# COURSE TITLE:

Introduction to Non-Linear Systems

# COURSE CODE:

EEE 566

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# Review of the last class

# An Electromechanical System

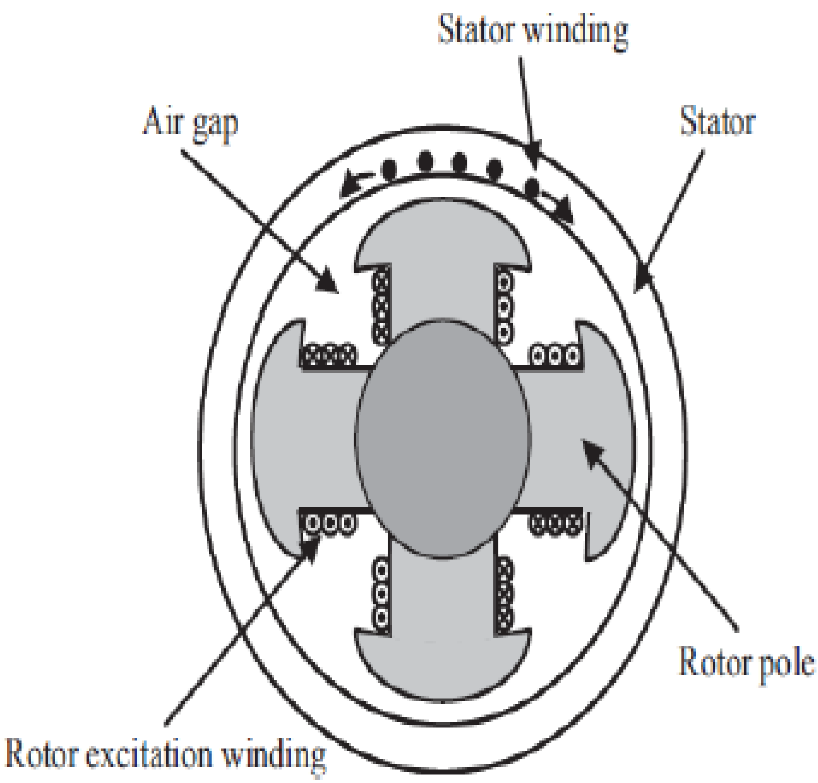
## System : Permanent-Magnet Synchronous Machines

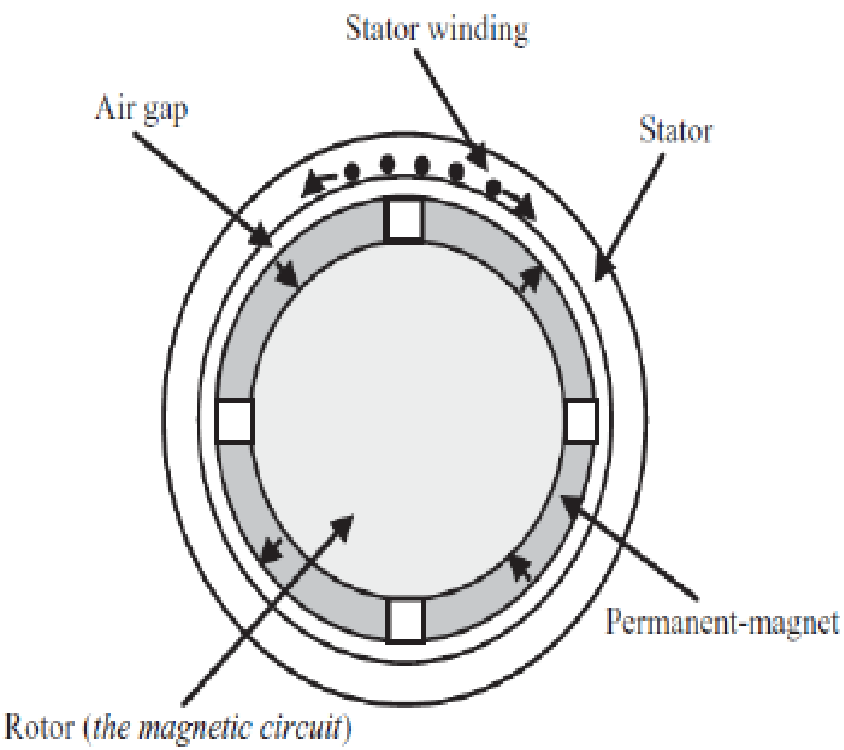
### Introductory Remarks

- Synchronous machines are machines that have the rotor speed and the speed of the rotating stator-generated magnetic field synchronized, hence the name.
- Synchronous machines are well known in applications requiring speed reversions and wide-range power variations.
- The stator is composed of three identical winding distributed in space such that any two successive windings has a space of $\mathbf{120^0}$ between them.
- When the stator windings are current-fed by a balanced three-phase AC supply, a turning field is generated along the air gap between the stator and the rotor.
- The turning field generated by the stator does not make the rotor to rotate.
- The rotor therefore needs to be excited separately to begin its own rotation.

# An Electromechanical System (cont'd)

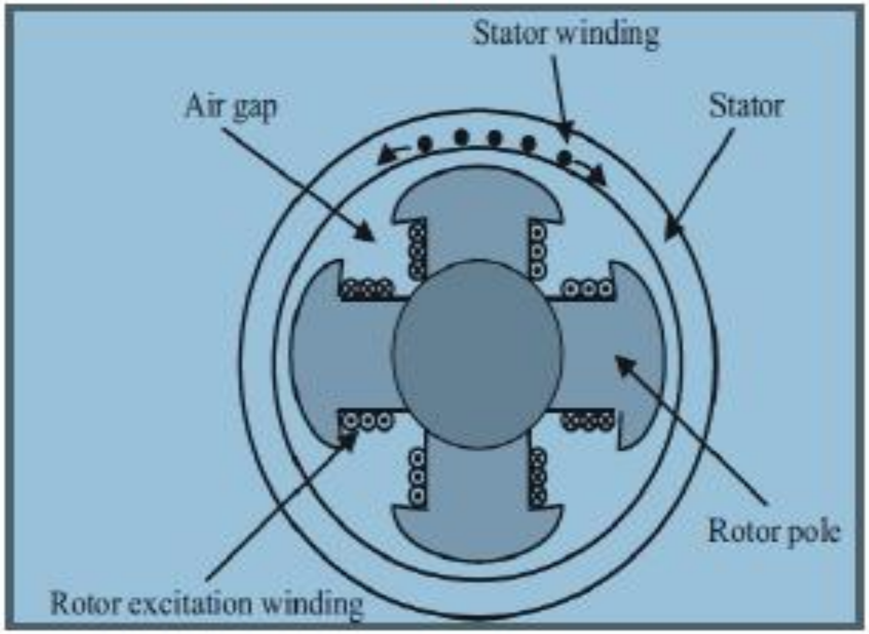
- Based on the source of this excitation, and hence the elements attached to, or associated with the rotor, synchronous machines exist in two variants i.e. **wound-rotor synchronous machines (WRSMs)** and **permanent-magnet synchronous machines (PMSMs)**.



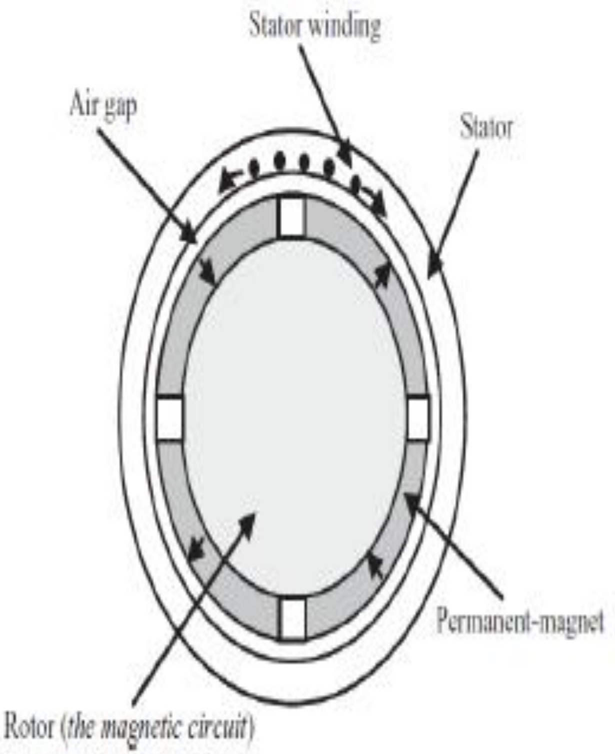



# An Electromechanical System (cont'd)

- In **WRSMs**, the rotor magnetic field is generated by windings fixed on the rotor.
- These windings are fed by a dc generator to create a magnetomotive force (MMF) along the air gap between the stator and the rotor.
- The interaction between the turning field created by the stator and the magnetomotive force created by the windings on the rotor generates an electromagnetic torque that gets applied to the rotor and generates a rotation.

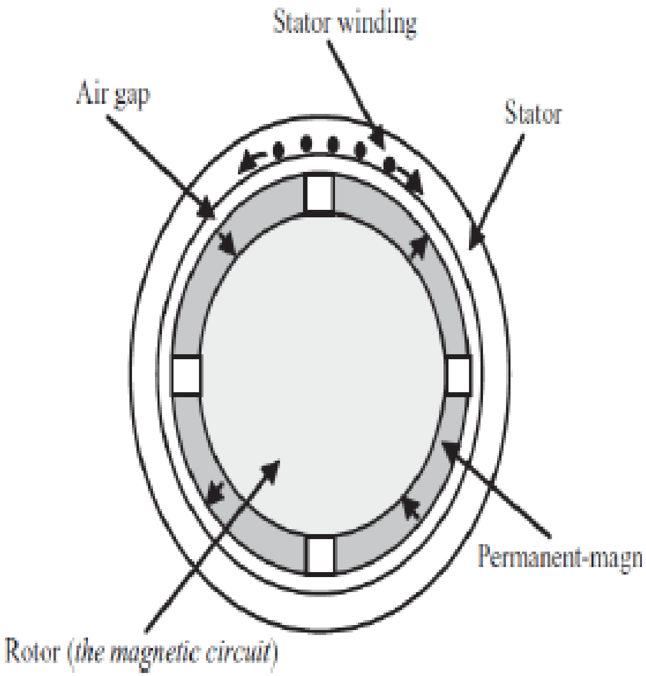


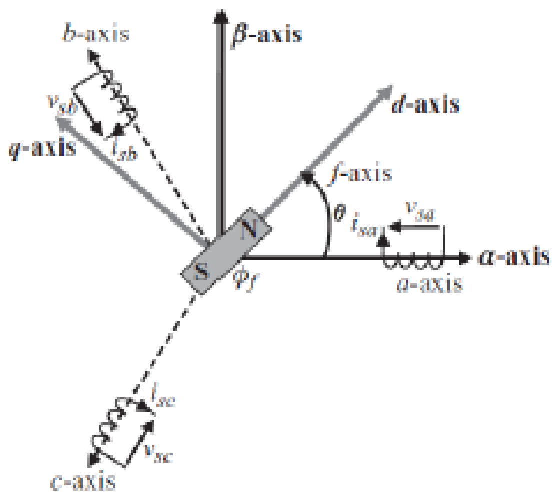
# An Electromechanical System (cont'd)




- In **PMSMs**, the rotor magnetic field is generated by permanent magnets fixed on the rotor.
- These magnets need no external excitation and generate a magnetomotive force (MMF) along the air gap between the stator and the rotor.
- Again, the interaction between the turning field created by the stator and the magnetomotive force created by the permanent magnets generates an electromagnetic torque that gets applied to the rotor and generates a rotation.
- The motions of the turning stator-generated magnetic field and the rotor reach steady-state when the rotor speed becomes equal to the speed of the turning field generated by the stator.

# An Electromechanical System (cont'd)



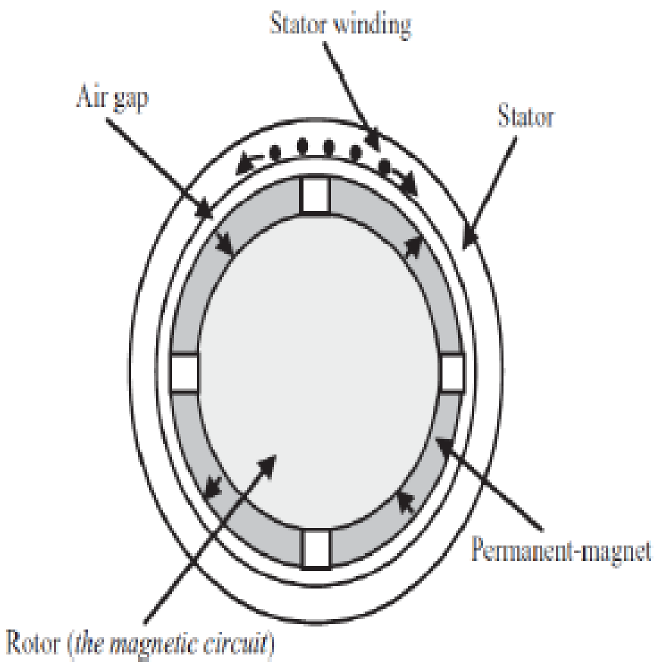

## Mathematical Modelling

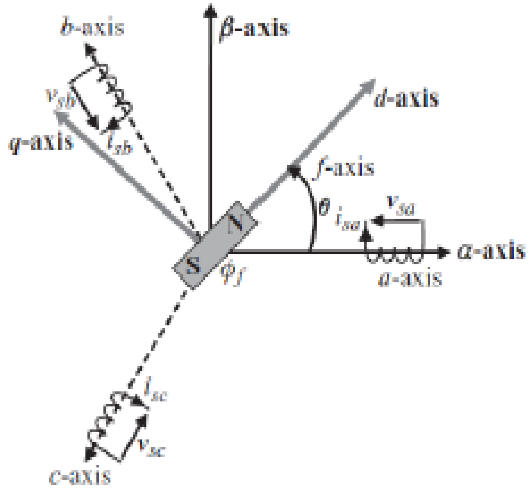
- Since we are dealing with three-phase systems, the balanced three-phase positive (or *abc*) phase sequence yields the following triplet of equations

$$x_a = A\cos(\omega t + \varphi)$$
$$x_b = A\cos\left(\omega t + \varphi - \frac{2\pi}{3}\right)$$
$$x_c = A\cos\left(\omega t + \varphi + \frac{2\pi}{3}\right)$$

  where $\boldsymbol{x}$ could represent, in this case, voltages or currents or magnetic fluxes.
- The three-coordinate frame above is usually stationary or stator-related.
- This frame is difficult to deal with when control-related applications are being considered. This is because the voltage expressions that take the derivatives of the respective fluxes in the system comprise expressions of self and mutual inductances, and the tri-dimensionality of the equations makes the equations unduly cumbersome.
- Also, there is a dependence of the fluxes on both time and rotor position.

# An Electromechanical System (cont'd)




- Because of these issues, a coordinate transformation system was developed by Park and Concordia to take the stator-related, position-dependent three-phase frame to an equivalent, lower-size, position-independent rotating direct-axis-quadrature axis (or $\boldsymbol{d}-\boldsymbol{q}$) frame.
- This frame has constant inductance terms and all signals are steady-state sinusoidal along the d- and q- axes.

- Going back to the abc frame, the application of Faraday's and Ohm's laws yields the following three-phase stator voltage equations:

$$\begin{bmatrix} v_{sa} \\ v_{sb} \\ v_{sc} \end{bmatrix} = \begin{bmatrix} R_s & 0 & 0 \\ 0 & R_s & 0 \\ 0 & 0 & R_s \end{bmatrix} \begin{bmatrix} i_{sa} \\ i_{sb} \\ i_{sc} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \emptyset_{sa} \\ \emptyset_{sb} \\ \emptyset_{sc} \end{bmatrix}$$

where

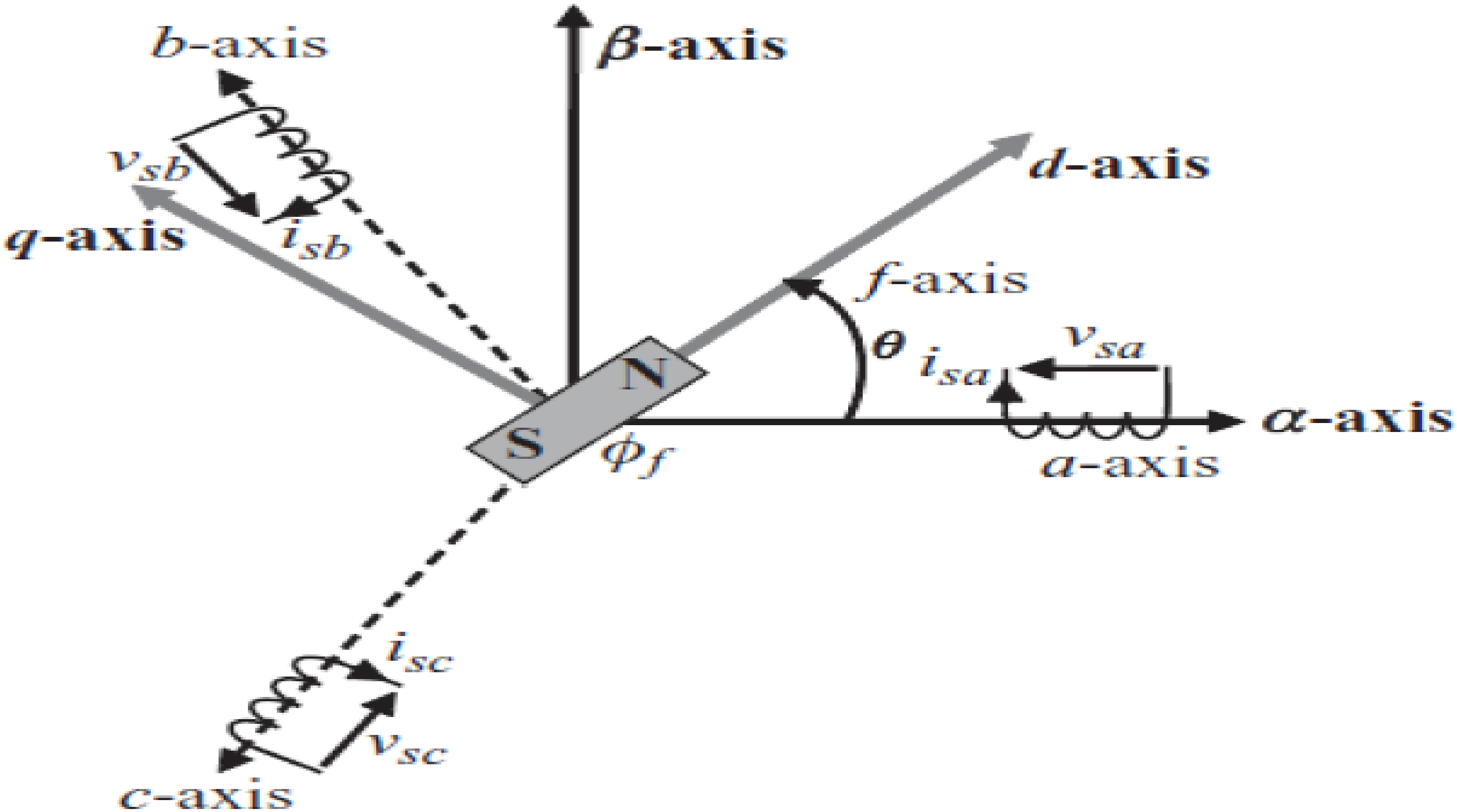
$\boldsymbol{v_{si}}(\boldsymbol{i}=\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ is the stator voltage for phase $\boldsymbol{i}$;

$\boldsymbol{i_{si}}(\boldsymbol{i}=\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ is the stator current for phase $\boldsymbol{i}$;

$\emptyset_{si}(\boldsymbol{i}=\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ is the induced flux in the stator windings for phase $\boldsymbol{i}$

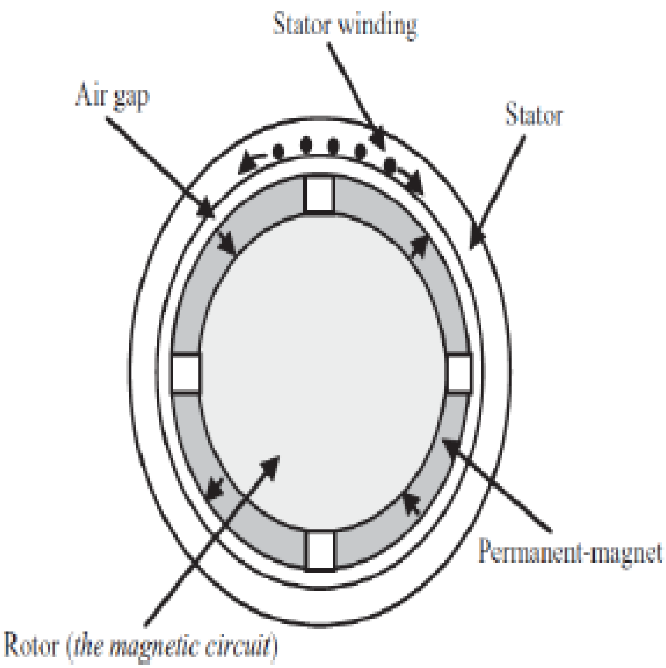
$\boldsymbol{R_s}$ is the stator winding resistance.

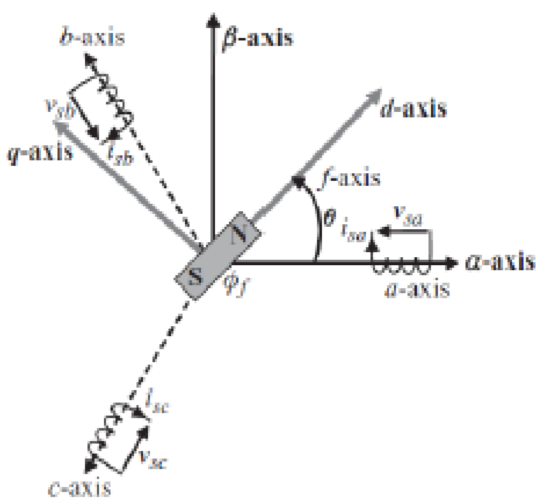
# An Electromechanical System (cont'd)




The Three-Phase ***abc***-Coordinate Frame, The ***αβ*** Stationary Two-Phase Coordinate Frame, and the ***dq*** Rotating Two-Phase Coordinate Frame for the PMSM

# An Electromechanical System (cont'd)




- We can write the above equation in shorthand form as

$$[v_{sabc}] = [R_s][i_{sabc}] + \frac{d}{dt}[\emptyset_{sabc}]$$

- In the rotor, a constant flux is created by the permanent magnets and a set of mutual fluxes is generated between the magnetic field of the rotor's permananet magnets and the rotating magnetic field generated by the stator.
- These fluxes can be written in the *abc*-frame as:

$$\emptyset_a = \emptyset_r \cos(p\theta)$$

$$\emptyset_b = \emptyset_r \cos\left(p\theta - \frac{2\pi}{3}\right)$$

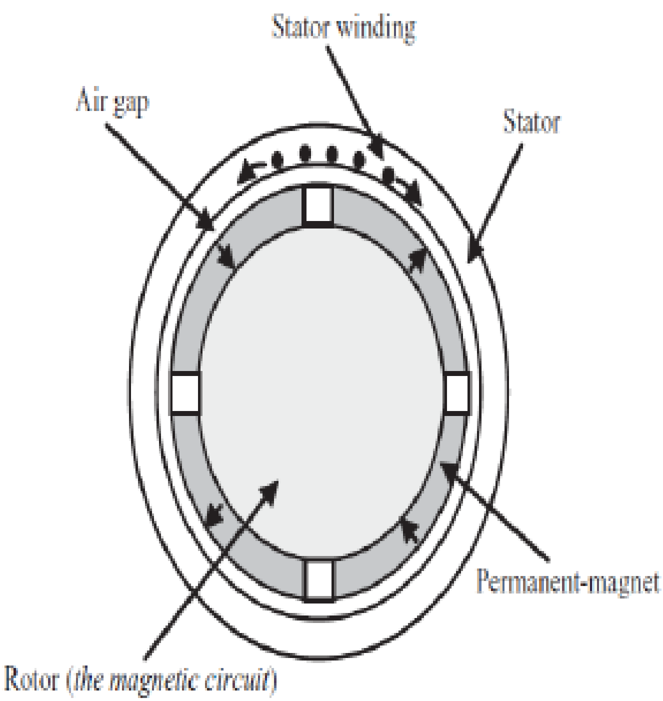
$$\emptyset_c = \emptyset_r \cos\left(p\theta + \frac{2\pi}{3}\right)$$

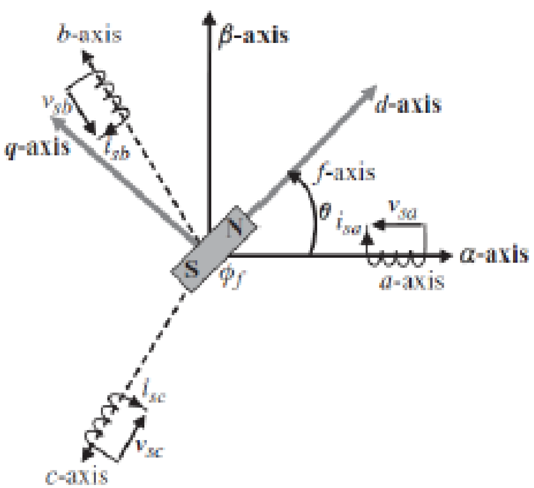
where $\emptyset_r$ is the amplitude of the flux produced by the magnets.

- We can therefore say that the flux through each of the stator windings is the sum of the flux induced by the rotor magnets and the flux produced by the currents carried by the stator phases, or

$$[\emptyset_{sabc}] = [L_{ss}][i_{sabc}] + [\emptyset_{rabc}]$$

# An Electromechanical System (cont'd)




- Thus, the stator voltage equation then becomes

$$[v_{sabc}] = [R_s][i_{sabc}] + \frac{d}{dt}[[L_{ss}][i_{sabc}] + [\emptyset_{rabc}]]$$

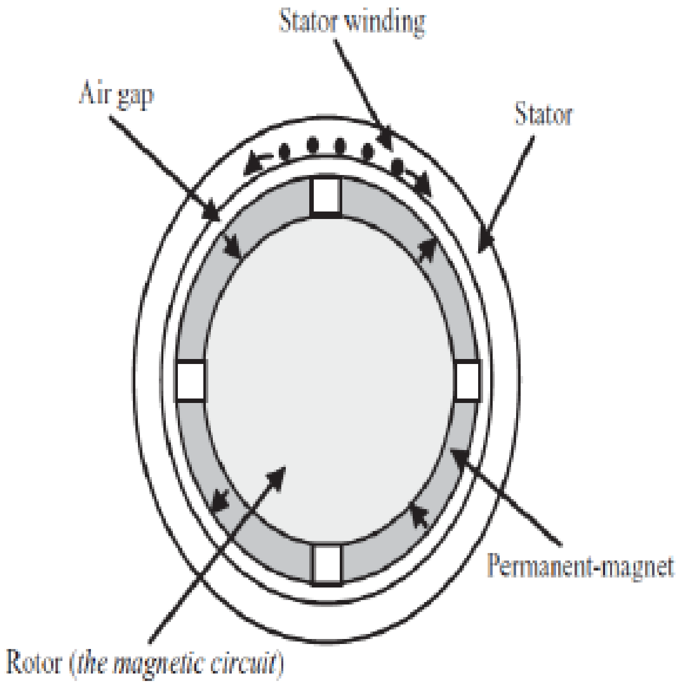
$$[v_{sabc}] = [R_s][i_{sabc}] + \frac{d}{dt}[[L_{ss}][i_{sabc}]] + \frac{d}{dt}[[\emptyset_{rabc}]]$$

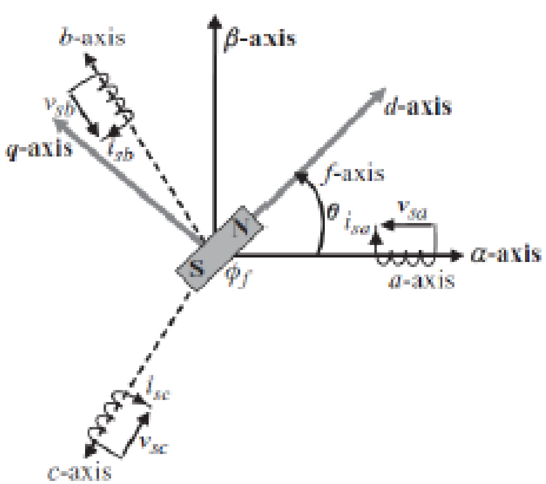
- Since ${d}/{dt}(\blacksquare) = {d}/{d\theta} \cdot {d\theta}/{dt}(\blacksquare) = {d\theta}/{dt} \cdot {d}/{d\theta}(\blacksquare)$ and ${d\theta}/{dt}$ represents the rotor speed, then the above equation can be re-written as

$$[v_{sabc}] = [R_s][i_{sabc}] + \frac{d}{dt}[[L_{ss}][i_{sabc}]] + \frac{d\theta}{dt}\frac{d}{d\theta}[[\emptyset_{rabc}]]$$

$$[v_{sabc}] = [R_s][i_{sabc}] + \frac{d}{dt}[[L_{ss}][i_{sabc}]] + \omega\frac{d}{d\theta}[[\emptyset_{rabc}]]$$

# An Electromechanical System (cont'd)




- It can be shown that through the use of the Concordia-Park transformation

$$\begin{bmatrix} x_d \\ x_q \end{bmatrix} = P(\rho)^T C_{32}{}^T \begin{bmatrix} x_a \\ x_b \\ x_c \end{bmatrix}$$

where

$$P(\rho) = \begin{bmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{bmatrix}$$

with $\rho$ representing the angular position of the rotating reference frame, and

$$C_{32} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & 0 \\ -{}^1\!/_2 & {}^{\sqrt{3}}\!/_2 \\ -{}^1\!/_2 & -{}^{\sqrt{3}}\!/_2 \end{bmatrix}$$

that the stator voltage equations in the dq-frame can be written as

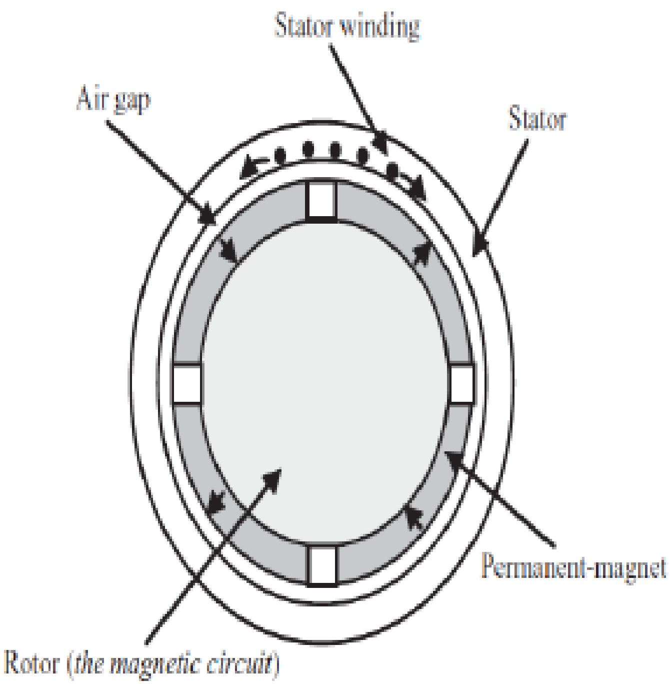
$$[v_{sdq}] = [R_s][i_{sdq}] + [L_{dq}]\frac{d}{dt}\big[[i_{sdq}]\big] + p\omega Q'[L_{dq}][i_{sdq}] + p\omega Q'[\phi_{rdq}]$$

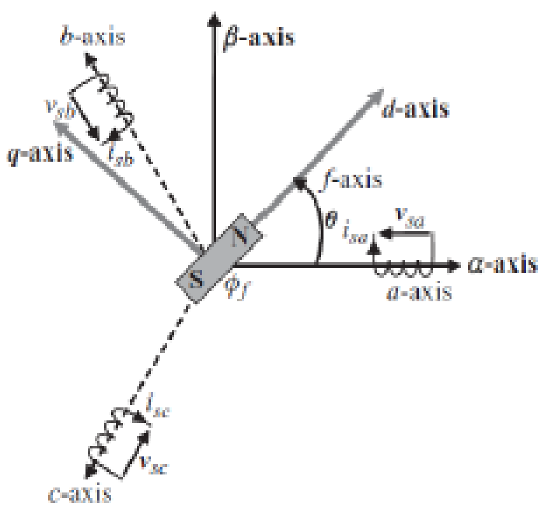
where

$p$ is a proportionality constant between the electrical equivalent of the angular position $\rho$ and the rotor angular displacement $\theta$; and

$$Q' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

# An Electromechanical System (cont'd)




- After appropriate substitutions, the eventual stator voltage equations in the $\boldsymbol{dq}$ frame can be written as

$$v_{sd} = R_s i_{sd} + L_{sd}\frac{di_{sd}}{dt} - p\omega L_{sq} i_{sq}$$

$$v_{sq} = R_s i_{sq} + L_{sq}\frac{di_{sq}}{dt} + p\omega L_{sd} i_{sd} + p\omega\sqrt{3/2}\,\emptyset_r$$

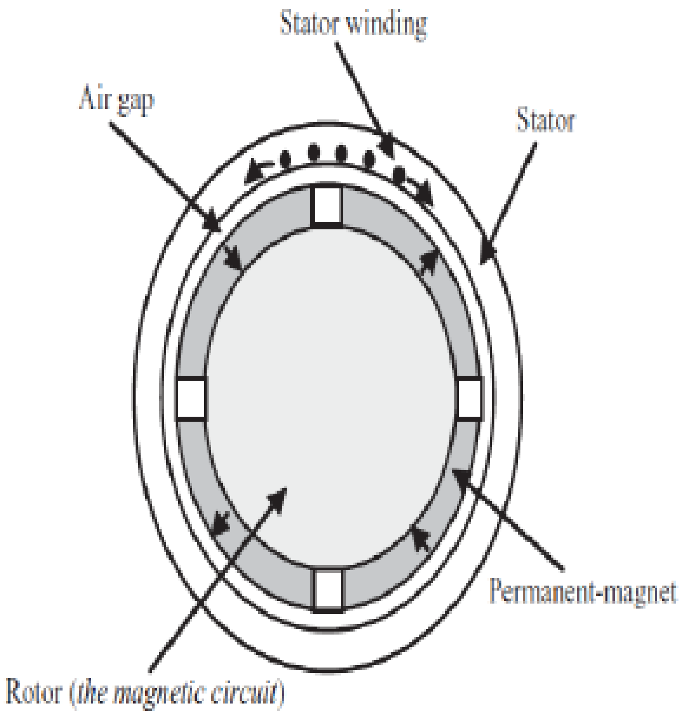
With the term $p\omega\sqrt{3/2}\,\emptyset_r$ being the voltage drop associated with the permanent-magnet flux.

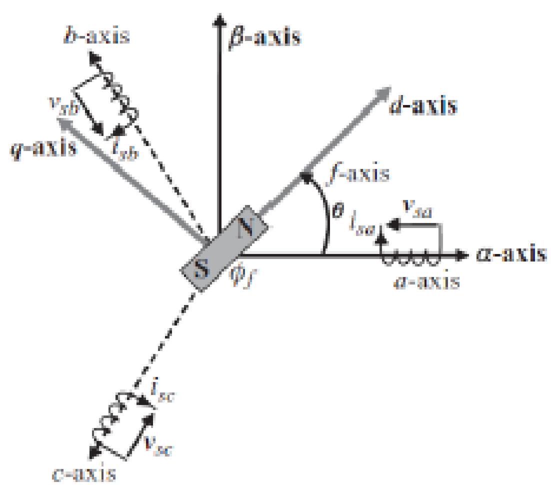
- Re-arranging the equations yields

$$\frac{di_{sd}}{dt} = \frac{1}{L_{sd}}\left[v_{sd} - R_s i_{sd} + p\omega L_{sq} i_{sq}\right]$$

$$\frac{di_{sq}}{dt} = \frac{1}{L_{sq}}\left[v_{sq} - p\omega L_{sd} i_{sd} - R_s i_{sq} - p\omega\sqrt{3/2}\,\emptyset_r\right]$$

# An Electromechanical System (cont'd)




- If we are doing current control, the two currents $\boldsymbol{i_{sd}}$ and $\boldsymbol{i_{sq}}$ becomes the outputs i.e.

$$y_1 = i_{sd}$$
$$y_2 = i_{sq}$$

- The two outputs are also the two states of the system, as seen in the differential equation above.
- The two currents $\boldsymbol{i_{sd}}$ and $\boldsymbol{i_{sq}}$ are manipulated by the corresponding voltages $\boldsymbol{v_{sd}}$ and $\boldsymbol{v_{sq}}$ respectively. Therefore, the inputs are the voltages $\boldsymbol{v_{sd}}$ and $\boldsymbol{v_{sq}}$.
- If we use standard state and input notations to replace the current and voltage symbols respectively, we have

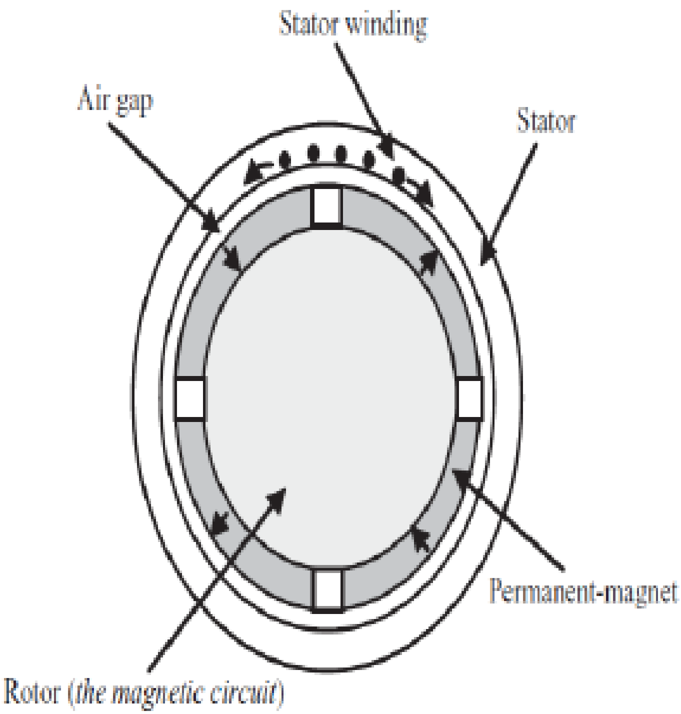
$$\frac{dx_1}{dt} = \frac{1}{L_{sd}}\left[-R_s x_1 + p\omega L_{sq} x_2 + u_1\right]$$

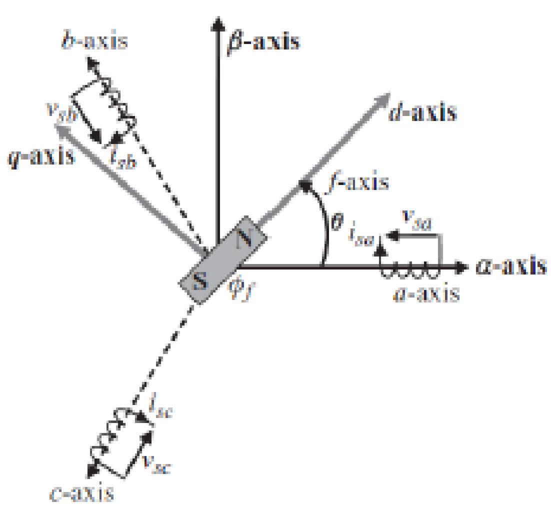
$$\frac{dx_2}{dt} = \frac{1}{L_{sq}}\left[-p\omega L_{sd} x_1 - R_s x_2 + u_2 - p\omega\sqrt{3/2}\,\phi_r\right]$$

and the output equations

$$y_1 = x_1$$
$$y_2 = x_2$$

# An Electromechanical System (cont'd)




- Thus, we can say

$$\dot{x}_1 = f_1(x_1, x_2, u_1, u_2)$$
$$\dot{x}_2 = f_2(x_1, x_2, u_1, u_2)$$
$$y_1 = g_1(x_1, x_2, u_1, u_2)$$
$$y_2 = g_2(x_1, x_2, u_1, u_2)$$

Where

$$f_1(x_1, x_2, u_1, u_2) = \frac{1}{L_{sd}}\left[-R_s x_1 + p\omega L_{sq} x_2 + u_1\right]$$

$$f_2(x_1, x_2, u_1, u_2) = \frac{1}{L_{sq}}\left[-p\omega L_{sd} x_1 - R_s x_2 + u_2 - p\omega\sqrt{3/2}\,\phi_r\right]$$

$$g_1(x_1, x_2, u_1, u_2) = x_1$$
$$g_2(x_1, x_2, u_1, u_2) = x_2$$

# An Electromechanical System (cont'd)

## Determination of Equilibrium Points

- Again, for equilibrium, the states must be steady i.e. the states must be unchanging with time i.e.

$$\dot{x}_1 = 0, \dot{x}_2 = 0$$

- This leads to

$$\frac{dx_1}{dt} = f_1(x_1, x_2, u_1, u_2) = 0$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2, u_1, u_2) = 0$$

- Again, we have 2 equations and 4 unknowns, this cannot be solved as it is.

- We therefore again assign nominal values to the 2 inputs for which we desire to determine steady-state information as we did for the two previous cases.

- Solving the equations for the nominal input values and for specific parameter values will give the values of the direct- and quadrature-axes currents at steady state.

# Continuation of Tangential Linearization of analytic Nonlinear Systems

# Tangential Linearization

- From the last class, we said that a tangent to a nonlinear curve at a particular point is a good representation of the curve if there are minimal excursions about the point.
- We called this point the **"operating point".**
- Our task is therefore to find the equation of the line that touches a specific given operating point along a nonlinear curve.
- The most common tool for this is the **"Taylor's Series Approximation".**

# Tangential Linearization……cont'd

**Taylor's Series Expansion:**

- We recall the Taylor's Series expansion of the single-input, single-output function

$$y = g(u)$$

with respect to the operating point $Y = g(U)$ as

$$y = g(U) + (u - U)g'(U) + \frac{(u-U)^2}{2!}g''(U) + \frac{(u-U)^3}{3!}g'''(U) + \cdots + \frac{(u-U)^k}{k!}g^k(U) + \cdots$$

or

$$y = g(U) + \Delta u\,[g'(U)] + \frac{(\Delta u)^2}{2!}[g''(U)] + \frac{(\Delta u)^3}{3!}g'''(U) + \cdots + \frac{(\Delta u)^k}{k!}g^k(U) + \cdots$$

# Tangential Linearization……cont'd

## Taylor's Series Expansion (2 Inputs, 1 Output):

- For the case of two inputs and an output

$$y = g(u_1, u_2)$$

with respect to the operating point $Y = g(U_1, U_2)$, we have

$$y = g(U_1, U_2) + (u_1 - U_1)\frac{dg(U_1, U_2)}{du_1} + (u_2 - U_2)\frac{dg(U_1, U_2)}{du_2}$$

$$+\frac{1}{2!}\left[(u_1 - U_1)^2 \frac{d^2 g(U_1, U_2)}{du_1{}^2} + 2(u_1 - U_1)(u_2 - U_2)\frac{d^2 g(U_1, U_2)}{du_1 du_2}\right.$$

# Tangential Linearization……cont'd

**Taylor's Series Expansion ($n$ Inputs, 1 Output):**

- For the case of $n$ inputs and an output

$$y = g(u_1, u_2, \dots, u_n)$$

with respect to the operating point $Y = g(U_1, U_2, \dots, U_n)$,

$$y = g(U_1, U_2, \dots, U_n) + \sum_{i=1}^{n} K_i\,(\Delta u_i)$$

$$+\frac{1}{2!}\left[\sum_{i=1}^{n}[K_{ii}(\Delta u_i)^2] + \sum_{i=1}^{n}\sum_{j=1}^{n}[K_{ij}\Delta u_i \Delta u_j]\right] + \cdots$$

where

$$K_i = \frac{\partial g(U_1, U_2, \dots, U_n)}{\partial\, u_i}$$

$$K_{ii} = \frac{\partial^2 g(U_1, U_2, \dots, U_n)}{\partial\, {u_i}^2}$$

$$K_{ij} = \frac{\partial^2 g(U_1, U_2, \dots, U_n)}{\partial\, u_i u_j}$$

# Tangential Linearization……cont'd

## Taylor's Series Expansion (n Inputs, 1 Output)(contd.):

- Since

$$y - g(U_1, U_2, \dots, U_n) = y - Y = \Delta y$$

we have

$$\Delta y = \sum_{i=1}^{n} K_i\,(\Delta u_i) + \frac{1}{2!}\left[\sum_{i=1}^{n}[K_{ii}(\Delta u_i)^2] + \sum_{i=1}^{n}\sum_{j=1}^{n}[K_{ij}\Delta u_i \Delta u_j]\right] + \cdots$$

# Tangential Linearization......cont'd

## Taylor's Series Expansion (Approximation for Small Excursions):

- Since excursions around the operating point are minimal, we argue that $\Delta u_i$ will be so small that

$$\Delta u_i{}^m \approx 0 \; (m \geq 2; m \in I; i = 1, 2, \dots, n)$$

- We therefore truncate all expressions containing $\Delta u_i{}^m$ for $m \geq 2$ to yield the equation

$$\Delta y \approx \sum_{i=1}^{n} K_i (\Delta u_i) = K_1 \Delta u_1 + K_2 \Delta u_2 + \cdots + K_n \Delta u_n$$

where

$$K_i = \frac{\partial y}{\partial u_i} \Big|_M$$

# Tangential Linearization……cont'd

## Taylor's Series Expansion (Multivariable Static Relations):

- For the case of a multivariable system of n inputs and q outputs

$$\begin{aligned} y_1 &= g_1(u_1, u_2, \dots, u_n) \\ y_2 &= g_2(u_1, u_2, \dots, u_n) \\ &\vdots \qquad\qquad \vdots \\ y_q &= g_q(u_1, u_2, \dots, u_n) \end{aligned}$$

with respect to the operating point

$$(U_1, U_2, \dots, U_n)$$

we linearize each of the $q$ output equations to get

$$\begin{aligned} \Delta y_1 &\approx K_{11}\Delta u_1 + K_{12}\Delta u_2 + \cdots + K_{1n}\Delta u_n \\ \Delta y_2 &\approx K_{21}\Delta u_1 + K_{22}\Delta u_2 + \cdots + K_{2n}\Delta u_n \\ &\vdots \qquad\qquad \vdots \\ \Delta y_q &\approx K_{q1}\Delta u_1 + K_{q2}\Delta u_2 + \cdots + K_{qn}\Delta u_n \end{aligned}$$

# Tangential Linearization……cont'd

## Taylor's Series Expansion

## (Multivariable Static Relations):

- In matrix form, the linearized expression becomes

$$\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \vdots \\ \Delta y_q \end{bmatrix} \approx \left[ \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & K_{22} & \dots & K_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ K_{q1} & K_{q1} & \dots & K_{qn} \end{bmatrix} \right] \left[ \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \vdots \\ \Delta u_q \end{bmatrix} \right]$$

where

$$K_{ij} = \frac{\partial y_i}{\partial u_j} \, |_M$$

- Scared? No reason to be!
- With examples, you'll see that it is all actually VERY simple!

# Examples of Tangential Linearization of analytic Nonlinear Systems

# Examples

## Example 1

## Question:

- Surge drums are important process control facilities with applications in such processes as mining, hydroelectric power generation and process industries.

- A surge drum is a standpipe or storage reservoir at downstream end of a feeder or dam or barrage pipe to absorb a sudden rise in pressure and provide extra fluid during a drop in pressure.

- In essence, a surge drum is a pressure regulator.

# Examples

## Example 1

## Question (contd.):

- For a particular gas surge drum in a chemical process, with a flow coefficient being $\boldsymbol{\beta}$ , the outlet molar flow rate $\boldsymbol{q}$ is related to the gas drum pressure $\boldsymbol{P}$ and the pressure of the downstream header piping $\boldsymbol{P_h}$ by the nonlinear relationship

$$q = \beta\sqrt{P - P_h}$$

- Taking a specific downstream header piping pressure $\boldsymbol{P_{hi}}$ and the corresponding molar flow rate $\boldsymbol{q_i}$ as the operating point values, linearize the nonlinear expression.

- Take the manipulated variable as $q$ and the controlled variable as $P_h$.
- Also take $P$ as a constant.

# Examples

**Example 1**

**Solution:**

- We recall the linearized expression

$$\Delta y \approx \sum_{i=1}^{n} K_i\,(\Delta u_i)$$

$$= K_1\Delta u_1 + K_2\Delta u_2 + \cdots + K_n\Delta u_n$$

where

$$K_i = \frac{\partial y}{\partial\, u_i}\;|_M$$

- Note that the "manipulated variable" is another name for the plant input, while the "controlled variable" is another name for the plant output.

- Let

$$u = q$$
$$y = P_h$$

- The nonlinear expression then becomes:

$$u = \beta\sqrt{P - y}$$

- We are dealing with 1 input and 1 output.

- Therefore,

$$n = 1$$

# Examples

**Example 1**

**Solution (contd.):**

- Thus, our linearized expression becomes

$$\Delta y \approx \sum_{i=1}^{1} K_i\,(\Delta u_i) = K_1 \Delta u_1$$

where

$$K_1 = \frac{\partial y}{\partial\, u_1} \;\; |_M$$

- Let us do away with the "1" subscript, since the number of inputs is 1 i.e.

$$\Delta y \approx K \Delta u = \left[\frac{\partial y}{\partial u} \;\; |_M\right] \Delta u$$

- We now proceed to find $\frac{\partial y}{\partial u}$.

- Let us make $y$ the subject of the formula in the original nonlinear expression i.e.

$$u = \beta\sqrt{P - y}$$

$$\frac{u}{\beta} = \sqrt{P - y}$$

$$\left(\frac{u}{\beta}\right)^2 = P - y$$

$$y = P - \left(\frac{u}{\beta}\right)^2$$

- Therefore,

$$\frac{\partial y}{\partial u} = -2\left(\frac{u}{\beta}\right)\left(\frac{1}{\beta}\right) = \frac{-2u}{\beta^2}$$

# Examples

**Example 1**

**Solution (contd.):**

- The alternative is to directly find $\frac{\partial u}{\partial y}$ and then invert the result to get $\frac{\partial y}{\partial u}$ i.e.

$$\frac{\partial u}{\partial y} = \beta\left(\frac{1}{2}(P-y)^{-\frac{1}{2}}(-1)\right)$$

$$\frac{\partial u}{\partial y} = -\frac{\beta}{2\sqrt{P-y}}$$

$$\frac{\partial y}{\partial u} = -\frac{2\sqrt{P-y}}{\beta}$$

- Since

$$u = \beta\sqrt{P-y}$$

$$\sqrt{P-y} = \frac{u}{\beta}$$

- Therefore,

$$\frac{\partial y}{\partial u} = -\frac{2\left(\frac{u}{\beta}\right)}{\beta}$$

$$\frac{\partial y}{\partial u} = \frac{-2u}{\beta^2}$$

- At the operating point,

$$u = q_i$$

- Therefore,

$$K = \frac{\partial y}{\partial u}\ |_M = \frac{-2q_i}{\beta^2}$$

- The linearized expression is therefore:

$$\Delta y \approx K\Delta u = \left[\frac{-2q_i}{\beta^2}\right]\Delta u$$

# Examples

**Example 1**

**Solution (contd.):**

- For consistency of notations, let us return the original variables in place of y and u i.e.

$$\Delta P_h \approx \left[\frac{-2q_i}{\beta^2}\right]\Delta q$$

- We can also decide to retain the y and u notations i.e.

$$\Delta y \approx \left[\frac{-2u_i}{\beta^2}\right]\Delta u$$

where

$$\Delta y = \Delta P_h$$
$$u_i = q_i$$
$$\Delta u = \Delta q$$

**Summary of Results**

**Example 1:**

$$\Delta P_h \approx \left[\frac{-2q_i}{\beta^2}\right]\Delta q$$

OR

$$\Delta y \approx \left[\frac{-2u_i}{\beta^2}\right]\Delta u$$

where

$$\Delta y = \Delta P_h$$
$$u_i = q_i$$
$$\Delta u = \Delta q$$

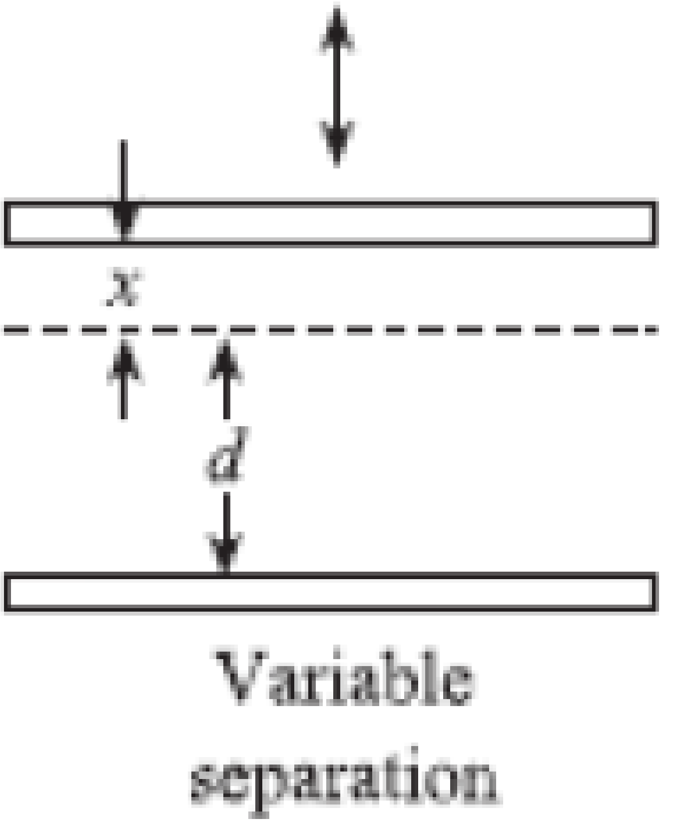
# Examples

## Example 2

## Question:

- Variable-separation displacement sensors are capacitive sensing elements that sense displacement changes by corresponding capacitance changes
- They utilize the relationship between the capacitance between plates and the distance between the plates to measure displacement.
- Thus, the displacement $x$ causes the plate separation to increase to $d + x$ so that the capacitance becomes

$$C = \frac{\varepsilon_0 \varepsilon A}{d + x}$$



# Examples

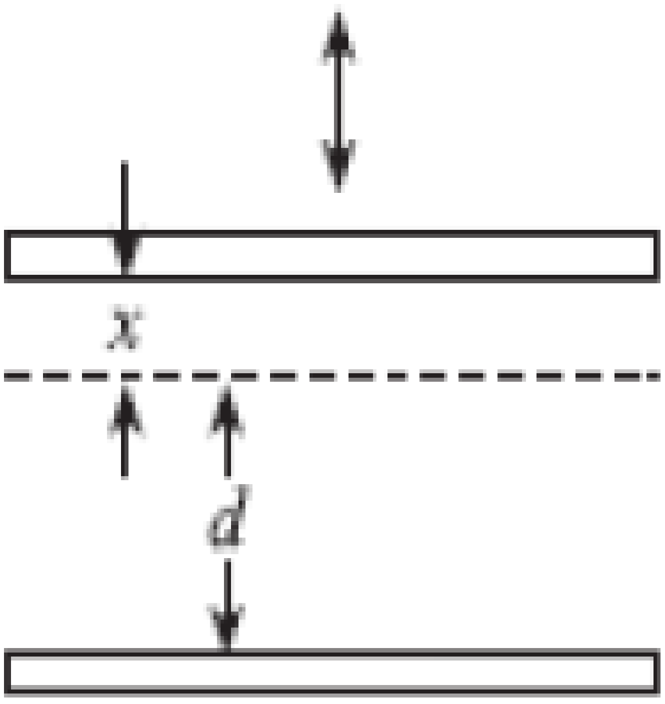
## Example 2

## Question:

- For minimal excursions about the operating point

$$x = X$$

find an approximate linear relationship between a displacement change and the corresponding capacitance change of the sensor using Taylor's Series approximation.




Variable separation

# Examples

**Example 2**

**Solution:**

- We again recall the linearized expression

$$\Delta y \approx \sum_{i=1}^{n} K_i\,(\Delta u_i)$$

$$= K_1 \Delta u_1 + K_2 \Delta u_2 + \cdots + K_n \Delta u_n$$

where

$$K_i = \frac{\partial y}{\partial\, u_i}\ |_M$$

- Again, we let

$$u = x$$

$$y = C$$

- The nonlinear expression then becomes:

$$y = \frac{\varepsilon_0 \varepsilon A}{d + u}$$

- We are again dealing with 1 input and 1 output.

- Therefore, as in Example 1,

$$n = 1$$

# Examples

**Example 1**

**Solution (contd.):**

- Thus, again, our linearized expression becomes

$$\Delta y \approx K\Delta u = \left[\frac{\partial y}{\partial u}\ |_M\right]\Delta u$$

- We again proceed to find $\frac{\partial y}{\partial u}$.

$$y = \frac{\varepsilon_0 \varepsilon A}{d+u}$$

$$\frac{\partial y}{\partial u} = \varepsilon_0 \varepsilon A(-1)(\mathrm{d}+\mathrm{u})^{-2}(1)$$

$$\frac{\partial y}{\partial u} = -\frac{\varepsilon_0 \varepsilon A}{(d+u)^2}$$

- At the operating point,

$$u = x = X$$

- Therefore,

$$K = \frac{\partial y}{\partial u}\ |_M = -\frac{\varepsilon_0 \varepsilon A}{(d+X)^2}$$

- The linearized expression is therefore:

$$\Delta y \approx K\Delta u = \left[-\frac{\varepsilon_0 \varepsilon A}{(d+X)^2}\right]\Delta u$$

- Again, for consistency of notations, let us return the original variables in place of $y$ and $u$ i.e.

$$\Delta C \approx \left[-\frac{\varepsilon_0 \varepsilon A}{(d+X)^2}\right]\Delta x$$

# Examples

## Example 2

## Solution (contd.):

- We can also decide to retain the $y$ and $u$ notations i.e.

$$\Delta y \approx \left[-\frac{\varepsilon_0 \varepsilon A}{(d+u_i)^2}\right]\Delta u$$

where

$$\Delta y = \Delta C$$
$$u_i = X$$
$$\Delta u = \Delta x$$

## Summary of Results

## Example 2:

$$\Delta C \approx \left[-\frac{\varepsilon_0 \varepsilon A}{(d+X)^2}\right]\Delta x$$

OR

$$\Delta y \approx \left[-\frac{\varepsilon_0 \varepsilon A}{(d+u_i)^2}\right]\Delta u$$

where

$$\Delta y = \Delta C$$
$$u_i = X$$
$$\Delta u = \Delta x$$

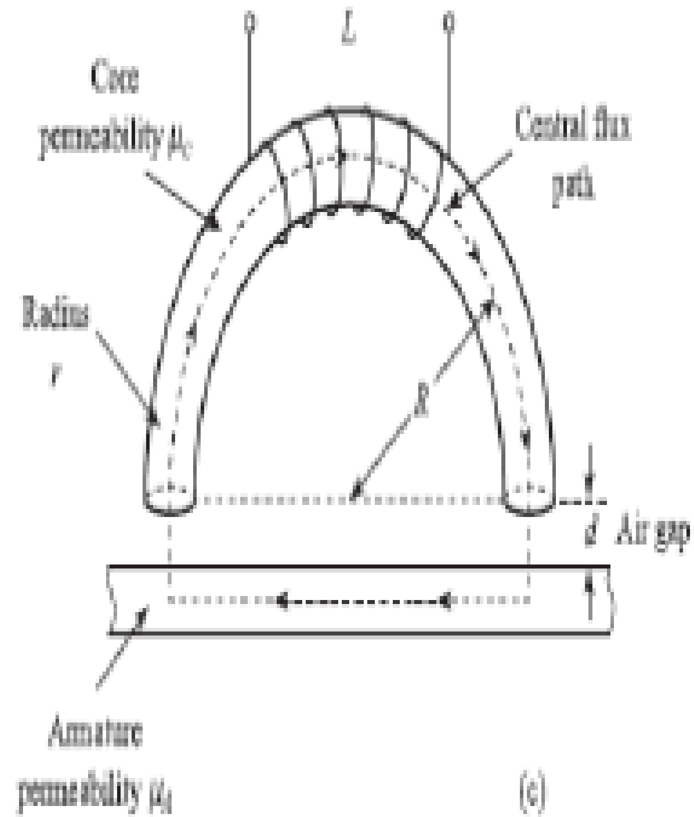
# Assignment 1

# Question:

- In place of the variable-separation capacitive displacement sensor of Example 2, we could decide to use the inductance principle to yield variable reluctance displacement sensors

- By considering the total reluctance of the magnetic circuit as a sum of the reluctances of the ferromagnetic core (toroid), the variable air gap and the ferromagnetic plate (armature), the relationship below is found to relate the displacement $x$ to the overall self-inductance $L$ of the sensing element:

$$L = \frac{n^2}{\Re_0 + k(d + x)}$$



# Assignment 1

# Question (contd.):

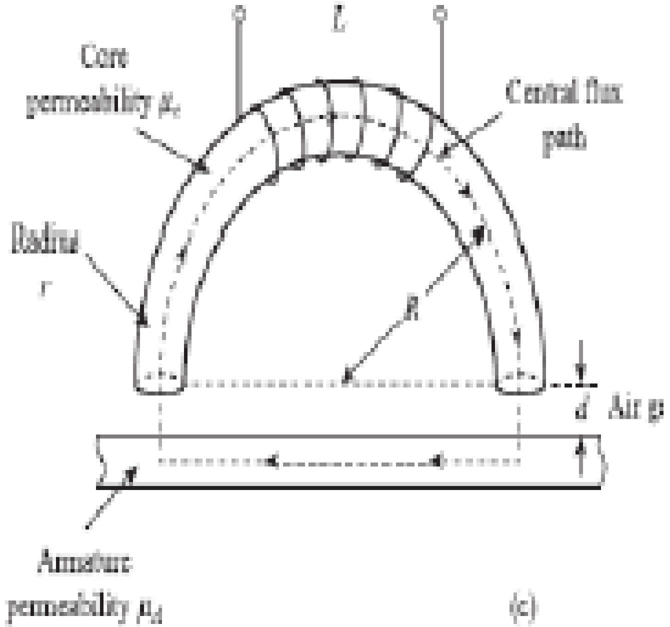
- $n$ is the number of turns of the coil on the ferromagnetic core
- $\Re_0$ is the reluctance at zero air gap, given mathematically by

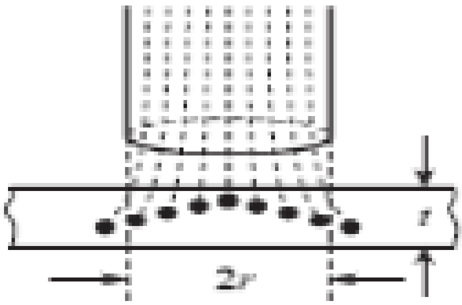
$$\Re_0 = \frac{R}{\mu_0 r}\left[\frac{1}{\mu_c r} + \frac{1}{\mu_A t}\right]$$

  With $R, \mu_0, r, \mu_c,$ and $\mu_A$ (as shown in the upper figure), and t being the armature height of flux concentration (also as shown in the lower figure)
- k is a constant given by

$$k = \frac{2}{\mu_0 \pi r^2}$$

- $d$ is the initial air-gap between core and armature



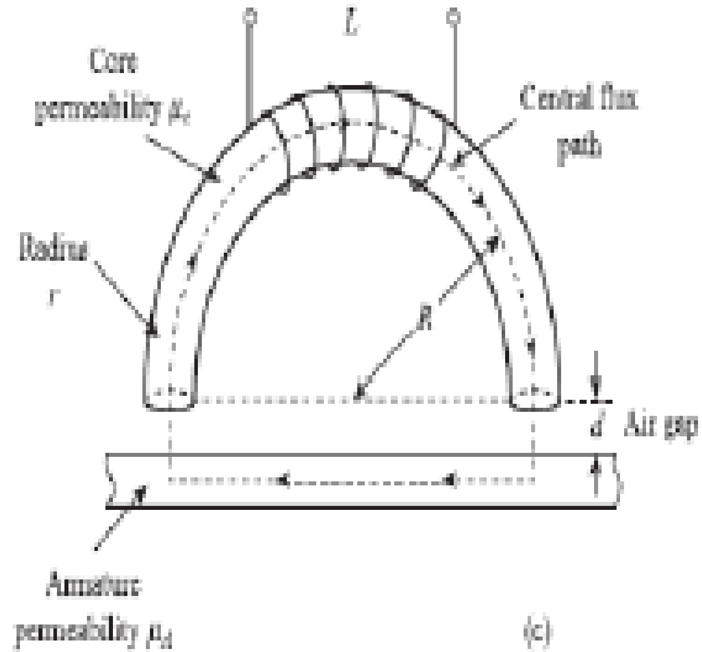
# Assignment 1
# Question (contd.):

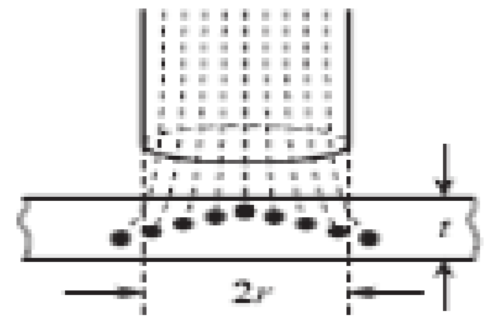
- Assume that the displacement being measured is small enough for a linear approximation to be performed

- For a specific displacement value

$$x = X$$

as the operating point, use Taylor's Series approximation to develop a linear model for the relationship between the input (displacement) and the output (self-inductance of the sensing element).



# Next Class?

- **We will look at Taylor's Series Approximation for Nonlinear Differential Equations and Nonlinear State Equations**
  - See you in the next class!