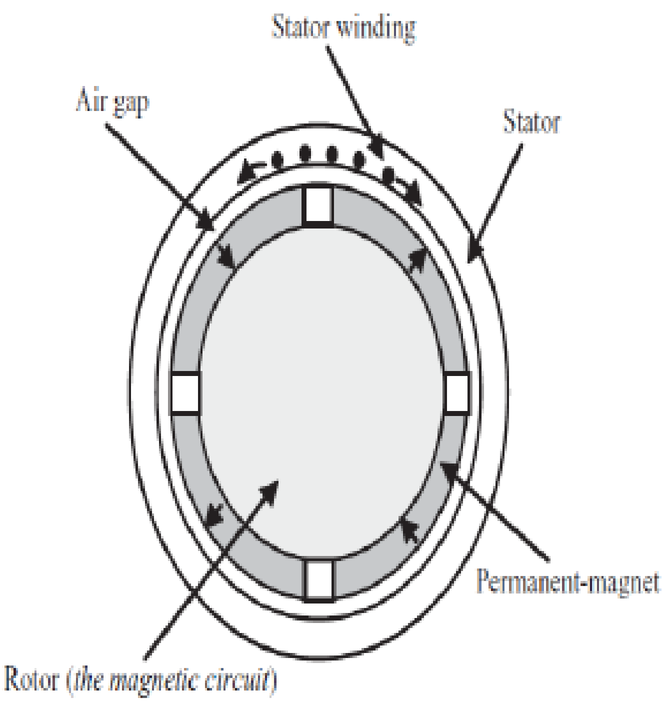
$$\emptyset_c = \emptyset_r \cos\left(p\theta + \frac{2\pi}{3}\right)$$

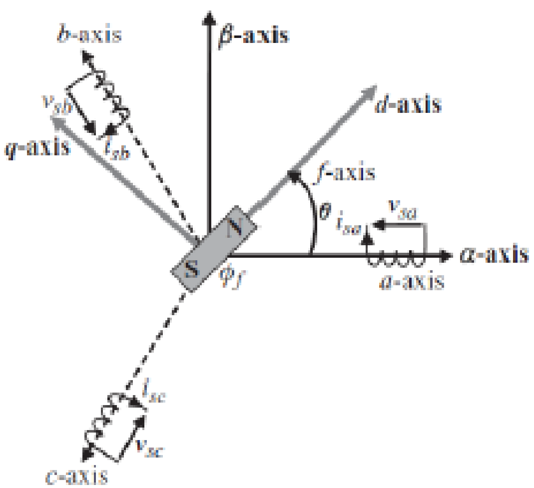
where $\emptyset_r$ is the amplitude of the flux produced by the magnets.

- We can therefore say that the flux through each of the stator windings is the sum of the flux induced by the rotor magnets and the flux produced by the currents carried by the stator phases, or

$$[\emptyset_{sabc}] = [L_{ss}][i_{sabc}] + [\emptyset_{rabc}]$$

An Electromechanical System (cont'd)





- Thus, the stator voltage equation then becomes

$$[\boldsymbol{v}_{sabc}] = [\boldsymbol{R}_s][\boldsymbol{i}_{sabc}] + \frac{\boldsymbol{d}}{\boldsymbol{dt}}[[\boldsymbol{L}_{ss}][\boldsymbol{i}_{sabc}] + [\emptyset_{rabc}]]$$

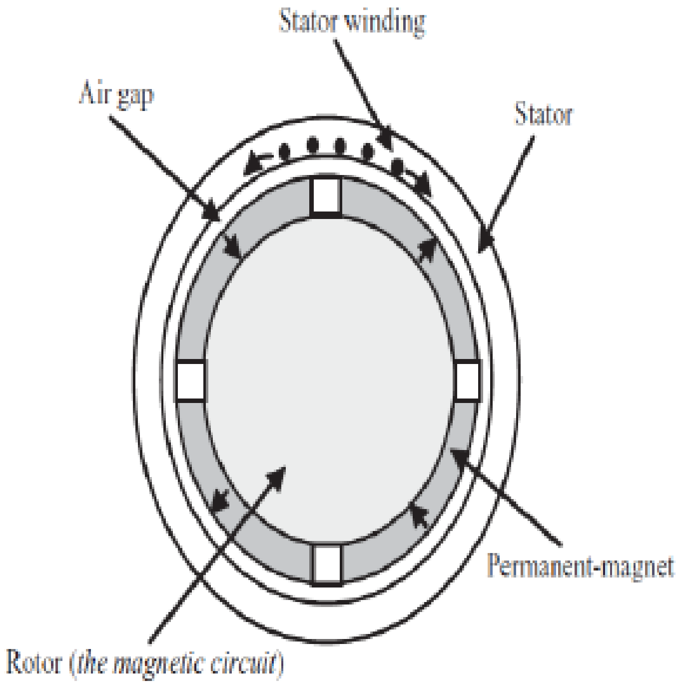
$$[\boldsymbol{v}_{sabc}] = [\boldsymbol{R}_s][\boldsymbol{i}_{sabc}] + \frac{\boldsymbol{d}}{\boldsymbol{dt}}[[\boldsymbol{L}_{ss}][\boldsymbol{i}_{sabc}]] + \frac{\boldsymbol{d}}{\boldsymbol{dt}}[[\emptyset_{rabc}]]$$

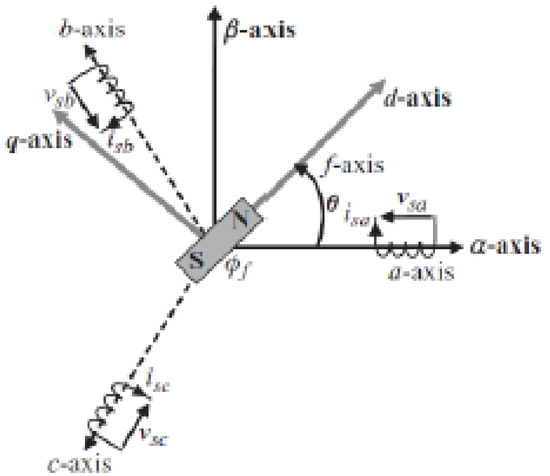
- Since ${}^{\boldsymbol{d}}/_{\boldsymbol{dt}}(\blacksquare) = {}^{\boldsymbol{d}}/_{\boldsymbol{d\theta}} \cdot {}^{d\theta}/_{dt}(\blacksquare) = {}^{\boldsymbol{d\theta}}/_{\boldsymbol{dt}} \cdot {}^{\boldsymbol{d}}/_{\boldsymbol{d\theta}}(\blacksquare)$ and ${}^{d\theta}/_{dt}$ represents the rotor speed, then the above equation can be re-written as

$$[\boldsymbol{v}_{sabc}] = [\boldsymbol{R}_s][\boldsymbol{i}_{sabc}] + \frac{d}{\boldsymbol{dt}}[[\boldsymbol{L}_{ss}][\boldsymbol{i}_{sabc}]] + \frac{\boldsymbol{d\theta}}{\boldsymbol{dt}}\frac{\boldsymbol{d}}{\boldsymbol{d\theta}}[[\emptyset_{rabc}]]$$

$$[\boldsymbol{v}_{sabc}] = [\boldsymbol{R}_s][\boldsymbol{i}_{sabc}] + \frac{\boldsymbol{d}}{\boldsymbol{dt}}[[\boldsymbol{L}_{ss}][\boldsymbol{i}_{sabc}]] + \boldsymbol{\omega}\frac{\boldsymbol{d}}{\boldsymbol{d\theta}}[[\emptyset_{rabc}]]$$

An Electromechanical System (cont'd)





- It can be shown that through the use of the Concordia-Park transformation

$$\begin{bmatrix} x_d \\ x_q \end{bmatrix} = P(\rho)^T C_{32}{}^T \begin{bmatrix} x_a \\ x_b \\ x_c \end{bmatrix}$$

where

$$P(\rho) = \begin{bmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{bmatrix}$$

with ρ representing the angular position of the rotating reference frame, and

$$C_{32} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & 0 \\ -{}^{1}/_{2} & {}^{\sqrt{3}}/_{2} \\ -{}^{1}/_{2} & -{}^{\sqrt{3}}/_{2} \end{bmatrix}$$

that the stator voltage equations in the dq-frame can be written as

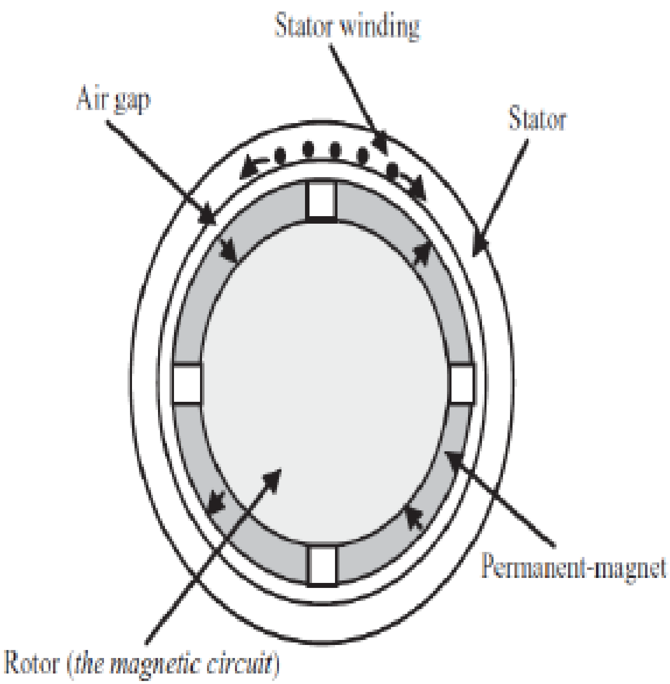
$$[v_{sdq}] = [R_s][i_{sdq}] + [L_{dq}]\frac{d}{dt}[[i_{sdq}]] + p\omega Q'[L_{dq}][i_{sdq}] + p\omega Q'[\emptyset_{rdq}]$$

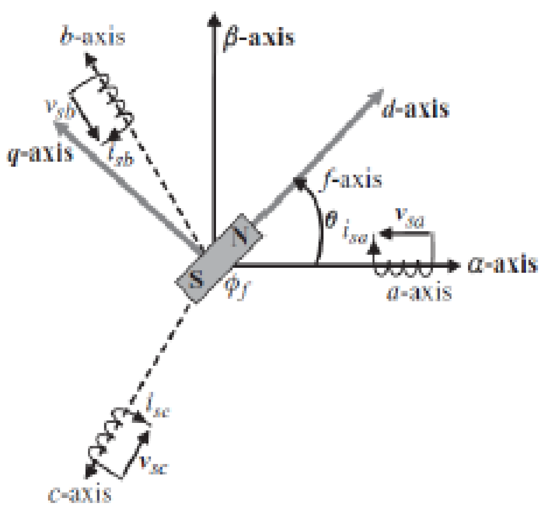
where

p is a proportionality constant between the electrical equivalent of the angular position ρ and the rotor angular displacement θ; and

$$Q' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

An Electromechanical System (cont'd)





- After appropriate substitutions, the eventual stator voltage equations in the ***dq*** frame can be written as

$$v_{sd} = R_s i_{sd} + L_{sd}\frac{di_{sd}}{dt} - p\omega L_{sq} i_{sq}$$

$$v_{sq} = R_s i_{sq} + L_{sq}\frac{di_{sq}}{dt} + p\omega L_{sd} i_{sd} + p\omega\sqrt{3/2}\,\emptyset_r$$

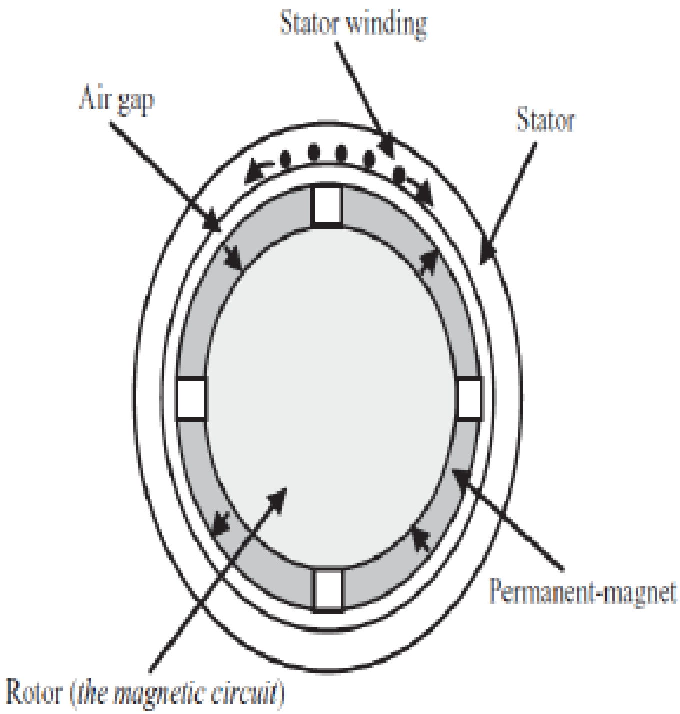
With the term $p\omega\sqrt{3/2}\,\emptyset_r$ being the voltage drop associated with the permanent-magnet flux.

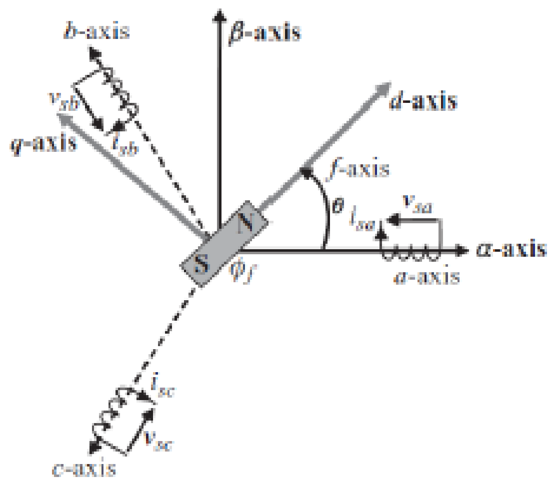
- Re-arranging the equations yields

$$\frac{di_{sd}}{dt} = \frac{1}{L_{sd}}\left[v_{sd} - R_s i_{sd} + p\omega L_{sq} i_{sq}\right]$$

$$\frac{di_{sq}}{dt} = \frac{1}{L_{sq}}\left[v_{sq} - p\omega L_{sd} i_{sd} - R_s i_{sq} - p\omega\sqrt{3/2}\,\emptyset_r\right]$$

An Electromechanical System (cont'd)





- If we are doing current control, the two currents $\boldsymbol{i_{sd}}$ and $\boldsymbol{i_{sq}}$ becomes the outputs i.e.

$$y_1 = i_{sd}$$
$$y_2 = i_{sq}$$

- The two outputs are also the two states of the system, as seen in the differential equation above.
- The two currents $\boldsymbol{i_{sd}}$ and $\boldsymbol{i_{sq}}$ are manipulated by the corresponding voltages $\boldsymbol{v_{sd}}$ and $\boldsymbol{v_{sq}}$ respectively. Therefore, the inputs are the voltages $\boldsymbol{v_{sd}}$ and $\boldsymbol{v_{sq}}$.
- If we use standard state and input notations to replace the current and voltage symbols respectively, we have

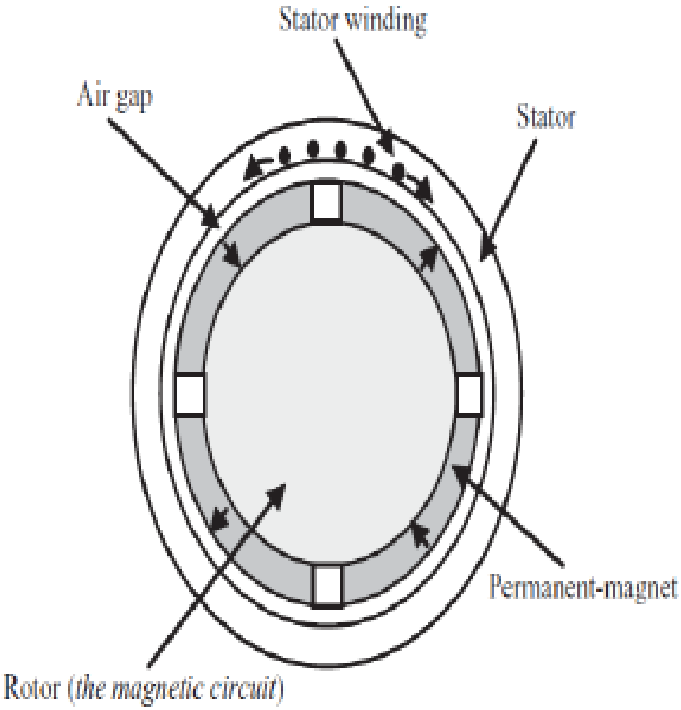
$$\frac{dx_1}{dt} = \frac{1}{L_{sd}}\left[-R_s x_1 + p\omega L_{sq} x_2 + u_1\right]$$

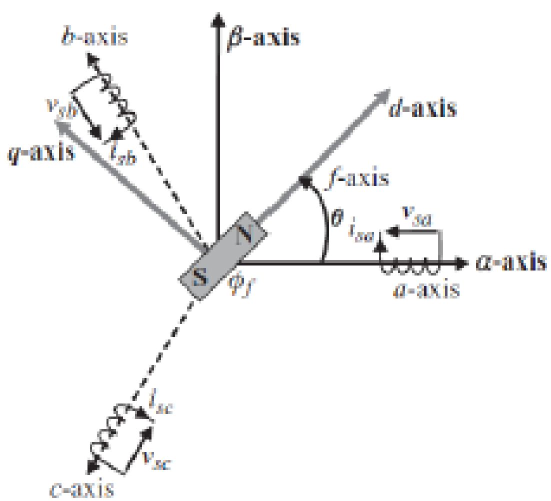
$$\frac{dx_2}{dt} = \frac{1}{L_{sq}}\left[-p\omega L_{sd} x_1 - R_s x_2 + u_2 - p\omega \sqrt{{}^{3}/_{2}}\, \phi_r\right]$$

and the output equations

$$y_1 = x_1$$
$$y_2 = x_2$$

An Electromechanical System (cont'd)





- Thus, we can say

$$\dot{x}_1 = f_1(x_1, x_2, u_1, u_2)$$
$$\dot{x}_2 = f_2(x_1, x_2, u_1, u_2)$$
$$y_1 = g_1(x_1, x_2, u_1, u_2)$$
$$y_2 = g_2(x_1, x_2, u_1, u_2)$$

Where

$$f_1(x_1, x_2, u_1, u_2) = \frac{1}{L_{sd}}\left[-R_s x_1 + p\omega L_{sq} x_2 + u_1\right]$$

$$f_2(x_1, x_2, u_1, u_2) = \frac{1}{L_{sq}}\left[-p\omega L_{sd} x_1 - R_s x_2 + u_2 - p\omega \sqrt{3/2}\, \phi_r\right]$$

$$g_1(x_1, x_2, u_1, u_2) = x_1$$
$$g_2(x_1, x_2, u_1, u_2) = x_2$$

An Electromechanical System (cont'd)

Determination of Equilibrium Points

- Again, for equilibrium, the states must be steady i.e. the states must be unchanging with time i.e.

$$\dot{x}_1 = 0, \dot{x}_2 = 0$$

- This leads to

$$\frac{dx_1}{dt} = f_1(x_1, x_2, u_1, u_2) = 0$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2, u_1, u_2) = 0$$

- Again, we have 2 equations and 4 unknowns, this cannot be solved as it is.

- We therefore again assign nominal values to the 2 inputs for which we desire to determine steady-state information as we did for the two previous cases.

- Solving the equations for the nominal input values and for specific parameter values will give the values of the direct- and quadrature-axes currents at steady state.

Common Methods of Analysis of Nonlinear Systems

Analysis Methods for NL Systems

- Unlike linear systems, there is no general theory applicable to all nonlinear systems.
- Therefore, analysis methods are used only for specific classes of nonlinearities.
- In more advanced studies of nonlinear systems, the use of complicated mathematical structures such as differential geometry, functional analysis, nonlinear differential equation theory, etc. is common.
- Because of the complexities, however, the use of approximate methods of analysis, with nonlinear characteristics substituted by idealized ones, is heavily in use.

Analysis Methods for NL Systems

- For this level, the common analysis tools for nonlinear systems are
  - Linearization Methods
    - Tangential or Time-Domain Linearization
    - Harmonic or Frequency-Domain Linearization
    - Statistical Linearization
    - Piecewise Linearization
    - Dual-Input
      - Systems with two sinusoidal inputs
      - Systems with single sinusoidal, single random inputs
      - Systems with two random inputs
  - Graphical Methods (Phase-Plane Analysis)
  - Lyapunov's Second Method of Stability Analysis

Linearization………/what?

- What exactly is "linearization"?
  - To "linearize" is simply to *represent something "nonlinear" with something "linear"*.
  - Obviously, linearization is an approximation, since the linear representation of the nonlinearity cannot be identical to the nonlinear representation in the large
  - Linearization can be in the **time domain** or in the **frequency domain**, for deterministic signals
  - Time domain linearization is often achieved either by the concept of "**tangential linearization**" or the concept of **"method of least-squares approximation"**
  - Frequency-domain linearization is often achieved using "**describing functions**" in a process called "**harmonic linearization**".

Linearization………/why?

- The reasons why we "linearize" are obvious:
  - Most of the "easy" available tools of analysis of systems in Control Engineering are linear (Bode plots, Nyquist Diagrams, Nichols Plots, Root Locus, etc.)
  - It has been shown (in Harmonic Linearization) that the assumptions made to achieve linearization can actually be valid ones.
  - It is actually possible to use the results of linearization about the steady state to get useful stability information about the actual nonlinear system in the vicinity of the steady state through a popular method called **Lyapunov's First Test Method** (more later).

Linearization……/drawbacks

- Some of the shortcomings of linearization are
  - Time-domain linearization is is not reliable for systems with varying operating regimes. A particular linear model is suitable for a specific operating point but loses validity of the operating point changes.
  - Time-domain linearization only applies to analytic nonlinearities (differentiable for all time) because differentiation is an important part of the process of arriving at the linearized models. Therefore, a saturation nonlinearity, for instance, cannot be linearized in the large.
  - Harmonic linearization involves some assumptions that are not always valid

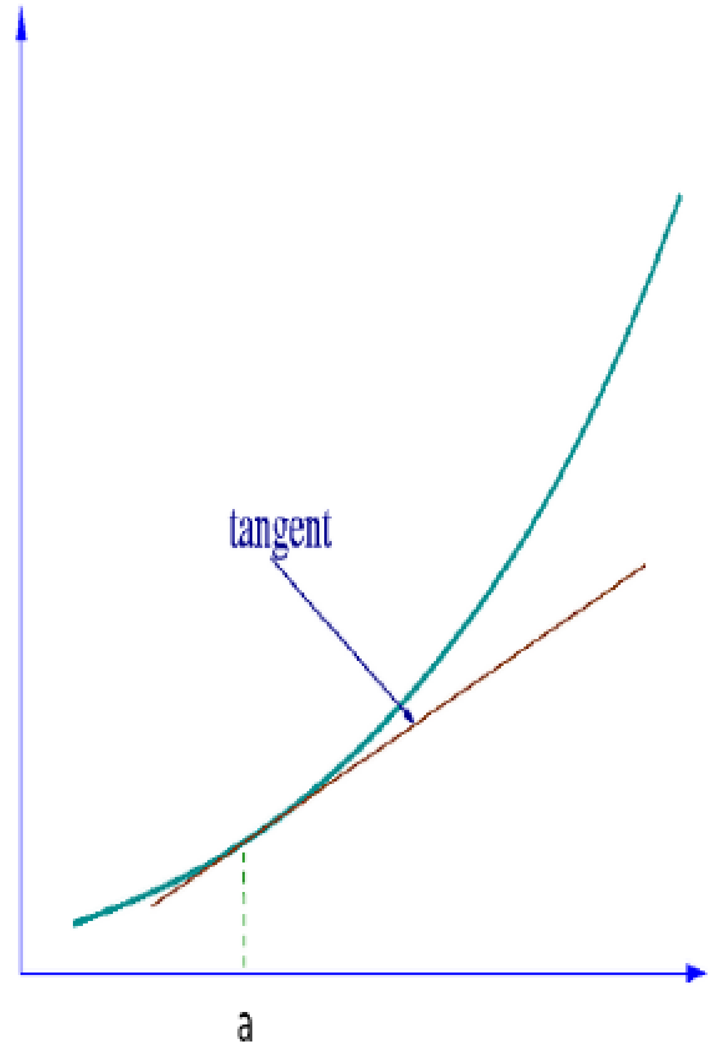
Linearization……./methods

Time-Domain Linearization

- Linearization in the time domain can be achieved by two common methods:
  - Tangential Linearization
  - Method of Least Squares

Tangential Linearization

- Tangential Linearization makes use of the concept of tangents
- A tangent is a straight line that touches a line at only a single point.
- For multi-dimensional systems, a tangent becomes a hyperplane intersecting hypersurface at a specific point

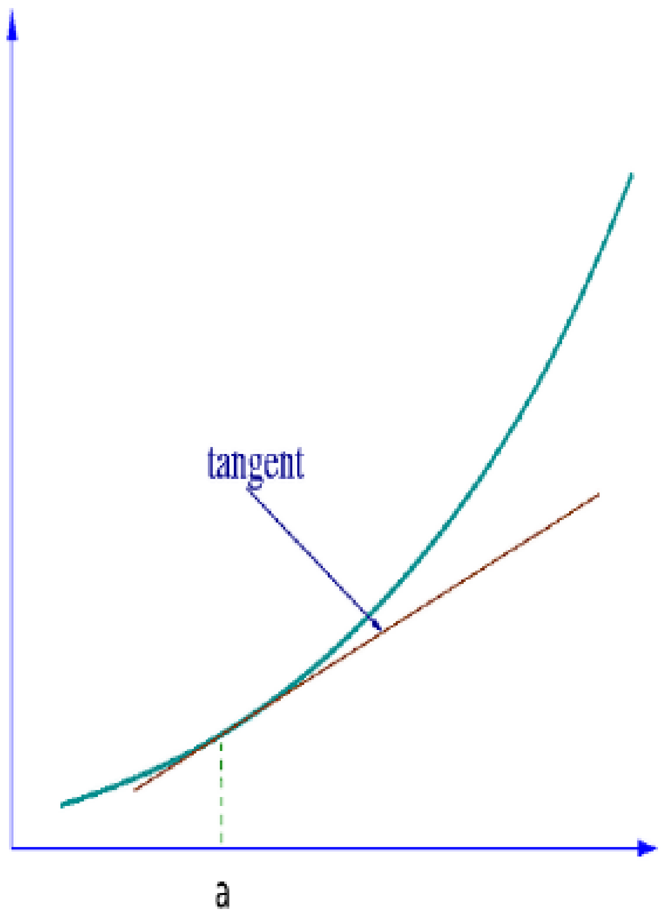


Tangential Linearization…./cont'd

- More specifically, we are interested in tangents to functions that pass through the origin.

- The idea is that a line that touches a curve at a point is an accurate representation of the function if excursions around the point of intersection are minimal.

- This point of intersection is called the "**operating point**" of the system.



Next Class?

- We will go further into the introductory aspects of Time-Domain Linearization using Tangential Linearization (Taylor's Series Approximation Technique)

  - See you in the next class!