COURSE TITLE:

Introduction to Non-Linear Systems

COURSE CODE:

EEE 566

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Introduction to Frequency-Domain/Harmonic Linearization : The Describing Function

- After the lecture, you should be able to:
  - understand the necessity for harmonic linearization of nonlinear systems and its merits and demerits as compared with time-domain linearization;
  - derive an expression for the describing function of a generalized nonlinearity;
  - use Fourier Series and the special characteristics of odd functions and half-wave-symmetric functions to get a simplified route to the derivation of describing functions for the generalized common nonlinearity; and
  - specifically derive the describing function of the well-known nonlinearities.

Limitation of Time-Domain Linearization

There is a little problem, however!

- While these methods are extremely effective in developing linear models for otherwise nonlinear systems, their effectiveness is limited to nonlinear functions that are differentiable everywhere.
- In situations where there are problems of discontinuities or "non-differentiability", time-domain linearization becomes ineffective or of restricted applicability.

Why Frequency-Domain Linearization?

Because of the aforementioned limitations,

- linearization in the frequency domain becomes an alternative to time-domain linearization.
- The technique used to achieve linearization in the frequency-domain is often called HARMONIC LINEARIZATION.
- It involves the use of a sinusoidal input in a system and a number of assumptions to validate the truncation of the components of the output that have "higher harmonics" of the input frequency, hence the name.

“Time-Domain” vs. “Frequency-Domain”?

The following points must be noted:

- Time-domain and frequency-domain linearization techniques are not competitive but complementary;
- Both techniques have their merits and demerits.
- For example, while time-domain linearization is restricted to analytic functions, harmonic linearization is restricted to systems with only sine-wave inputs.
- Harmonic Linearization is also based on a number of assumptions that may not always hold in practice.
- Therefore, both have their limitations as well as their well-documented strong points.
- There is however no such method as a **perfect method** applicable for all situations to solve all problems of nonlinear control.

Introduction to Frequency-Domain Linearization

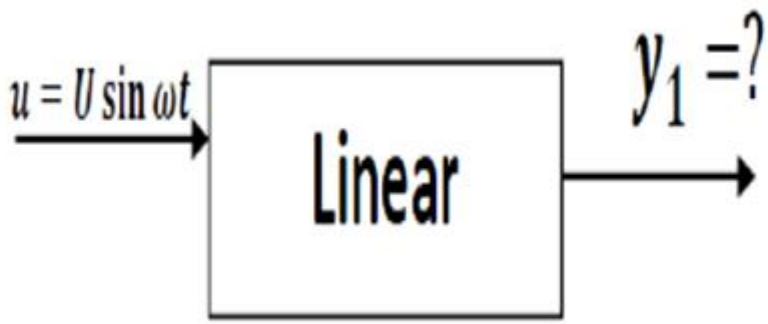
Comparison of Responses of Linear and Nonlinear Systems to same Sinusoidal Input

- From elementary control theory that the output $\boldsymbol{y_1}$ of the linear system will possess a transient part $\boldsymbol{y_{1t}}$ (which dies out with time) and a steady-state part $\boldsymbol{y_{1ss}}$

$$y_1(t) = y_{1t}(t) + y_{1ss}(t)$$

- The steady-state part is a sinusoidal signal that is different in magnitude from the input signal $\boldsymbol{u}$ and displaced in phase by an angle $\boldsymbol{\phi}$.

$$y_{1ss}(t) = Y_1 \sin(\omega t + \phi)$$



- The output $\boldsymbol{y_1(t)}$ can be written as

$$y_1(t) = y_{1t}(t) + Y_1 \sin(\omega t + \phi)$$

- The plots of $\frac{Y_1}{U}$ against $\boldsymbol{\omega}$, on one hand, and $\boldsymbol{\phi}$ against $\boldsymbol{\omega}$, on the other, respectively give the magnitude and phase plots in Bode and Nyquist diagrams that we are familiar with.

Introduction to Frequency-Domain Linearization

Comparison of Responses of Linear and Nonlinear Systems to same Sinusoidal Input

- For a nonlinear system, however, the expression may be both input-magnitude-dependent and input-frequency dependent

$$y_2(t) = y_{2transient}(t, U, \omega) + a_0(t, U, \omega) + y_{2ssfund-freq}(t, U, \omega) + y_{2higher-harmonics}(t, U, \omega) + y_{2sub-harmonics}(t, U, \omega)$$

$y_{2transient}$ is the transient component of the output;

a_0 is the dc (non-sinusoidal) component of the output,

$y_{2ssfund-freq}$ is the steady-state fundamental-frequency component of the output,

$y_{2higher-harmonics}$ is the component of the output containing terms with integral multiples of the input frequency.

Introduction to Frequency-Domain Linearization

Comparison of Responses of Linear and Nonlinear Systems to same Sinusoidal Input

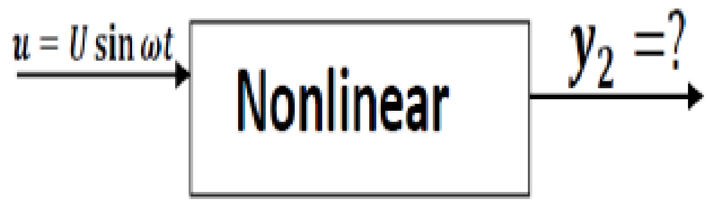
- Assuming time invariance, we can write

  $$y_{2ssfund-freq}(t,U,\omega) = Y_2(U,\omega)(\sin(\omega t + \phi(U,\omega));$$

  $$y_{2higher-harmonics}(t,U,\omega) = \sum_{i=2}^{\infty}(A_i \cos i\omega t + B_i \sin i\omega t)$$

- $y_2(t)$ can then be re-written as

  $$y_2(t) = y_{2t}(U,\omega) + a_0 + Y_2(U,\omega)(\sin(\omega t + \phi(U,\omega)) + \sum_{i=2}^{\infty}(A_i \cos i\omega t + B_i \sin i\omega t) + y_{2sub-harmonics}$$



- Clearly, the nonlinear expression is a more complicated expression than the linear one.

- There has to be some form of **justifiable** reduction of the nonlinear expression into a form that can be analyzed more easily.

Introduction to Frequency-Domain Linearization

Assumptions for Harmonic Linearization

- The sub-harmonics component $y_{2sub-harmonics}$ is assumed to be absent.

- The dc component of the output is also assumed to be **0**.

- The nonlinear element is in a feedback set-up with at least a linear element in it (the linear element/elements acts/act as low-pass filter/filters to filter out the high-frequency components of the output).

- With these assumptions, the new expression becomes
  $$y_2(t) \approx Y_2(U,\omega)(\sin\left(\omega t + \phi(U,\omega)\right)$$

- The expression above can be written as
  $$y_2(t) \approx A_1 \cos \omega t + B_1 \sin \omega t$$

Introduction to Frequency-Domain Linearization

Relating "$A_1 \cos \omega t + B_1 \sin \omega t$" and "$Y_2(U, \omega)(\sin(\omega t + \phi(U, \omega))$"

- Let $A_1 \cos \omega t + B_1 \sin \omega t = C_1 \sin(\omega t + \phi)$

$$A_1 \cos \omega t + B_1 \sin \omega t = C_1[\sin(\omega t) \cos \phi + \cos(\omega t) \sin \phi]$$

$$A_1 \cos \omega t + B_1 \sin \omega t = [C_1 \sin \phi] \cos(\omega t) + [C_1 \cos \phi] \sin(\omega t)$$

- Therefore,

$$A_1 = C_1 \sin \phi; \; B_1 = C_1 \cos \phi;$$

- Squaring both sides of the two equations above and add the results, we get

$$C_1{}^2 \sin^2 \phi + C_1{}^2 \cos^2 \phi = A_1{}^2 + B_1{}^2$$

$$C_1{}^2 [\sin^2 \phi + \cos^2 \phi] = A_1{}^2 + B_1{}^2$$

$$C_1{}^2 = A_1{}^2 + B_1{}^2$$

$$C_1 = \sqrt{A_1{}^2 + B_1{}^2}$$

- Dividing the first equation by the second equation above gives

$$\frac{C_1 \sin \phi}{C_1 \cos \phi} = \frac{A_1}{B_1} \Rightarrow \frac{\sin \phi}{\cos \phi} = \frac{A_1}{B_1} \Rightarrow \tan \phi = \frac{A_1}{B_1} \Rightarrow \phi = \tan^{-1}\left(\frac{A_1}{B_1}\right)$$

Introduction to Frequency-Domain Linearization

Relating "$A_1 \cos \omega t + B_1 \sin \omega t$" and "$Y_2(U,\omega)(\sin(\omega t + \phi(U,\omega))$"

- Therefore, if we put $C_1 = Y_2(U,\omega)$,

$$A_1 \cos \omega t + B_1 \sin \omega t = C_1(\sin(\omega t + \phi)$$

$$A_1 \cos \omega t + B_1 \sin \omega t = \sqrt{{A_1}^2 + {B_1}^2}(\sin\left(\omega t + \tan^{-1}\left(\frac{A_1}{B_1}\right)\right)$$

- Using phasor representation,

$$A_1 \cos \omega t + B_1 \sin \omega t = C_1 \langle \phi = \sqrt{{A_1}^2 + {B_1}^2} \langle \left[\tan^{-1}\left(\frac{A_1}{B_1}\right)\right]$$

- The describing function, written as $N(U, j\omega)$, is simply the phasor ratio of the output and input sinusoids

$$N(U, j\omega) = \frac{C_1}{U} \langle (\phi - 0) = \frac{\sqrt{{A_1}^2 + {B_1}^2}}{U} \langle \left[\tan^{-1}\left(\frac{A_1}{B_1}\right)\right]$$

- The main task in getting the describing function, therefore, is to find A_1 and B_1 from the output $y_2(t)$ of the nonlinearity and then use the above equation to calculate the describing function.

Introduction to Frequency-Domain Linearization

About the Describing Function

- It is called a describing function because it gives a description of the frequency-domain performance of a nonlinear system in a feedback system that comprises at least a linear element.

- It does not quite merit the tag “transfer function”. However, as the transfer function gives frequency-response information (and hence stability information) for a linear system, so does the describing function of a nonlinear element of a feedback system.

- We will demonstrate the use of the describing function for stability analysis of feedback systems comprising nonlinear elements later.

Even Functions

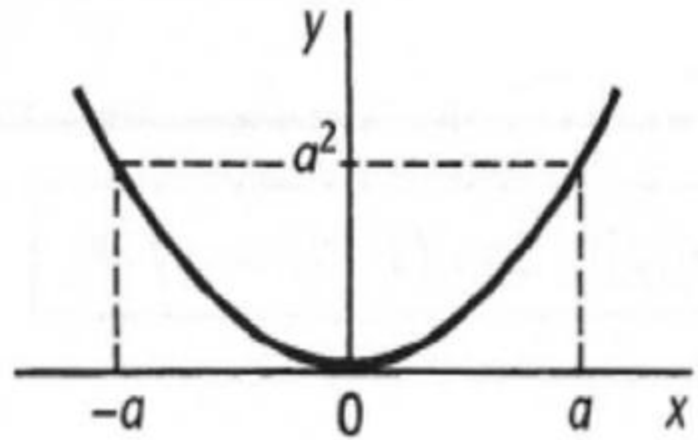
What is an even function?

- For two variables $\boldsymbol{u}$ and $\boldsymbol{y}$, a function $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{u})$ is said to be EVEN if

$$\boldsymbol{y}(\boldsymbol{u}) = \boldsymbol{y}(-\boldsymbol{u})$$

- In other words, the function value for a particular negative value of $\boldsymbol{u}$ is the same as that for the corresponding positive value of $\boldsymbol{u}$

- The function $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{x}^2$ is an example of an even function



- For $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{x}^2$ above, we see that:
  - * when

$$\boldsymbol{x} = \boldsymbol{a}, \quad \boldsymbol{y} = \boldsymbol{a}^2$$

  - * when

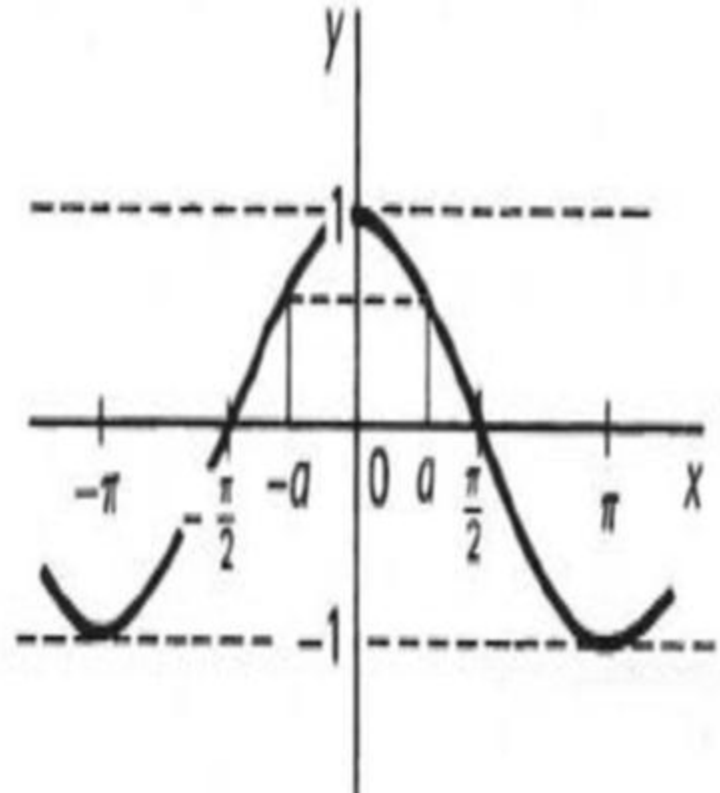
$$\boldsymbol{x} = -\boldsymbol{a}, \quad \boldsymbol{y} = (-\boldsymbol{a})^2 = \boldsymbol{a}^2$$

Even Functions (contd.)

Another even function...

- The function $\boldsymbol{y = y(x) = \cos x}$ is another example of an even function.

- For $\boldsymbol{y = y(x) = cos\ x}$, we see that:
  * when
    $\boldsymbol{x = \pi,}$
    $\boldsymbol{y = cos\ (\pi) = -1}$
  * when
    $\boldsymbol{x = -\pi,}$
    $\boldsymbol{y = cos\ (-\pi) = cos\ (\pi) = -1}$



Odd Functions

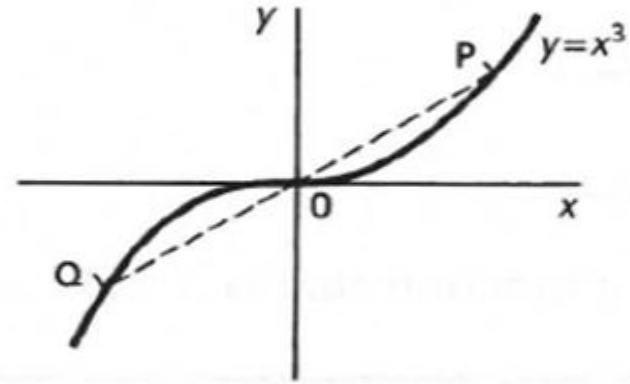
What is an odd function?

- For two variables $\boldsymbol{u}$ and $\boldsymbol{y}$, a function $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{u})$ is said to be ODD if

$$\boldsymbol{y}(\boldsymbol{u}) = -\boldsymbol{y}(-\boldsymbol{u})$$

- In other words, the function value for a particular negative value of $\boldsymbol{u}$ is numerically equal to that for the corresponding positive value of $\boldsymbol{u}$, but is opposite in sign.

- The function $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{x}^3$ is an example of an odd function.



- For $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{x}^3$ above, we see that:
  - * when

$$\boldsymbol{x} = \boldsymbol{a},$$
$$\boldsymbol{y} = \boldsymbol{a}^3$$

  - * when

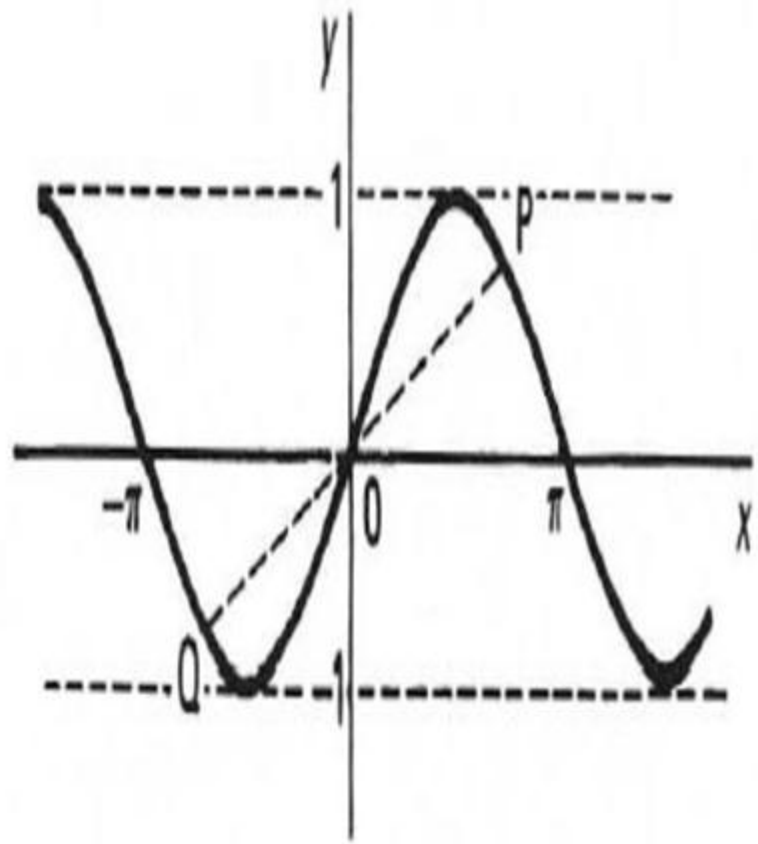
$$\boldsymbol{x} = -\boldsymbol{a},$$
$$\boldsymbol{y} = (-\boldsymbol{a})^3 = -\boldsymbol{a}^3$$

Odd Functions (contd.)

Another odd function...

- The function $y = y(x) = \sin x$ is another example of an odd function

- For $y = y(x) = \sin x$, we see that:
  - when $x = \frac{\pi}{2}$, $y = \sin\left(\frac{\pi}{2}\right) = 1$
  - when $x = -\frac{\pi}{2}$, $y = \sin\left(-\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$



Half-Wave Symmetry

What is "half-wave symmetry"?

- A periodic function of time

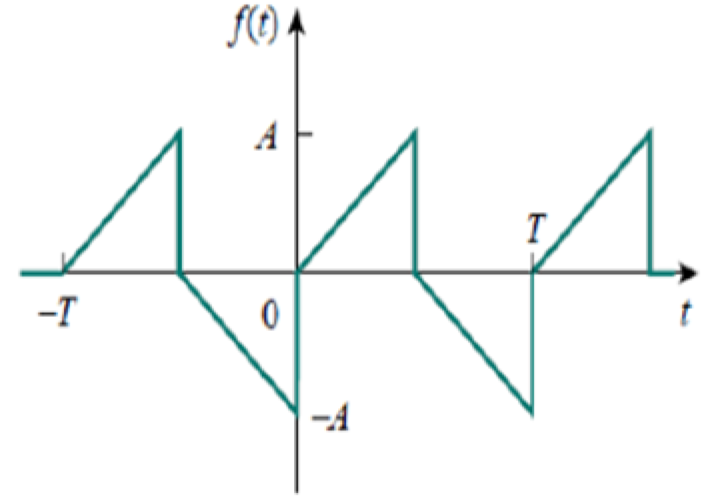
$$y = y(t)$$

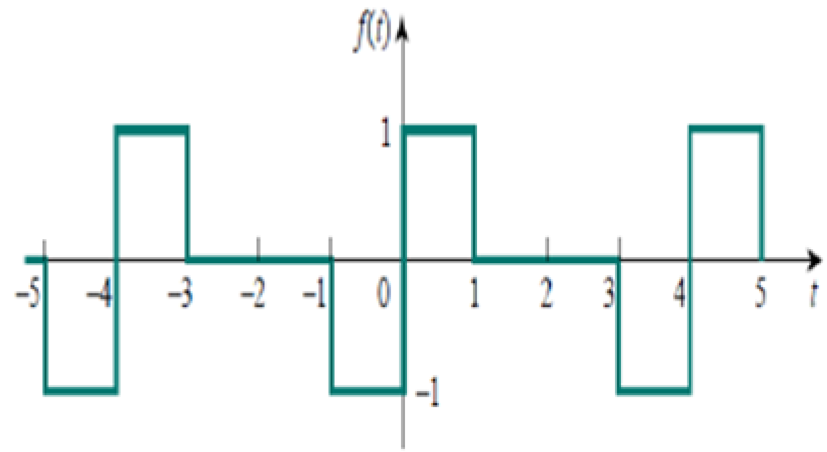
exhibits half-wave symmetry if

$$y(t) = -y\left(t + \frac{T}{2}\right)$$

where T is the period of the waveform

- In words, each half-cycle is a mirror-image of the next half-cycle
- The functions on the right are half-wave-symmetrical functions





Half-Wave Symmetry

What is "half-wave symmetry"?

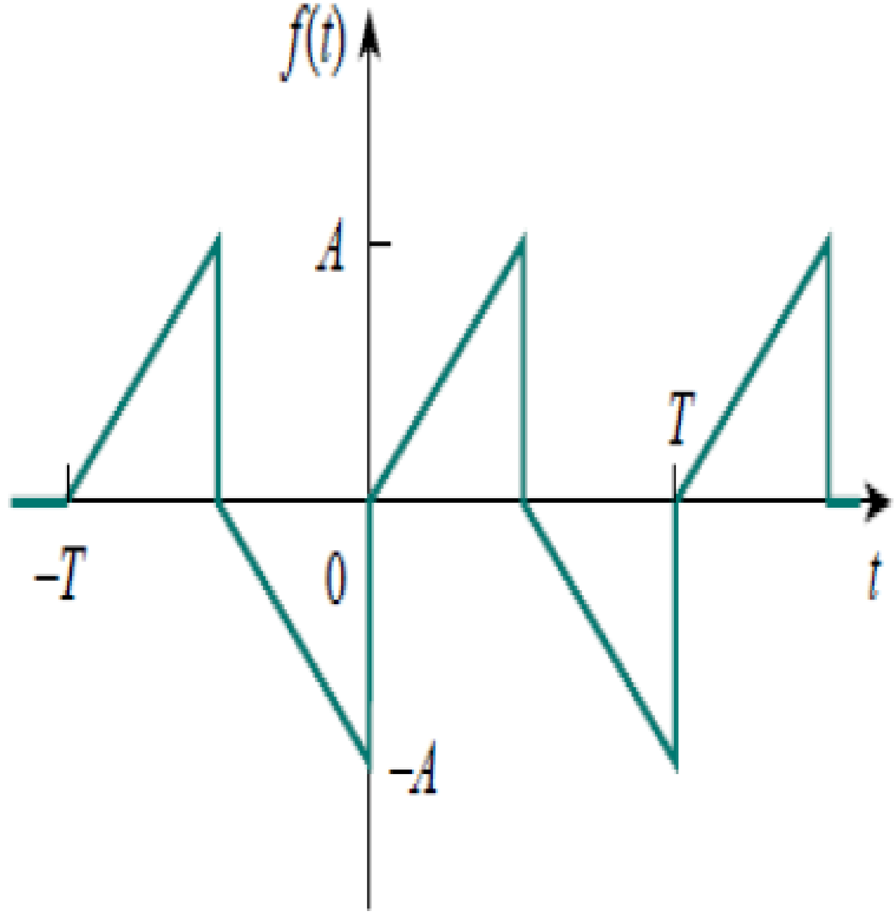
- Taking the first of the two periodic functions, we have

$$f(t) = \frac{2A}{T}t, 0 \le t < \frac{T}{2}$$

$$f(t) = -\frac{2A}{T}t + A, -\frac{T}{2} \le t < T$$

- Let us take $t = \frac{T}{4}$

$$f(t)|_{t=\frac{T}{4}} = \frac{2A}{T}\left(\frac{T}{4}\right) = \frac{A}{2}$$



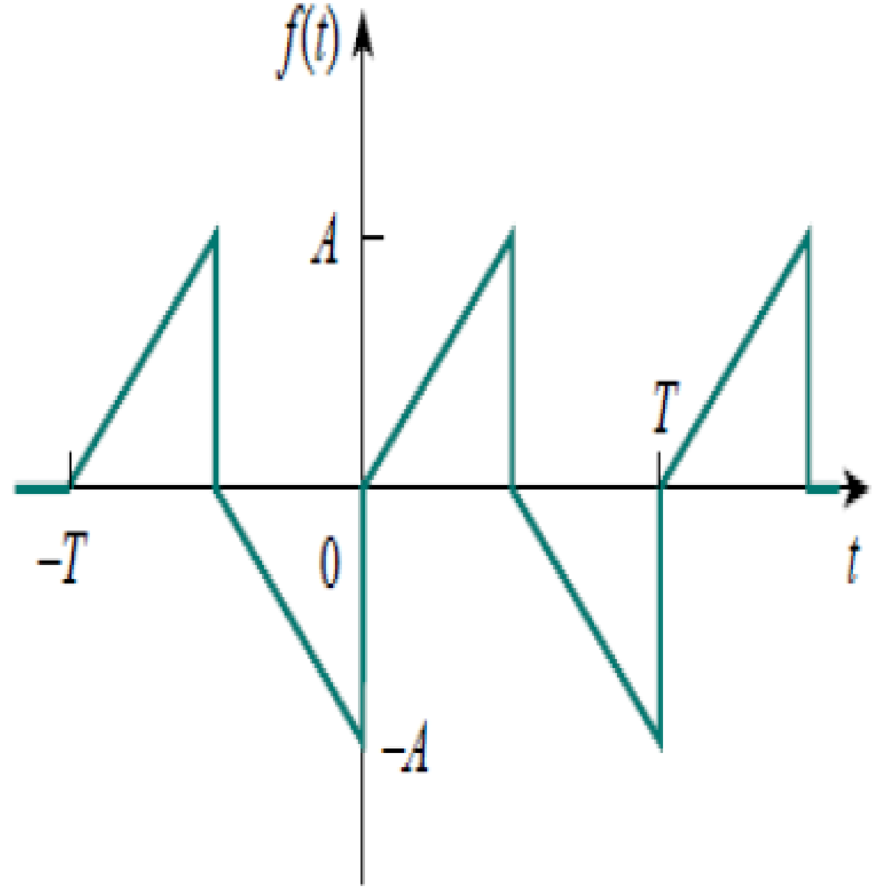
Half-Wave Symmetry

$$f\left(t+\frac{T}{2}\right)|_{t=\frac{T}{4}} = f\left(\frac{T}{4}+\frac{T}{2}\right)$$

$$f\left(t+\frac{T}{2}\right)|_{t=\frac{T}{4}} = f\left(\frac{3T}{4}\right)$$

$$f\left(t+\frac{T}{2}\right)|_{t=\frac{T}{4}} = -\frac{2A}{T}\left(\frac{3T}{4}\right)+A$$

$$f\left(t+\frac{T}{2}\right)|_{t=\frac{T}{4}} = -\frac{3A}{2}+A = -\frac{A}{2}$$



- Thus

$$f(t) = -f\left(t+\frac{T}{2}\right)$$

- The periodic function is therefore half-wave symmetrical.

Significance of Oddness and Half-Wave Symmetry

Why all these discussions on odd, even, and half-wave symmetrical functions?

- We are about to use Fourier Series expansion to determine the values of $\boldsymbol{A_1}$ and $\boldsymbol{B_1}$ so as to be able to calculate the describing functions of common nonlinearities.
- The expressions, from good old Fourier Series, tend to cover full periods of oscillation of periodic signals.
- The significance of oddness of a nonlinearity is in the fact that one only needs to consider half of a full period in the calculation and then multiply the result by 2.
- The significance of half-wave symmetry is that we can even go further and calculate only for a quarter of a wavelength and then multiply the result by 4.

Significance of Oddness and Half-Wave Symmetry

Why all these discussions on odd, even, and half-wave symmetrical functions? (contd.)

- It therefore greatly simplifies our calculations to use the knowledge of oddness and half-wave symmetry (where they apply) to calculate the describing functions of nonlinearities.

- Let us now go into the mathematical details of the determination of $\boldsymbol{A_1}$ and $\boldsymbol{B_1}$ for the calculation of the describing function.

Describing Function Calculation Using Fourier Series Approximation

- Recall from Elementary Circuit Analysis that a function

$$y = y(t)$$

can be expanded into the form

$$y(t) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t)$$

Where

$$A_n = \frac{2}{T}\int_0^T y(t)\ \cos n\omega t\, dt$$

and

$$B_n = \frac{2}{T}\int_0^T y(t)\ \sin n\omega t\, dt$$

Describing Function Calculation Using Fourier Series Approximation (contd.)

- If we decide to integrate with respect to ωt instead of t and to make the period of oscillation to be such that

$$\omega T = 2\pi$$

then,

$$d(\omega t) = \omega dt$$

$$dt = \frac{1}{\omega} d(\omega t)$$

and

$$T = \frac{2\pi}{\omega}$$

- Therefore, we have

$$A_n = \frac{2}{T} \int_{t=0}^{t=T} y(t) \cos n\omega t \, dt;$$

$$A_n = \frac{2}{\left(\frac{2\pi}{\omega}\right)} \int_{\omega t=0}^{\omega t=\omega T=2\pi} y(t) \cos n\omega t \left[\frac{1}{\omega} d(\omega t)\right]$$

$$A_n = \frac{2\omega}{2\pi} \frac{1}{\omega} \int_{\omega t=0}^{\omega t=2\pi} y(t) \cos n\omega t \, [d(\omega t)];$$

$$A_n = \frac{1}{\pi} \int_{0}^{2\pi} y(t) \cos n\omega t \, [d(\omega t)]$$

Describing Function Calculation Using Fourier Series Approximation (contd.)

- In a similar manner, we have, for B_n:

$$B_n = \frac{2}{T}\int_{t=0}^{t=T} y(t)\ \sin n\omega t\, dt;$$

$$B_n = \frac{2}{\left(\frac{2\pi}{\omega}\right)}\int_{\omega t=0}^{\omega t=\omega T=2\pi} y(t)\sin n\omega t\left[\frac{1}{\omega}d(\omega t)\right];$$

$$B_n = \frac{2\omega}{2\pi}\frac{1}{\omega}\int_{\omega t=0}^{\omega t=2\pi} y(t)\ \sin n\omega t\,[d(\omega t)];$$

$$B_n = \frac{1}{\pi}\int_{0}^{2\pi} y(t)\sin n\omega t\,[d(\omega t)]$$

- Putting together our expressions, with the variable of integration changed to ωt, gives

$$A_n = \frac{1}{\pi}\int_{0}^{2\pi} y(t)\cos n\omega t\,[d(\omega t)]$$

$$B_n = \frac{1}{\pi}\int_{0}^{2\pi} y(t)\sin n\omega t\,[d(\omega t)]$$

- The preference for ωt is because of the need to work with angles more than time in the integration.

Odd Functions and Describing Function Calculation

- We recall from Fourier Series analysis that for odd functions,

$$A_n = 0 \text{ for all } n \in \mathbf{I}$$

- We also recall that for odd functions,

$$B_n \neq 0 \text{ for all } n \in \mathbf{I}$$

- Since almost all the common nonlinearities (ideal relay, saturation, relay with dead-band, saturation with dead-band, etc.) are odd functions, then the above equations hold for them.

- Also, since n in the above equations includes $\mathbf{0}$, then $A_0 = 0$ and the assumption of zero dc-component is therefore valid for the common nonlinearities.

A Small Point on Even Functions …

- I believe we know that

$$\boldsymbol{A_n \neq 0} \text{ for all } \boldsymbol{n \in \mathbf{I}}$$

and

$$\boldsymbol{B_n = 0} \text{ for all } \boldsymbol{n \in \mathbf{I}}$$

- We will not deal with even functions to any great extent because our common nonlinearities are mostly odd functions

Back to Odd Functions and Describing Function Calculation…

- Another important point for odd nonlinearities is that it is not necessary to integrate over a complete period.

- Thus, we can simply integrate over half of a period and then multiply the result by $\mathbf{2}$.

- Thus, we have

$$A_n = 0$$

and

$$B_n = \frac{2}{\pi}\int_0^{\pi} y(t)\ \sin n\omega t\, d\omega t \neq 0$$

- Thus, if $n = 1$, we have

$$A_1 = 0$$

and

$$B_1 = \frac{2}{\pi}\int_0^{\pi} y(t)\ \sin \omega t\, d\omega t \neq 0$$

- We are only interested in A_1 and B_1 (and hence B_1 since A_1 is zero).

- Thus $N(U, j\omega)\ =$

$$\frac{\sqrt{{A_1}^2+{B_1}^2}}{U}\langle tan^{-1}\left(\frac{A_1}{B_1}\right)$$

$$N(U, j\omega)\ = \frac{\sqrt{0+{B_1}^2}}{U}\langle tan^{-1}\left(\frac{0}{B_1}\right)$$

$$N(U, j\omega)\ = \frac{B_1}{U}\langle 0^0$$

On Odd Functions and Describing Function…

- For an odd nonlinearity, the describing function is given by

$$\boldsymbol{N(U, j\omega) = \frac{B_1}{U}\langle 0^0}$$

where $\boldsymbol{B_1}$ is given by

$$\boldsymbol{B_1 = \frac{2}{\pi}\int_0^{\pi} y(t)\ \sin \omega t\, d\omega t \neq 0}$$

- We will see shortly that for the function $\boldsymbol{y(t)}$ that exhibits half-wave symmetry, there is a further simplification of the above definite integral.

Half-Wave Symmetry and Describing Function Calculation

- If an oddly symmetric nonlinearity (input-output relationship) leads to a half-wave symmetric output-time relationship (more on this later), then the integration for the determination of B_n can be done over a quarter of a period rather than half of the period as was done for odd symmetry.

- Thus, B_n can be calculated by integrating over a quarter of a period and multiplying the result by 4.

- Thus, B_n becomes

$$B_n = 2.\frac{2}{\pi}\int_0^{\frac{\pi}{2}} y(t)\ \sin n\omega t\, d(\omega t)$$

$$= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} y(t)\ \sin n\omega t\, d(\omega t)$$

and by implication,

$$B_1 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} y(t)\ \sin \omega t\, d(\omega t)$$

- Covering only a quarter of a period hugely simplifies the describing function calculation for functions exhibiting half-wave symmetry.

On Half-Wave-Symmetrical Functions and Describing Function…

- For an odd function/nonlinearity that yields a half-wave symmetric output as a function of time, the describing function is given by

$$\boldsymbol{N(U, j\omega) = \frac{B_1}{U} \langle 0^0}$$

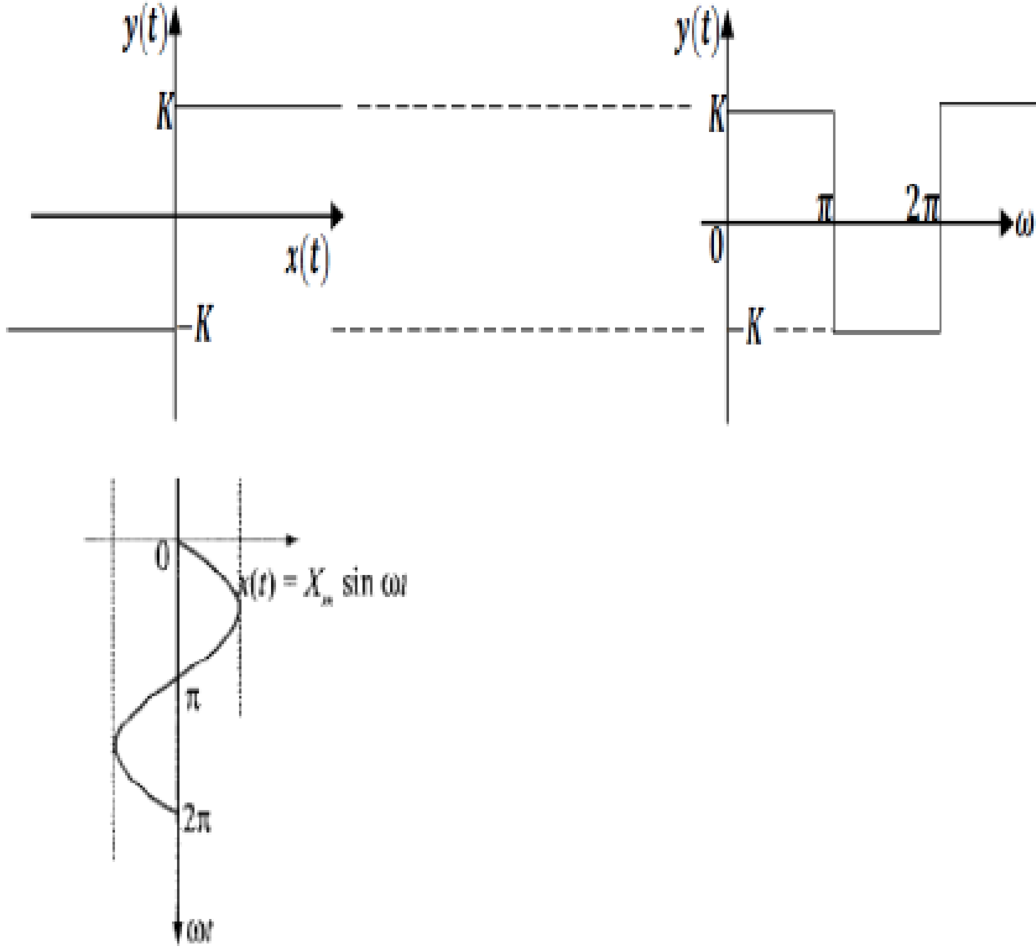
where B_1 is given by

$$\boldsymbol{B_1 = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} y(t)\ \sin \omega t \, d\omega t \neq 0}$$

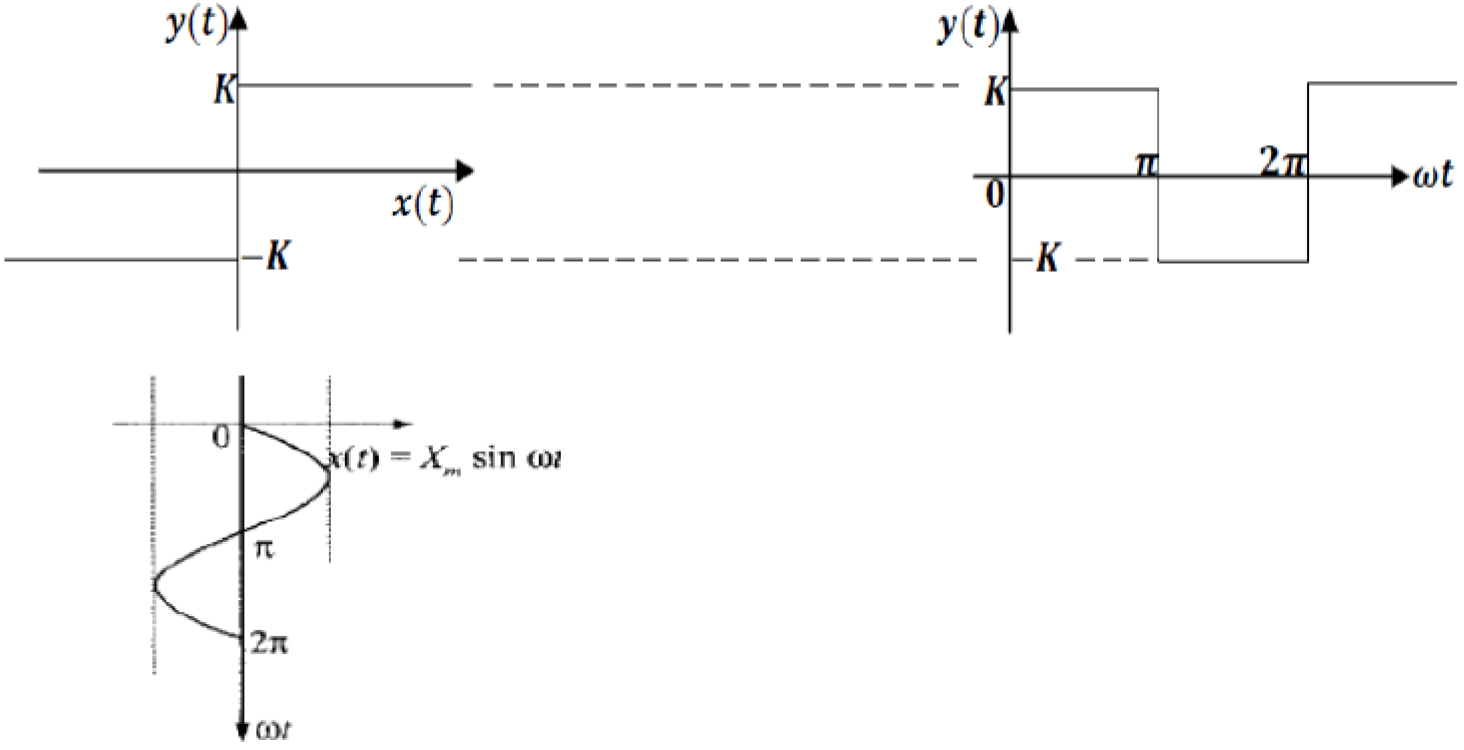
- This, as we will see, is a simpler procedure than integration over the full period of a periodic function representing a nonlinear system.

Procedure for Describing Function Calculation

- **Step 1**: Position the nonlinearity (input-output relationship, ***y*** as a function of ***u***) on the ***top left*** part of the page, and the sinusoidal input (input-time relationship, ***u*** as a function of ***ωt***) in the ***bottom left*** part of the page. These two plots can then be combined to draw the output waveform (output-time relationship, ***y*** as a function of ***ωt***) on the **top right** part of the page.



Bigger Image of Diagram for Step 1 of Describing Function Analysis



Procedure for Describing Function Calculation (contd.)

- **Step 2**: Write out the output-time equation based on the graphical plot and the mathematical relationships given.

- **Step 3**: Inspect the graphs and equations for odd symmetry and half-wave symmetry. If both exist, use

$$B_1 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} y(t)\ \sin\omega t\, d\omega t \neq 0$$

to get the expression for B_1. If not, integrate over a full period and calculate A_1 and B_1

Procedure for Describing Function Calculation (contd.)

- **Step 4**: For odd nonlinearity and half-wave symmetrical output, use

$$N(U, j\omega) = \frac{B_1}{U} \langle 0^0$$

to calculate the describing function.

For non-odd functions, use

$$N(U, j\omega) = \frac{\sqrt{{A_1}^2 + {B_1}^2}}{U} \langle tan^{-1}\left(\frac{A_1}{B_1}\right)$$

for the calculation.

Example 1 on Describing Function Calculation

- Calculate the describing function $\boldsymbol{N(X, j\omega)}$ for the Ideal Relay described by the equation

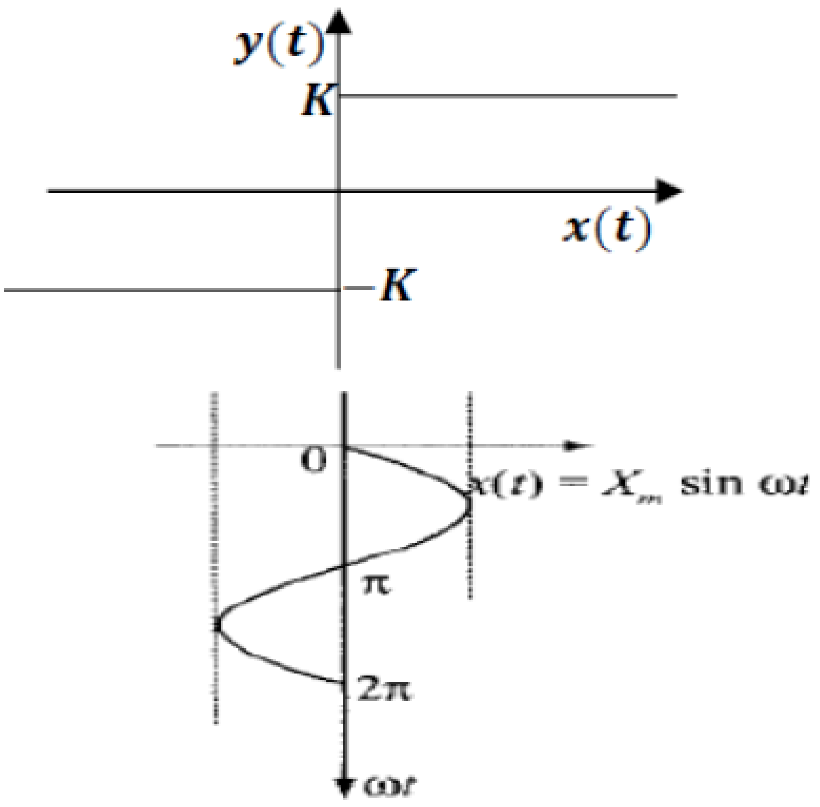
$$\left.\begin{aligned} y(t) &= -K, \; x < 0 \\ y(t) &= K, \; x \geq 0 \end{aligned}\right\}$$

if the input to the nonlinear element is a sinusoidal input given by

$$x = X_m \sin \omega t$$

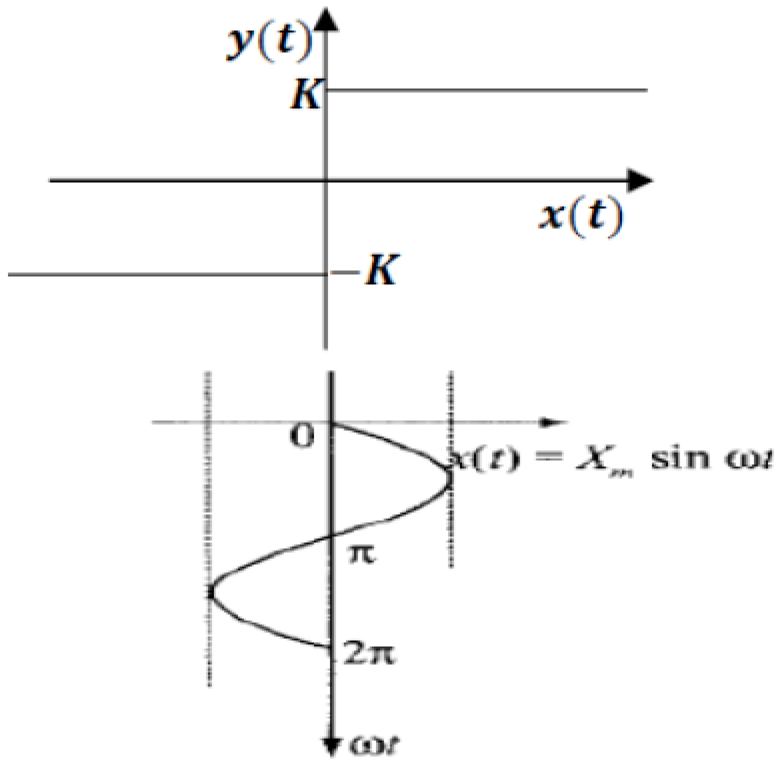
- **Step 1:**

We start by drawing $\boldsymbol{y(t)}$**v** $\boldsymbol{x(t)}$ (the nonlinearity) on the top left and $\boldsymbol{x(t)}$**v** $\boldsymbol{t}$ directly below it (i.e. at the bottom left) as shown in the diagram on the right.



- **Step 1 (contd):**

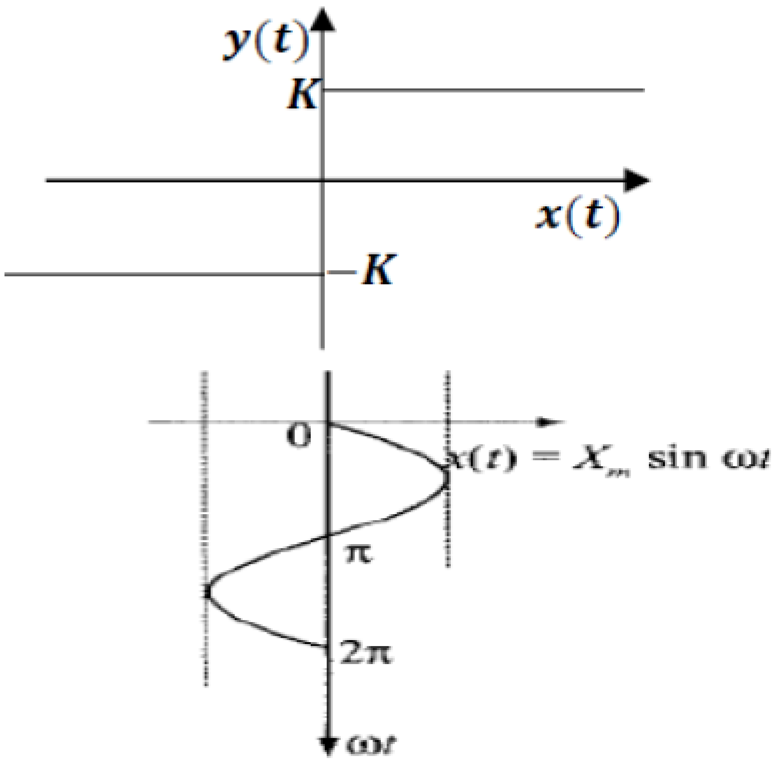
The next thing we do is to draw out the output waveform using the two plots in the previous slide. This process involves "travelling" along the plot of $\boldsymbol{x(t)}$ **v** $\boldsymbol{\omega t}$ and looking at the corresponding situations on the plot of $\boldsymbol{y(t)}$ **v** $\boldsymbol{x(t)}$.



Step 1 (contd):

- The idea is to use the $x(t)$v ωt to know how ωt changes and how $x(t)$ changes with it.

- This variation of $x(t)$ is then used together with the plot of $y(t)$v $x(t)$ to know how $y(t)$ changes with respect to ωt.

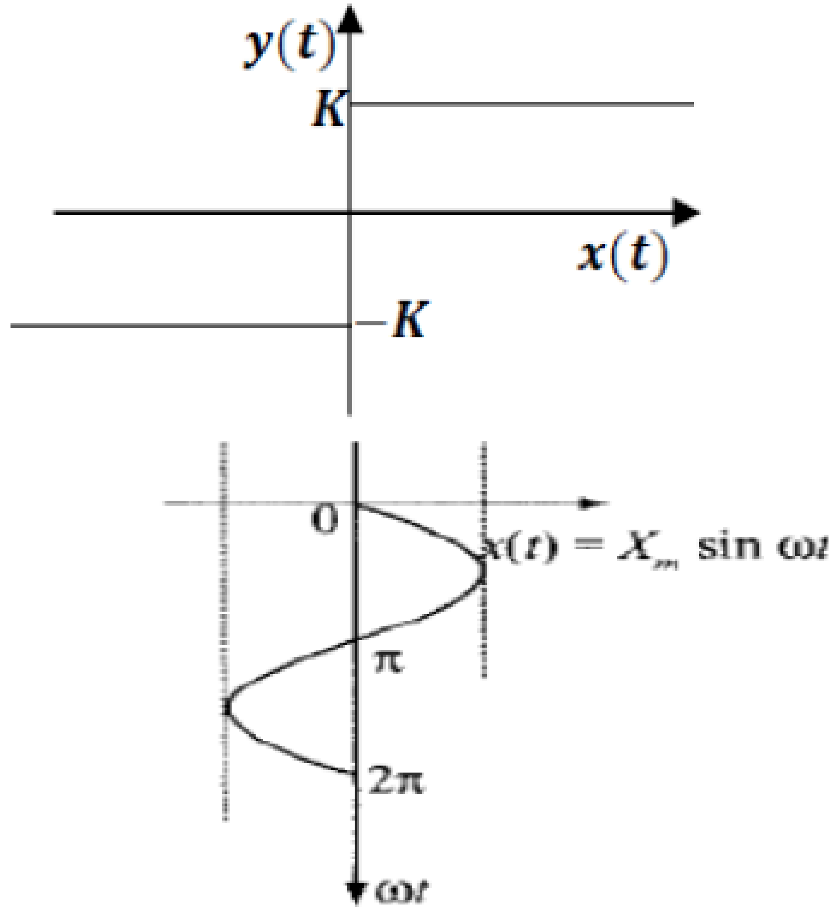


Step 1 (contd):

- When ωt changes from 0 to π, $x(t)$ increases sinusoidally from 0 to X_m and then decreases back to 0.

- For the entire time that $x(t)$ is increasing to X_m and then decreasing to 0, $y(t)$ stays at a constant value of K.

- Therefore, the graph of y vs ωt (for ωt changing from 0 to π) is simply a constant of value K.

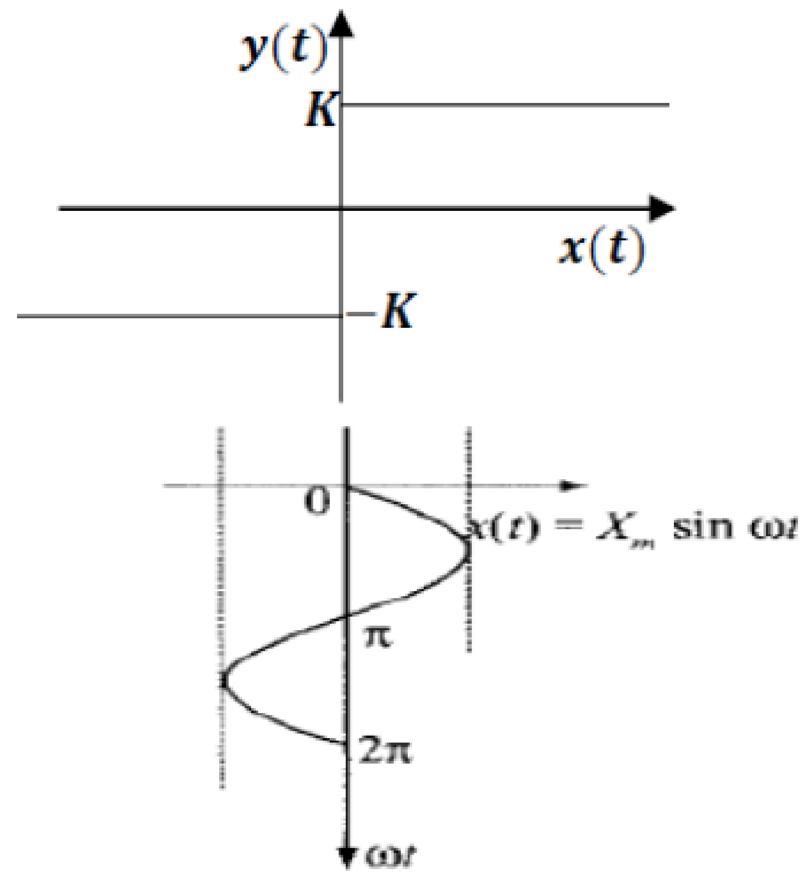


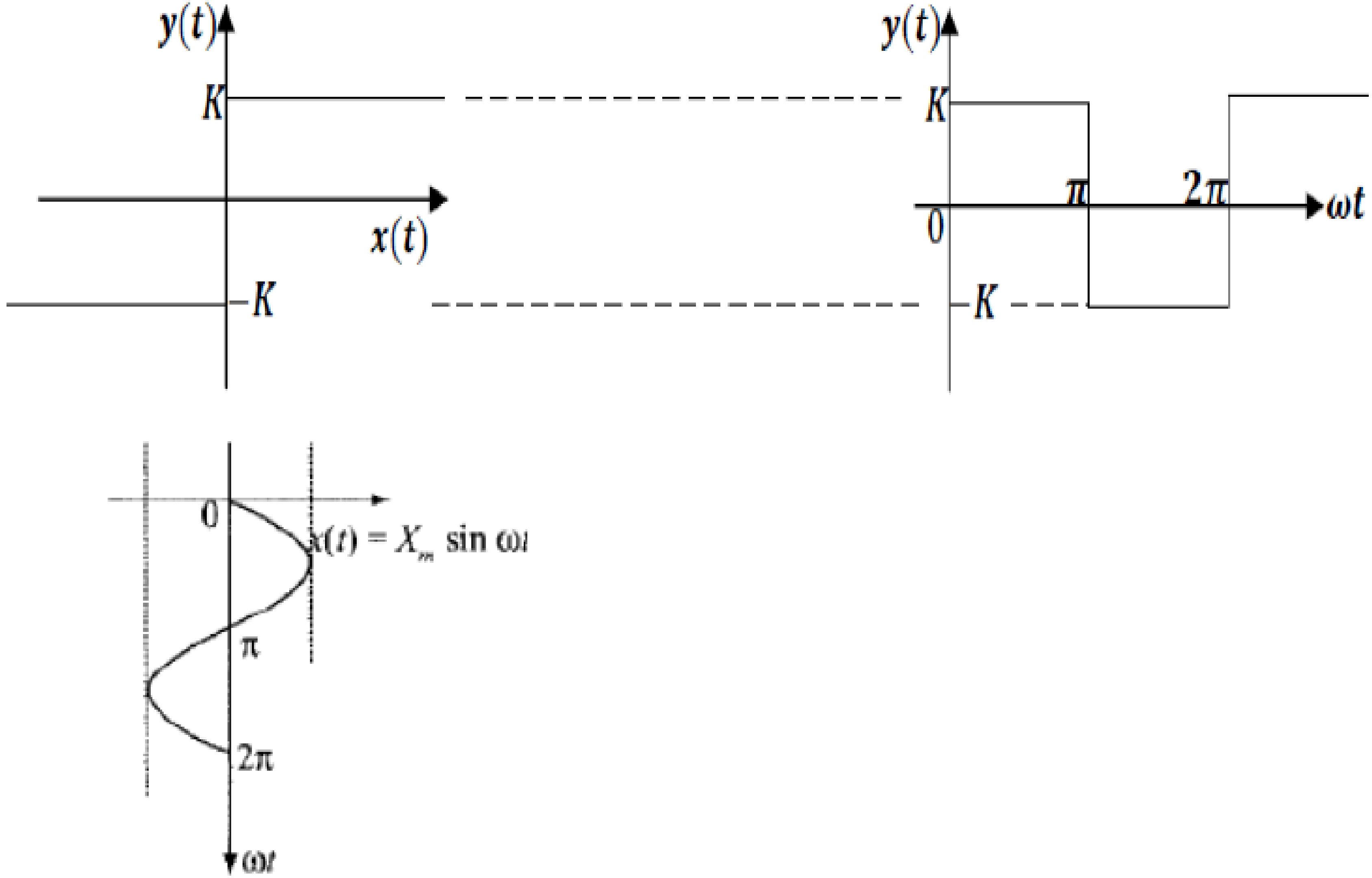
Step 1 (contd):

- When $\boldsymbol{\omega t}$ changes from $\boldsymbol{\pi}$ **to** $\boldsymbol{2\pi}$, $\boldsymbol{x}(\boldsymbol{t})$ decreases sinusoidally from $\mathbf{0}$ to $-\boldsymbol{X_m}$ and then increases back to $\mathbf{0}$.

- For the entire time that $\boldsymbol{x}(\boldsymbol{t})$ is decreasing to $-\boldsymbol{X_m}$ and then increasing to $\mathbf{0}$, $\boldsymbol{y}(\boldsymbol{t})$ again stays at another constant value of $-\boldsymbol{K}$.

- Therefore, the graph of $\boldsymbol{y}$ vs $\boldsymbol{\omega t}$ (for $\boldsymbol{\omega t}$ changing from $\boldsymbol{\pi}$ to $2\boldsymbol{\pi}$) is simply a constant of value $-\boldsymbol{K}$.





y(t)
K
x(t)
−K
y(t)
K
π
2π
ωt
0
−K
0
x(t) = X_m sin ωt
π
2π
ωt

- **Step 2:**

We now write out the mathematical relationship between $y(t)$ and ωt using the $y(t)$ vs ωt plot in the top right corner of the figure in the previous slide.

- The relationship between $y(t)$ and ωt for a full period (from $\omega t = 0$ to $\omega t = 2\pi$) is given by

$$y(t) = K, \qquad 0 \le \omega t < \pi$$
$$y(t) = -K, \qquad \pi \le \omega t < 2\pi$$

- **Step 3:**

We now check for odd symmetry (in the $\boldsymbol{y(t)}$ v $\boldsymbol{x(t)}$ plot) and half-wave symmetry (in the $\boldsymbol{y(t)}$ v $\boldsymbol{\omega t}$ plot)

- **Odd Symmetry Check**

  As seen from the Ideal relay equations and the figures for $\boldsymbol{y}$ against $\boldsymbol{x}$, the nonlinearity is oddly symmetric since

$$\boldsymbol{y(x) = -y(-x)}$$

- **Half-Wave Symmetry Check**

  Also, examining the plot of $\boldsymbol{y(t)}$ v $\boldsymbol{\omega t}$, we find by inspection that the waveform exhibits half-wave symmetry.

- **Step 3 (contd.):**

$$B_1 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} y(t)\ \sin\omega t\, d\omega t \neq 0$$

- From $\omega t = 0$ to $\omega t = \frac{\pi}{2}$, $y(t) = K$.
- Therefore,

$$B_1 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} K\ \sin\omega t\, d\omega t$$

$$B_1 = \frac{4}{\pi}[-K\cos\omega t]\,\Big|_{\omega t=0}^{\omega t=\frac{\pi}{2}}$$

$$B_1 = \frac{-4K}{\pi}\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$B_1 = \frac{4K}{\pi}$$

- **Step 4 (contd.):**

Calculate the describing function of the nonlinearity using

$$N(U, j\omega) = \frac{B_1}{U} \langle 0^0$$

In this case,

$$U = X$$

Therefore,

$$N(U, j\omega) = \frac{B_1}{U} \langle 0^0$$

$$N(X, j\omega) = \frac{4K}{\pi X} \langle 0^0$$

Example 2 on Describing Function Calculation

- A nonlinear element in a feedback system has the following input-output equations for a sinusoidal input $x = X_m \sin \omega t$

$$
\begin{aligned}
y &= -K, & x &< -d \\
&= \frac{K}{d-b}(x+b), & -d &< x < -b \\
&= 0, & -b &< x < b \\
&= \frac{K}{d-b}(x-b), & b &< x < d \\
&= K, & x &> d
\end{aligned}
$$

i) Identify this nonlinearity

ii) With all diagrams and illustrations as appropriate, and assuming that $b < X_m < d$, derive the describing function, given that all the conditions for the validity of the describing function are satisfied.

Solution to Example 2

- We go again according to the specified steps
- However, because the nonlinearity is given in equations, it has to be identified first.
- Let us take the nonlinearity in segments.

Segment 1: $y = -K, \ x < -d$

- This is straightforward to sketch

Segment 2: $y = \frac{K}{d-b}(x+b), \quad -d < x < -b$

- This is not straightforward to sketch. We'll come back to this.

Segment 3: $y = 0, \ -b < x < b$

- This is also straightforward.

Segment 4: $y = \frac{K}{d-b}(x-b), \ b < x < d$

This is not straightforward to sketch. We'll also come back to this.

Segment 5: $y = K, x > d$

This is also straightforward.

Segment 2: $y = \frac{K}{d-b}(x+b), \quad -d < x < -b$

- Using the equation $y = mx + c$, we have

$$m = \frac{K}{d-b}; c = \frac{Kb}{d-b}$$

- When $y = 0$, $0 = \frac{K}{d-b}(x+b) \Rightarrow \frac{K}{d-b}x = -\frac{Kb}{d-b} \Rightarrow x = -b$
- Since $b < d$ from the question $d - b$ is positive and therefore $m = \frac{K}{d-b}$ and $c = \frac{Kb}{d-b}$ are both positive.

- The line of the segment has a positive gradient $\frac{K}{d-b}$ and passes through the point $x = -b$ (when $y = 0$)

Segment 4: $y = \frac{K}{d-b}(x - b), \quad b < x < d$

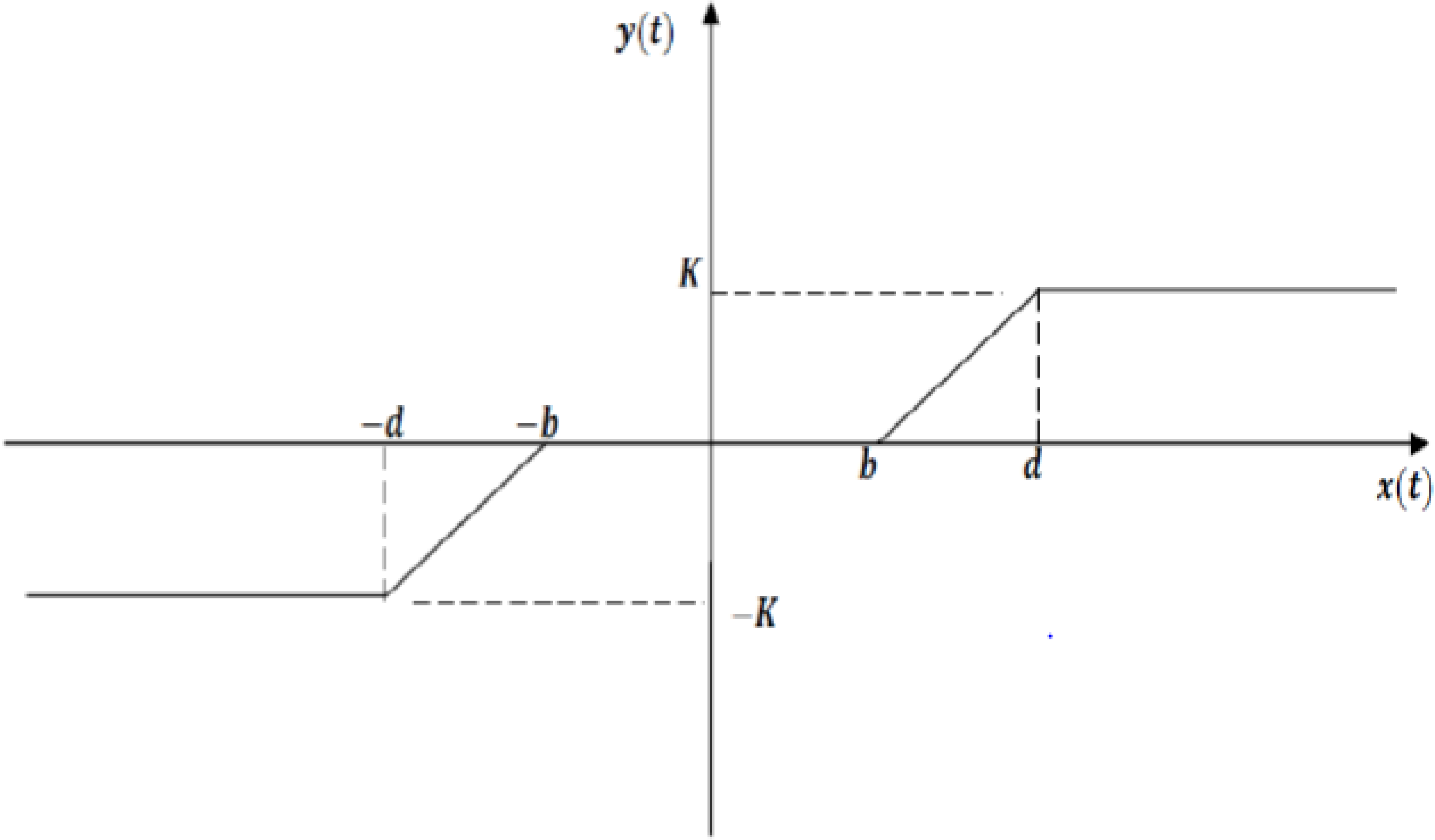
- Using the equation $y = mx + c$, we have

$$m = \frac{K}{d - b}; c = \frac{-Kb}{d - b}$$

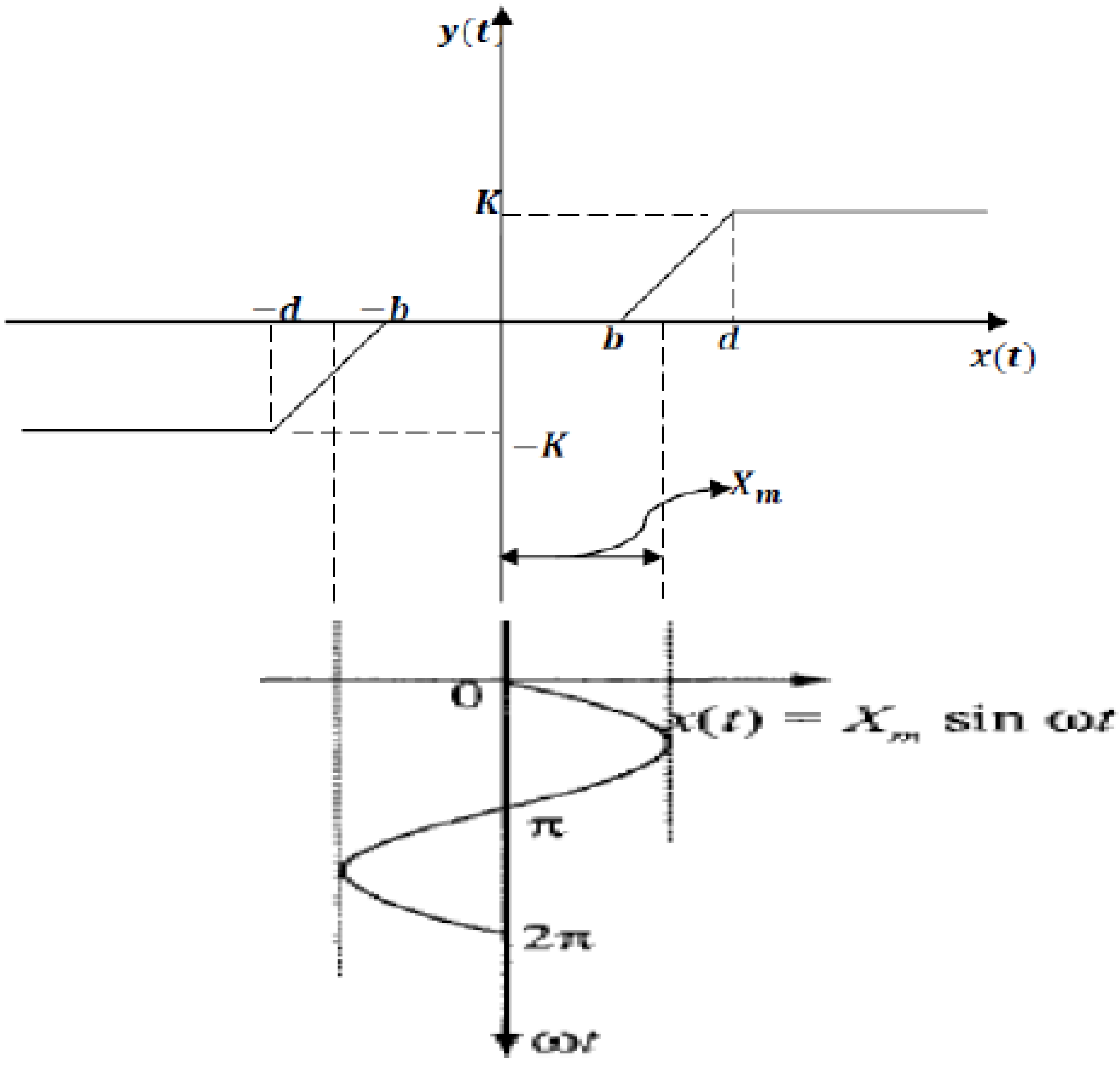
- When $y = 0$, $0 = \frac{K}{d-b}(x - b) \Rightarrow \frac{K}{d-b}x = \frac{Kb}{d-b} \Rightarrow x = b$
- Since $b < d$ from the question $d - b$ is positive.
- Therefore $m = \frac{K}{d-b}$ **is positive** and $c = -\frac{Kb}{d-b}$ is negative.

- The line of the segment has a positive gradient $\frac{K}{d-b}$ and passes through the point $x = b$ (when $y = 0$)

Putting the information together, we have a SATURATION-WITH DEADBAND NONLINEARITY.



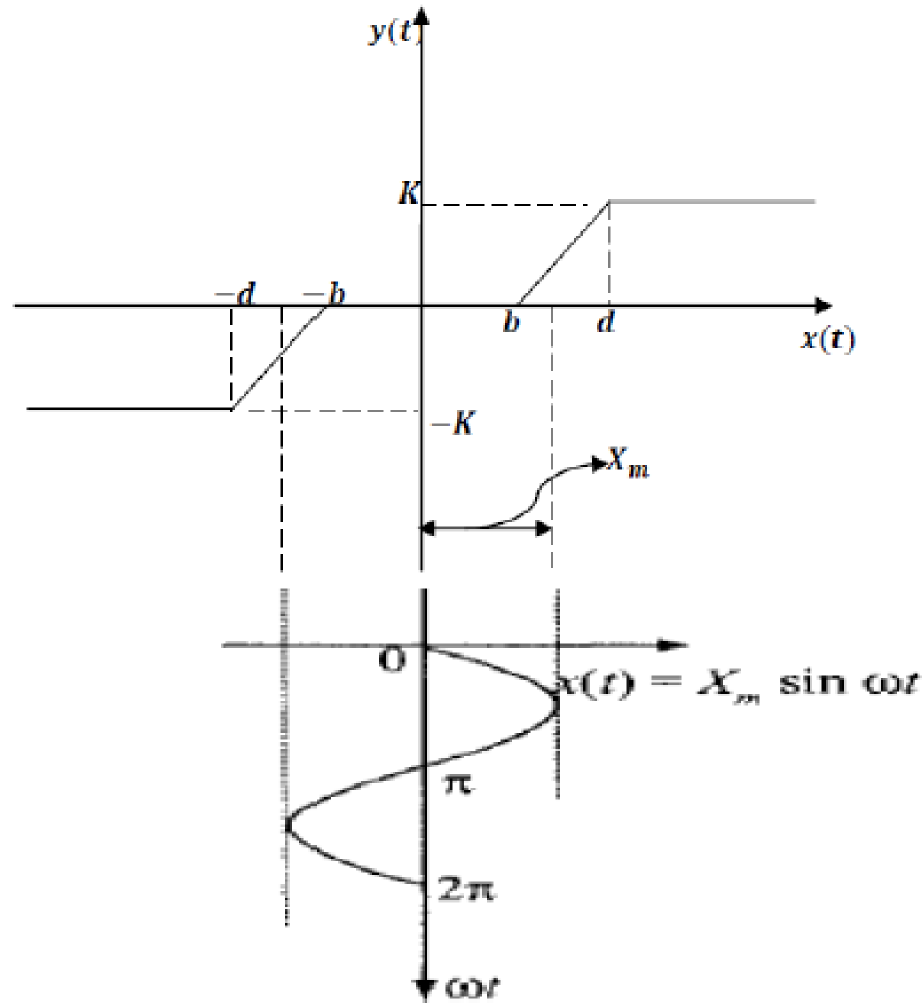
- **Step 1:**
  - Again, we start by drawing $y(t)$v $x(t)$ (the nonlinearity) on the top left and $x(t)$v t directly below it (i.e. at the bottom left) as shown in the diagram on the right.
  - We recall that we are given that $b < X_m < d$.
  - The implication of this is that the maximum value of the input sinusoid (i.e. X_m) lies between b and d



Solution to Example 2 (contd.)

Step 1 (contd):

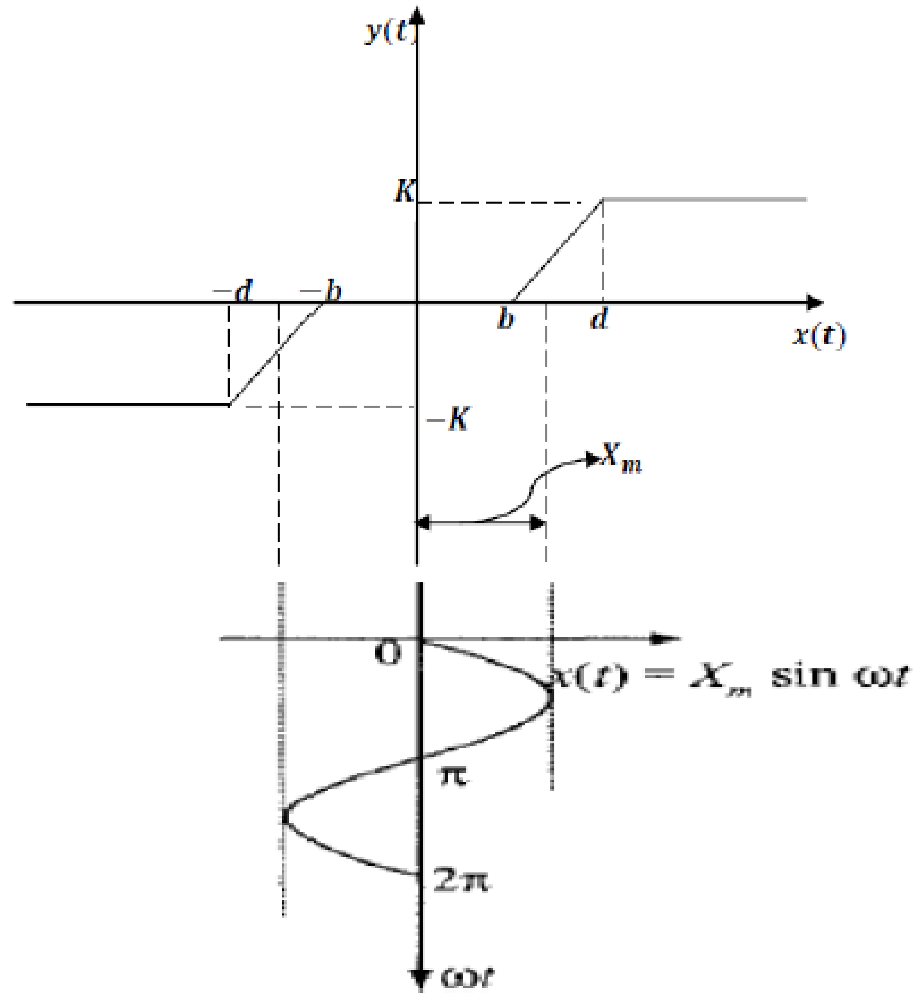
- When $\boldsymbol{\omega t}$ changes from $\mathbf{0}$ to $\boldsymbol{\pi}$, $\boldsymbol{x}(\boldsymbol{t})$ increases sinusoidally from $\mathbf{0}$ to $\boldsymbol{X_m}$ and then decreases back to $\mathbf{0}$.
- As $\boldsymbol{x}(\boldsymbol{t})$ is increasing from 0 initially, $\boldsymbol{y}(\boldsymbol{t})$ initially stays "dead" i.e. at $\boldsymbol{y} = \mathbf{0}$ until we reach $\boldsymbol{x} = \boldsymbol{b}$.
- Therefore, the graph of $\boldsymbol{y}$ vs $\boldsymbol{\omega t}$ (for $\boldsymbol{x}(\boldsymbol{t})$ changing from $\mathbf{0}$ to $\boldsymbol{b}$ initially) is simply 0.



Step 1 (contd):

- When $x(t)$ goes from $x = b$ to $x = X_m (at\ \omega t = \frac{\pi}{2})$, we notice by looking at the $y(t)$ vs $x(t)$ plot that the $x(t)$ sinusoid is simply scaled by the linear part of the nonlinearity.

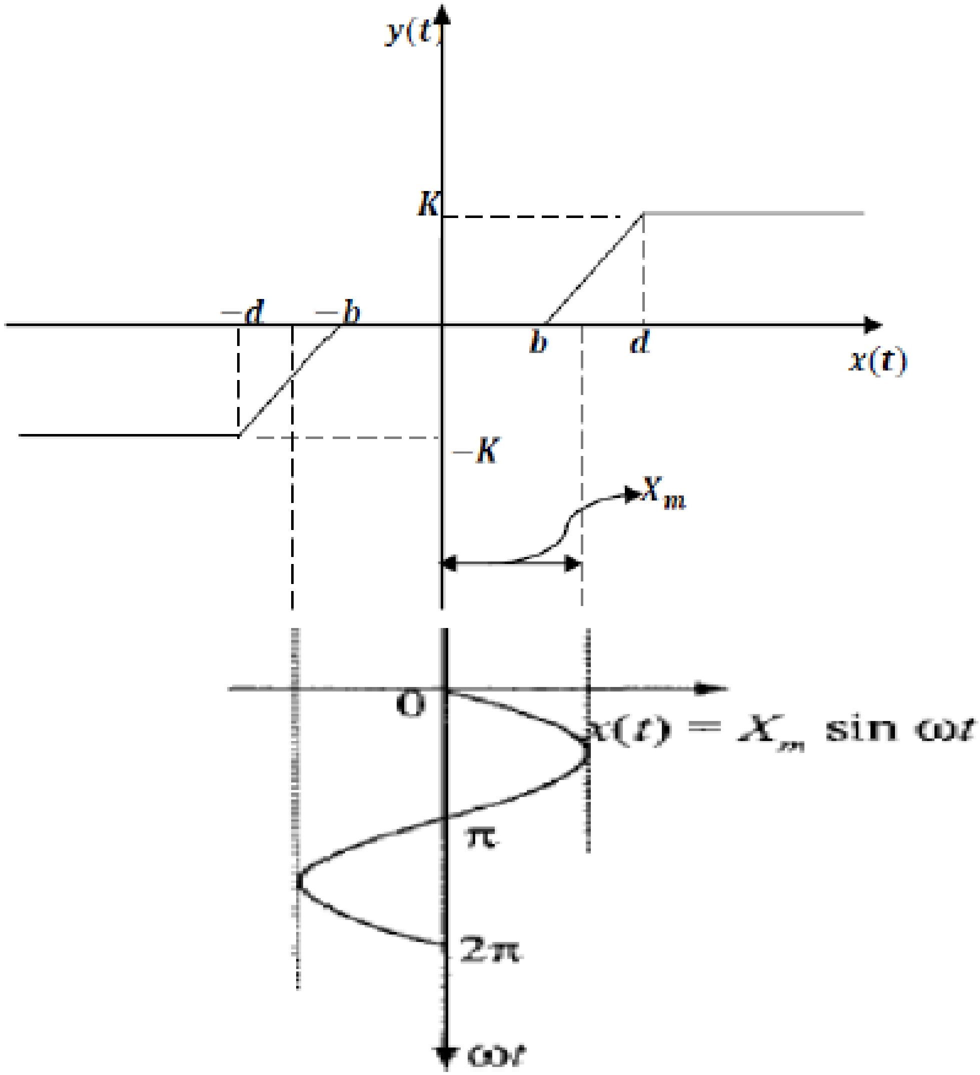
- Therefore, the $y(t)$ vs ωt plot will be sinusoidal in this region, starting from a value $y(t) = 0$ at $x = b$.



Step 1 (contd):

- When $\boldsymbol{x(t)}$ goes from $\boldsymbol{x} = \boldsymbol{X_m}$ $(\boldsymbol{at\ \omega t} = \frac{\pi}{2})$ to $\boldsymbol{x} = \boldsymbol{b}$ in the direction of increasing $\boldsymbol{\omega t}$ along the input sinusoid, the $\boldsymbol{x(t)}$ sinusoid remains scaled by the linear part of the nonlinearity.

- Therefore, the $\boldsymbol{y(t)}$ vs $\boldsymbol{\omega t}$ plot remains sinusoidal in this region.

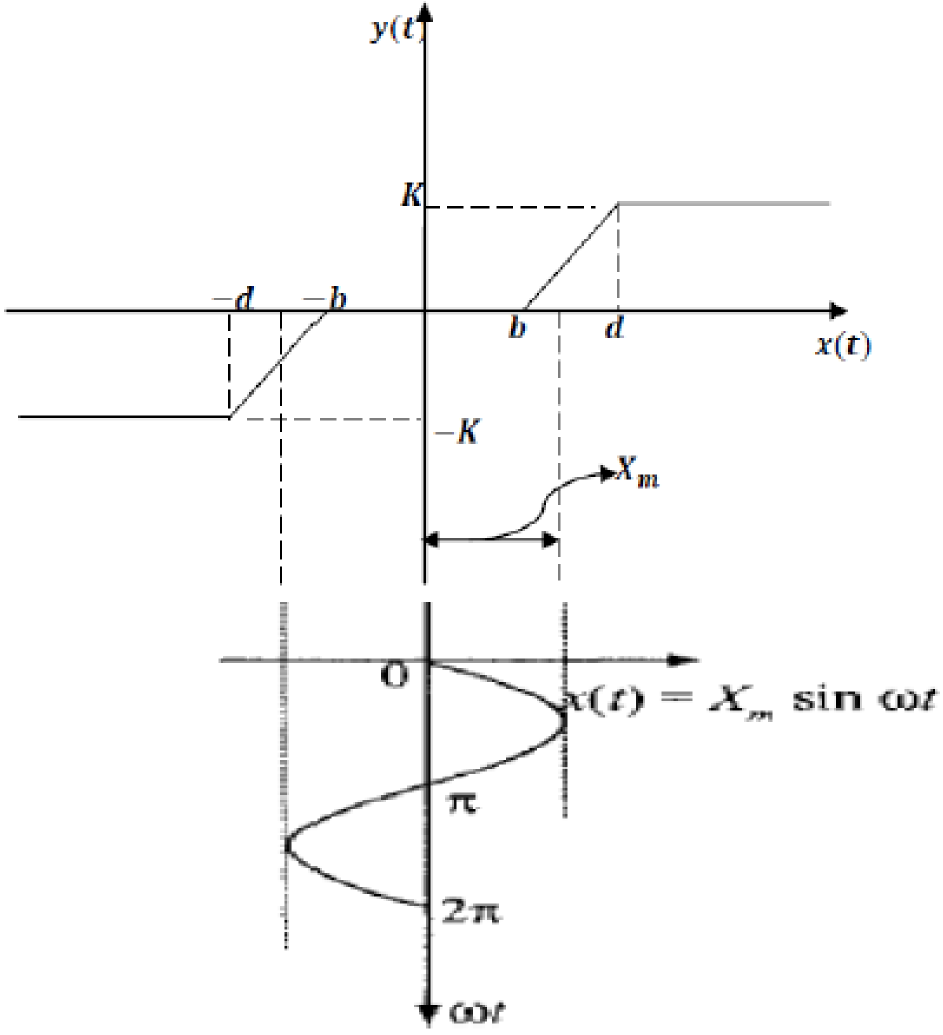


Step 1 (contd):

- When $\boldsymbol{x(t)}$ goes below $\boldsymbol{x = b}$, the corresponding value of y becomes "dead" again i.e. 0.

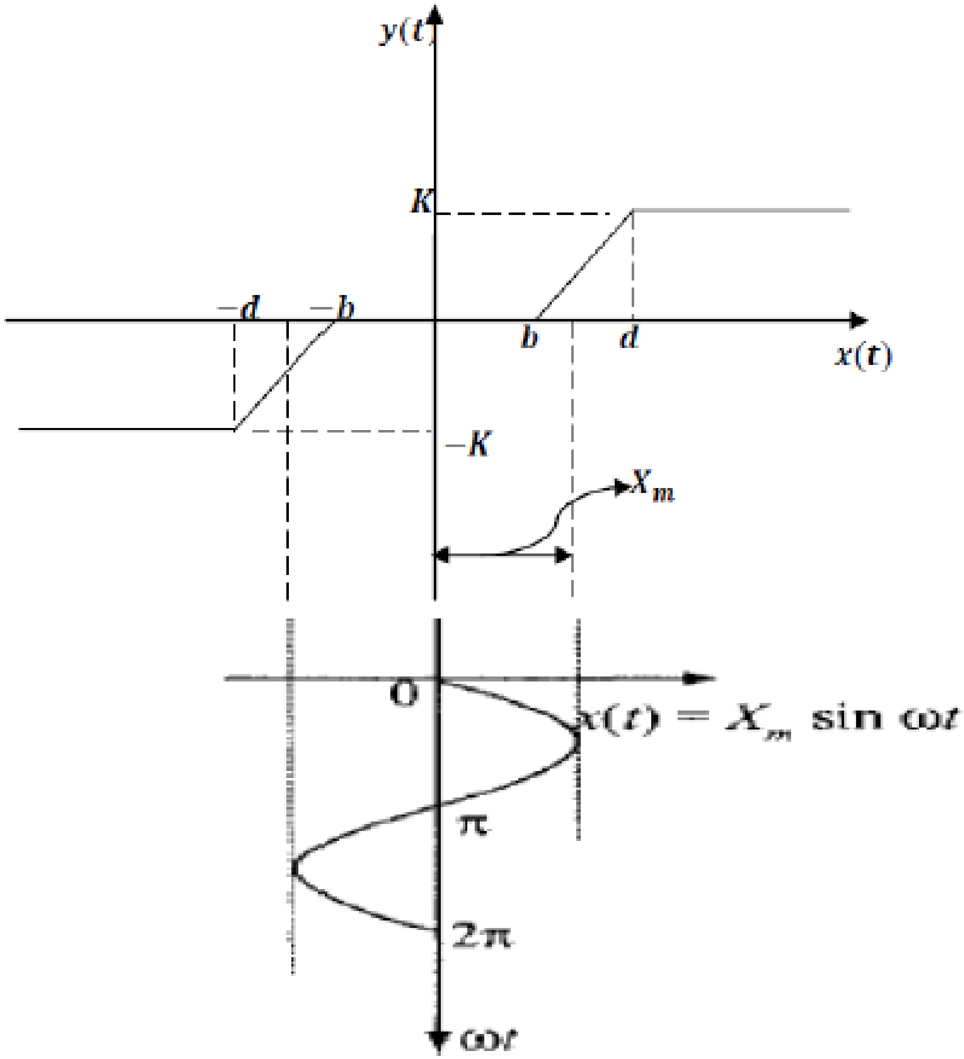
- Therefore, the $\boldsymbol{y(t)}$ vs $\boldsymbol{\omega t}$ plot changes to 0.

- This remains the case as $\boldsymbol{x(t)}$ goes from $\boldsymbol{x = b}$ through $\boldsymbol{x = 0}$ $(\boldsymbol{at\ \omega t = \pi})$ to $\boldsymbol{x = -b}$ along the input sinusoid.



- When $x(t)$ goes from $x = -b$ through $x = -X_m(at\ \omega t = \frac{3\pi}{2})$ and back to $x = -b$, we again have a scaling of the $x(t)$ sinusoid.

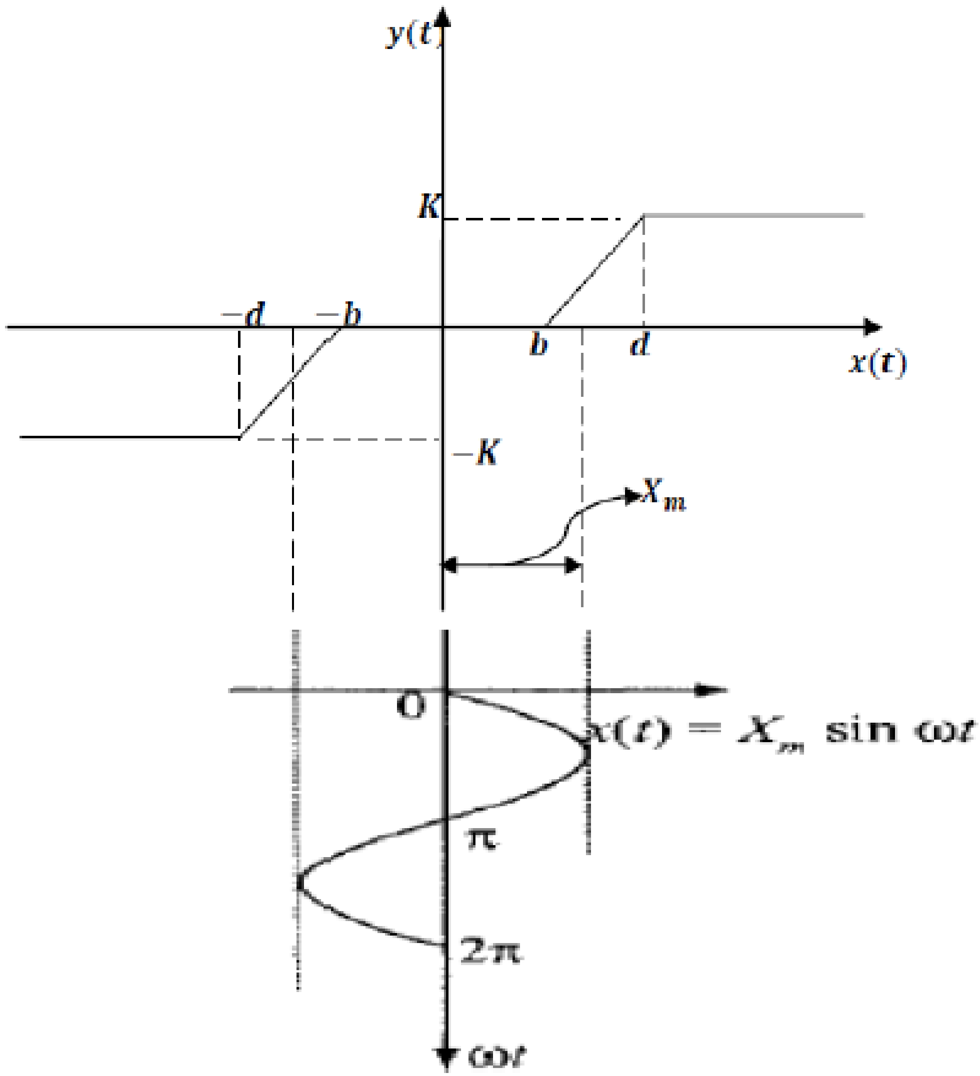
- Therefore, the $y(t)$ vs ωt plot will be sinusoidal in this region.

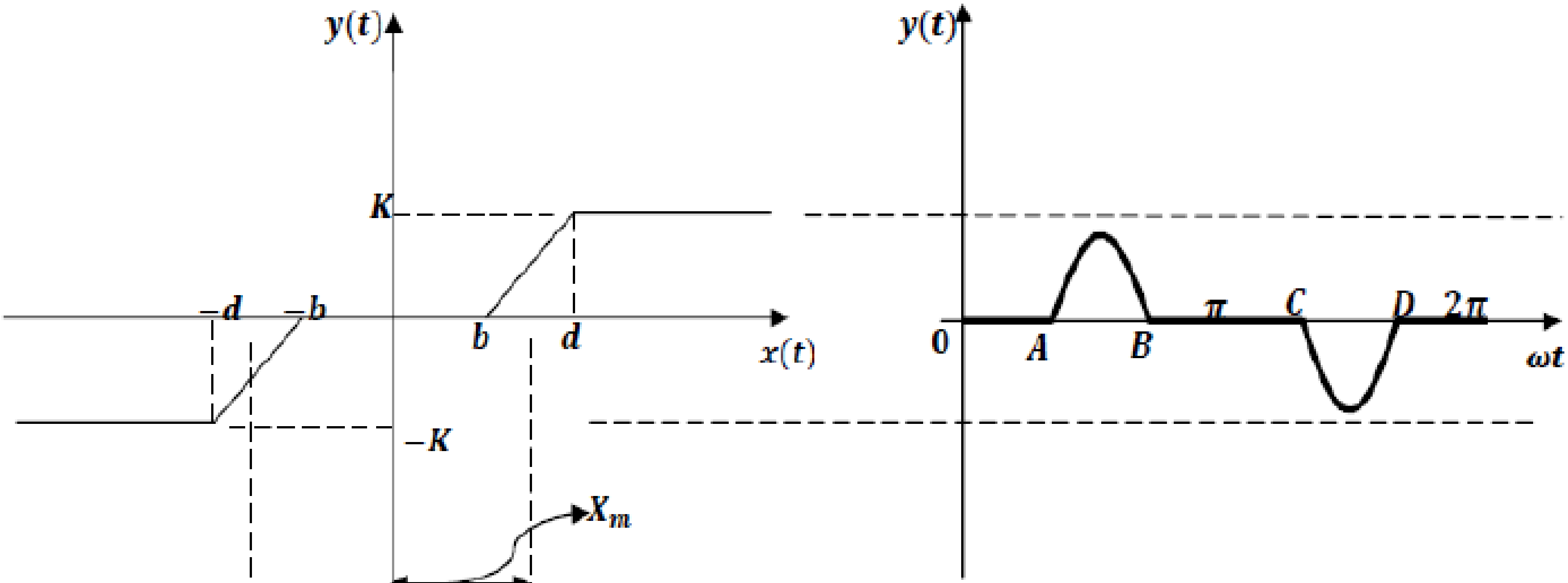


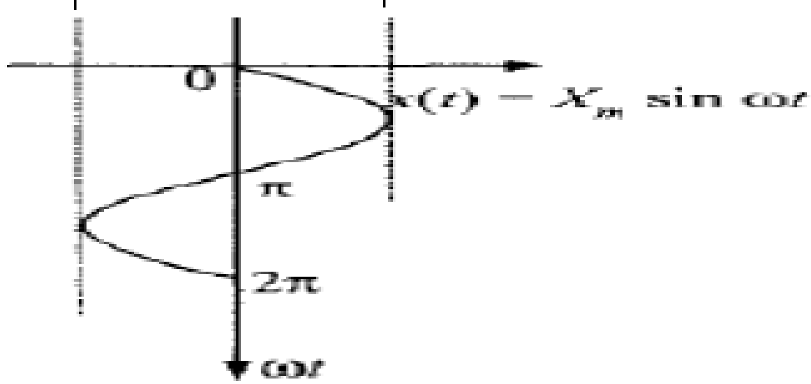
- Finally, when $\boldsymbol{x(t)}$ goes from $\boldsymbol{x = -b}$ to $\boldsymbol{x = 0}$ $(at\ \boldsymbol{\omega t = 2\pi})$, the corresponding value of y becomes "dead" again i.e. 0.

- Therefore, the $\boldsymbol{y(t)}$ vs $\boldsymbol{\omega t}$ plot changes to 0.

- The $\boldsymbol{y}$ vs $\boldsymbol{\omega t}$ plot $\boldsymbol{(\omega t}$ from $\boldsymbol{0}$ to $2\boldsymbol{\pi)}$ is clearly different from that of the Ideal Relay.







A is the point $x = b \left(0 < \omega t < \frac{\pi}{2}\right)$

B is the point $x = b \left(\frac{\pi}{2} < \omega t < \pi\right)$

C is the point $x = -b \left(\pi < \omega t < \frac{3\pi}{2}\right)$

D is the point $x = -b \left(\frac{3\pi}{2} < \omega t < 2\pi\right)$

- **Step 2:**

We now write out the mathematical relationship between $\boldsymbol{y(t)}$ and $\boldsymbol{\omega t}$ using the $\boldsymbol{y(t)}$ vs $\boldsymbol{\omega t}$ plot in the top right corner of the figure in the previous slide.

- Before writing the relationship, let us simplify our calculations by writing

$$M = \frac{K}{d - b}$$

- The relationship between $\boldsymbol{y(t)}$ and $\boldsymbol{\omega t}$ for a full period (from $\boldsymbol{\omega t = 0}$ to $\boldsymbol{\omega t = 2\pi}$) is therefore given by:

$$\begin{aligned}
y(t) &= 0, & 0 \le \omega t < \alpha_1 \\
y(t) &= M(X \sin \omega t - b), & \alpha_1 < \omega t < \alpha_2 \\
y(t) &= 0, & \alpha_2 < \omega t < \alpha_3 \\
y(t) &= M(X \sin \omega t + b), & \alpha_3 < \omega t < \alpha_4 \\
y(t) &= 0, & \alpha_4 < \omega t < 2\pi
\end{aligned}$$

- where $\boldsymbol{\alpha_1}$, $\boldsymbol{\alpha_2}$, $\boldsymbol{\alpha_3}$, and $\boldsymbol{\alpha_4}$ are the $\boldsymbol{\omega t}$ values corresponding to the points A, B, C, and D in the figure.

- **Step 3:**

We now check for odd symmetry (in the $\boldsymbol{y(t)}$ v $\boldsymbol{x(t)}$ plot) and half-wave symmetry (in the $\boldsymbol{y(t)}$ v $\boldsymbol{\omega t}$ plot)

- **Odd Symmetry Check**

  As seen from the Ideal relay equations and the figures for $\boldsymbol{y}$ against $\boldsymbol{x}$, the nonlinearity is oddly symmetric since

$$y(x) = -y(-x)$$

- **Half-Wave Symmetry Check**

  Also, examining the plot of $\boldsymbol{y(t)}$ v $\boldsymbol{\omega t}$, we find by inspection that the waveform exhibits half-wave symmetry.

- Step 3 (contd.):

$$B_1 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} y(t)\ \sin\omega t\, d\omega t \neq 0$$

- The point $\omega t = \frac{\pi}{2}$ lies between points A and B.

- Therefore,

$$y(t) = 0, 0 \leq \omega t < \alpha_1$$

$$= M(X_m\ \sin\omega t - b), \quad \alpha_1 < \omega t < \frac{\pi}{2}$$

- Thus, B_1 becomes: ,

$$B_1 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} y(t)\ \sin\omega t\, d\omega t$$

$$B_1 = \frac{4}{\pi}\left[\int_0^{\alpha_1} 0\ \sin\omega t\, d\omega t + \int_{\alpha_1}^{\frac{\pi}{2}} (M(X_m\ \sin\omega t - b))\ \sin\omega t\, d\omega t\right]$$

$$B_1 = \frac{4}{\pi}\left[\int_{\alpha_1}^{\frac{\pi}{2}} (MX_m \sin^2\omega t - Mb\ \sin\omega t) d\omega t\right]$$

We recall from Compound Angle Formulae

$$\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

- Therefore, making appropriate substitutions gives

$$B_1 = \frac{4}{\pi}\int_{\alpha_1}^{\frac{\pi}{2}} \left(MX_m\left[\frac{1-\cos 2\omega t}{2}\right] - Mb\ \sin\omega t\right)d\omega t$$

$$B_1 = \frac{4}{\pi}\int_{\alpha_1}^{\frac{\pi}{2}} \left(\frac{MX_m}{2}[1-\cos 2\omega t] - Mb\ \sin\omega t\right)d\omega t$$

$$B_1 = \frac{2MX_m}{\pi}\left[\omega t - \frac{\sin 2\omega t}{2}\right]\Big|_{\omega t=\alpha_1}^{\omega t=\frac{\pi}{2}} - \frac{4Mb}{\pi}[-\cos\omega t]\Big|_{\omega t=\alpha_1}^{\omega t=\frac{\pi}{2}}$$

$$B_1 = \frac{2MX_m}{\pi}\left[\frac{\pi}{2} - \frac{\sin 2\left(\frac{\pi}{2}\right)}{2} - \alpha_1 + \frac{\sin 2\alpha_1}{2}\right] + \frac{4Mb}{\pi}\left[\cos\frac{\pi}{2} - \cos\alpha_1\right]$$

$$B_1 = \frac{2MX_m}{\pi}\left[\frac{\pi}{2} - \frac{\sin\pi}{2} - \alpha_1 + \frac{\sin 2\alpha_1}{2}\right] - \frac{4Mb}{\pi}\cos\alpha_1$$

$$B_1 = \frac{2MX}{\pi}\left[\frac{\pi}{2} - \alpha_1 + \frac{\sin 2\alpha_1}{2}\right] - \frac{4Mb}{\pi}\cos\alpha_1$$

$$B_1 = MX_m - \frac{2MX_m\alpha_1}{\pi} + \frac{MX_m\sin 2\alpha_1}{\pi} - \frac{4Mb}{\pi}\cos\alpha_1$$

- Now, defining α_1

$$x = b, \omega t = \alpha_1$$

$$x = X_m \sin \omega t$$

$$b = X_m \sin \alpha_1$$

$$\alpha_1 = \sin^{-1}\left(\frac{b}{X_m}\right)$$

- But we know that

$$\sin^2\theta + \cos^2\theta = 1$$

- Therefore, we have

$$\cos\theta = \sqrt{1 - \sin^2\theta}$$

$$\cos\alpha_1 = \sqrt{1 - \sin^2\alpha_1}$$

$$\cos\alpha_1 = \sqrt{1 - (\frac{b}{X_m})^2}$$

$$\cos\alpha_1 = \frac{\sqrt{X_m{}^2 - b^2}}{X_m}$$

- Substituting for $cos\ \alpha_1$ in the expression for B_1:

$$B_1 = MX_m - \frac{2MX_m\alpha_1}{\pi} + \frac{MX_m sin\ 2\alpha_1}{\pi} - \frac{4Mb}{\pi}\frac{\sqrt{X_m{}^2 - b^2}}{X_m}$$

- The describing function is therefore given by

$$N(X_m, j\omega) = \frac{B_1}{X_m}\langle 0^0$$

$$N(X_m, j\omega) = \frac{1}{X_m}\left[MX_m - \frac{2MX_m\alpha_1}{\pi} + \frac{MX_m sin\ 2\alpha_1}{\pi} - \frac{4Mb}{\pi}\frac{\sqrt{X_m{}^2 - b^2}}{X_m}\right]\langle 0^0$$

$$N(X_m, j\omega) = \left[M - \frac{2M\alpha_1}{\pi} + \frac{M sin\ 2\alpha_1}{\pi} - \frac{4Mb}{\pi}\frac{\sqrt{X_m{}^2 - b^2}}{X_m{}^2}\right]\langle 0^0$$

- Now, we can re-write $\frac{4Mb}{\pi}\frac{\sqrt{X_m{}^2 - b^2}}{X_m{}^2}$ as

$$\frac{4Mb}{\pi}\frac{\sqrt{X_m{}^2 - b^2}}{X_m{}^2} = \frac{4M}{\pi}.\frac{b}{X_m}.\frac{\sqrt{X_m{}^2 - b^2}}{X_m} = \frac{4M}{\pi}.sin\ \alpha_1 . cos\ \alpha_1$$

- We know from elementary trigonometry that

$$\sin 2\alpha_1 = 2\sin\alpha_1 . \cos\alpha_1$$

- Then

$$\frac{4Mb}{\pi}\frac{\sqrt{{X_m}^2-b^2}}{{X_m}^2} = \frac{2M}{\pi}.2\sin\alpha_1 . \cos\alpha_1 = \frac{2M\sin 2\alpha_1}{\pi}$$

- Therefore, replacing $\frac{4Mb}{\pi}\frac{\sqrt{{X_m}^2-b^2}}{{X_m}^2}$ with $\frac{2M\sin 2\alpha_1}{\pi}$ in the expression for $N(X_m, j\omega)$:

$$N(X_m, j\omega) = \left[M - \frac{2M\alpha_1}{\pi} + \frac{M\sin 2\alpha_1}{\pi} - \frac{2M\sin 2\alpha_1}{\pi}\right]\langle 0^0$$

$$N(X_m, j\omega) = \left[M - \frac{2M\alpha_1}{\pi} - \frac{M\sin 2\alpha_1}{\pi}\right]\langle 0^0$$

$$\boldsymbol{N(X_m, j\omega) = \frac{2M}{\pi}\left[\frac{\pi}{2} - \alpha_1 - \frac{\sin 2\alpha_1}{2}\right]\langle 0^0}$$

- The student is left to verify that if

$$X_m > d$$

then the describing function is given by

$$N(X_m, j\omega) = \frac{2M}{\pi}\left[\beta - \alpha_1 - \frac{\sin 2\alpha_1 - \sin 2\beta}{2}\right]\langle 0^0$$

where

$$\alpha_1 = \sin^{-1}\left(\frac{\mathrm{b}}{X_m}\right)$$

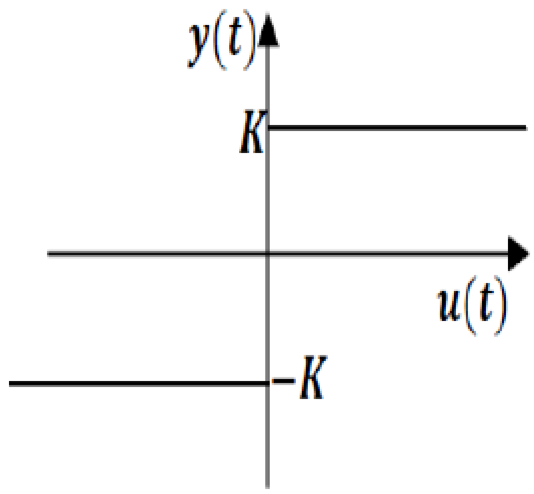
and

$$\beta = \sin^{-1}\left(\frac{\mathrm{d}}{X_m}\right)$$

Describing Functions of Some Common Nonlinearities

The following are the describing functions of some of the common nonlinearities:

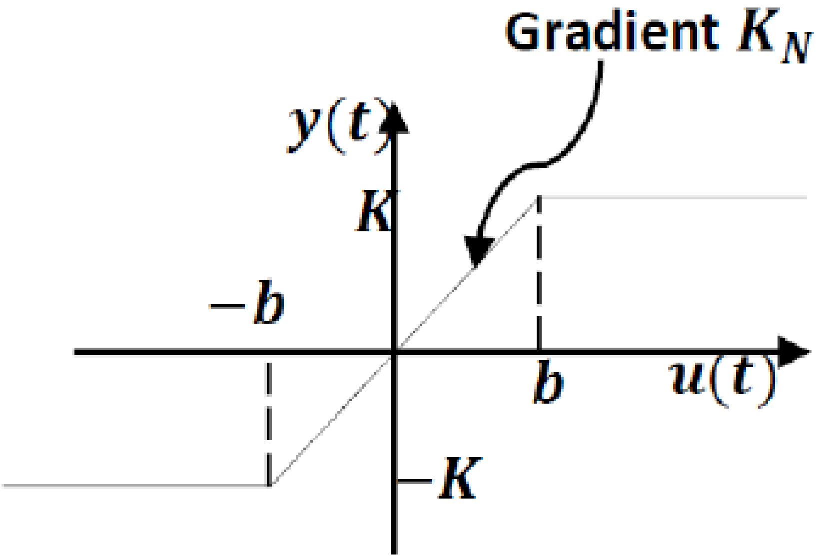
- **Nonlinearity 1:**
  Ideal Relay Nonlinearity



- The describing function for the Ideal-Relay nonlinearity is given by

$$N(U, j\omega) = \frac{4K}{\pi U} \langle 0^0$$

- **Nonlinearity 2:** Saturation/Limitation



- The describing function for the Saturation/Limitation nonlinearity is given by

$$\boldsymbol{N(U,j\omega) = K_N \langle 0^0} \quad \text{if } \boldsymbol{U} < b$$

$$\boldsymbol{N(U,j\omega) = \frac{2K_N}{\pi}\left[\alpha_1 + \frac{\sin 2\alpha_1}{2}\right] \langle 0^0} \quad \text{if } \boldsymbol{U} > b$$

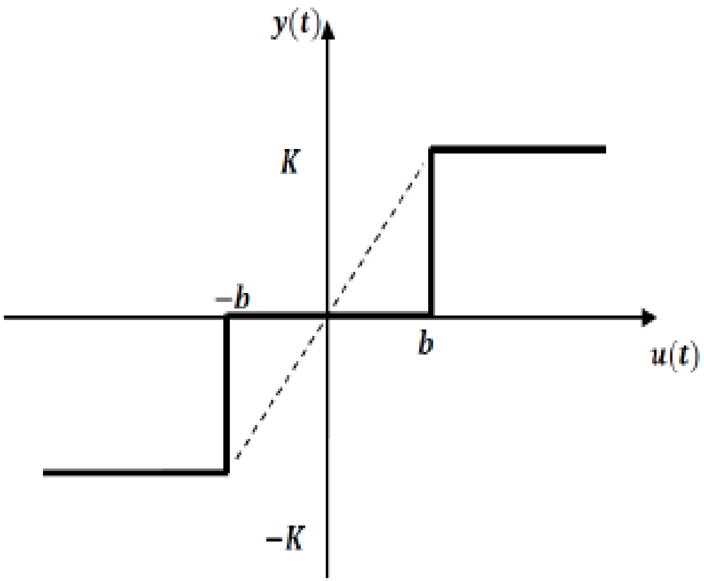
where

$$\boldsymbol{\alpha_1 = \sin^{-1}\left(\frac{b}{U}\right)}$$

and the gradient $\boldsymbol{K_N}$ is given by

$$\boldsymbol{K_N = \frac{K}{b}}$$

- **Nonlinearity 3:** Three-Position Relay with Dead Band



- The describing function for the Saturation/Limitation nonlinearity is given by

$$N(U, j\omega) = 0 \langle 0^0 \quad \text{if } U < b$$

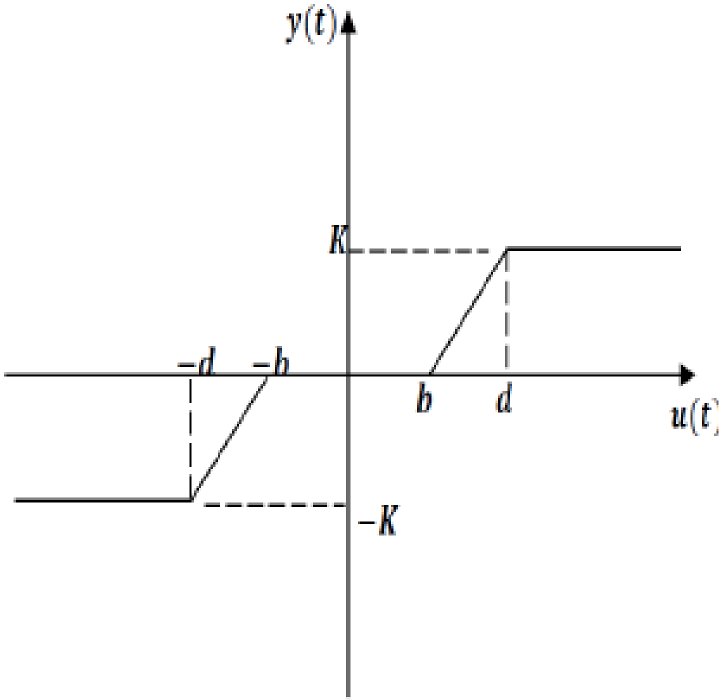
$$N(U, j\omega) = \frac{4K}{\pi U}\sqrt{\left[1 - \left[\frac{b}{U}\right]^2\right]} \langle 0^0$$

if $U > b$

where

$$\alpha_1 = \sin^{-1}\left(\frac{b}{U}\right)$$

- **Nonlinearity 4:** Saturation with Dead Band



- The describing function for the Saturation-with-Dead-Band nonlinearity is given by

$$N(U, j\omega) = 0 \langle 0^0 \quad \text{if } U < b;$$

$$N(U, j\omega) = \frac{2K_N}{\pi}\left[\frac{\pi}{2} - \alpha - \frac{\sin 2\alpha}{2}\right] \langle 0^0 \quad \text{if } b < U < d$$

$$N(U, j\omega) = \frac{2K_N}{\pi}\left[\beta - \alpha - \frac{\sin 2\alpha - \sin 2\beta}{2}\right] \langle 0^0 \text{ if } U > d$$

where α and β are respectively given by

$$\alpha = \sin^{-1}\left(\frac{b}{U}\right)$$

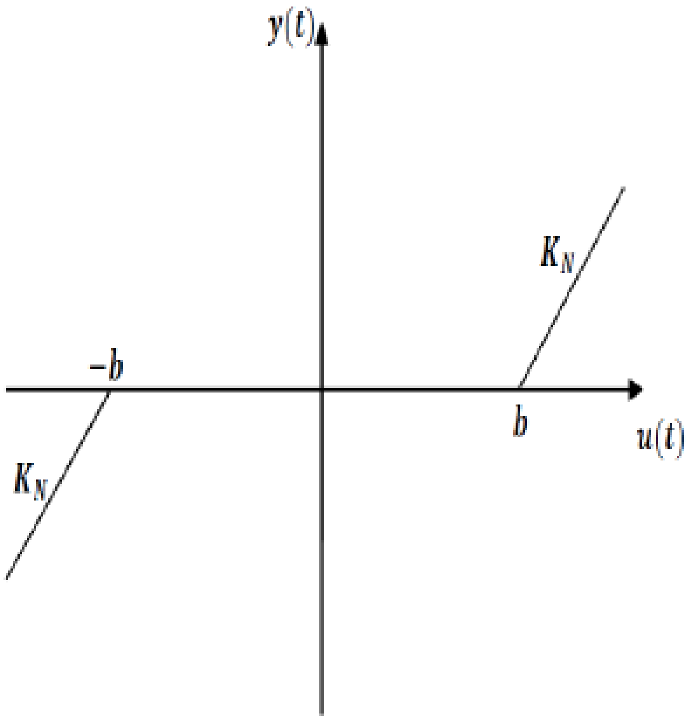
and

$$\beta = \sin^{-1}\left(\frac{d}{U}\right)$$

and the gradient K_N is given by

$$K_N = \frac{K}{d - b}$$

- **Nonlinearity 5:** Dead Band I/Dead Zone I/Threshold I



- The describing function for the Dead-Band-I/Dead-Zone-I/Threshold-I nonlinearity is given by

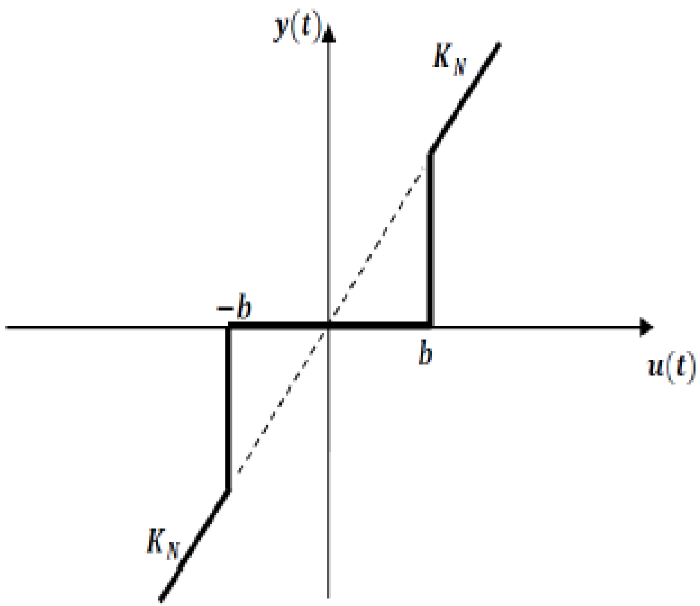
$$N(U,j\omega) = 0\,\langle 0^0 \qquad \text{if } U < b$$

$$N(U,j\omega) = \frac{2K_N}{\pi}\left[\frac{\pi}{2} - \alpha - \frac{\sin 2\alpha}{2}\right]\langle 0^0 \qquad \text{if } U > b$$

where α is given by

$$\alpha = \sin^{-1}\left(\frac{b}{U}\right)$$

- **Nonlinearity 6:**
  Dead Band II/Dead Zone II/Threshold II



- The describing function for the Dead-Band-II/Dead-Zone-II/Threshold-II nonlinearity is given by

$$\boldsymbol{N(U,j\omega)} = \boldsymbol{0\,\langle 0^0} \qquad \text{if } \boldsymbol{U} < b$$

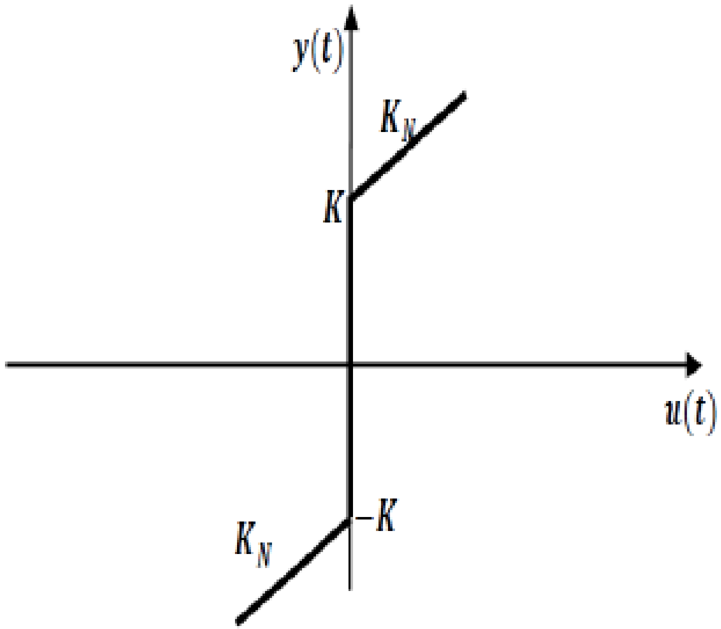
$$\boldsymbol{N(U,j\omega)} = \frac{\boldsymbol{2K_N}}{\boldsymbol{\pi}}\left[\frac{\boldsymbol{\pi}}{\boldsymbol{2}} - \boldsymbol{\alpha} + \frac{\mathbf{sin\,2}\boldsymbol{\alpha}}{\mathbf{2}}\right]\boldsymbol{\langle 0^0}$$

if $\boldsymbol{U > b}$

where $\boldsymbol{\alpha}$ is given by

$$\boldsymbol{\alpha} = \mathbf{sin}^{-1}\left(\frac{\mathbf{b}}{\boldsymbol{U}}\right)$$

- **Nonlinearity 7:**
  Negative Deficiency

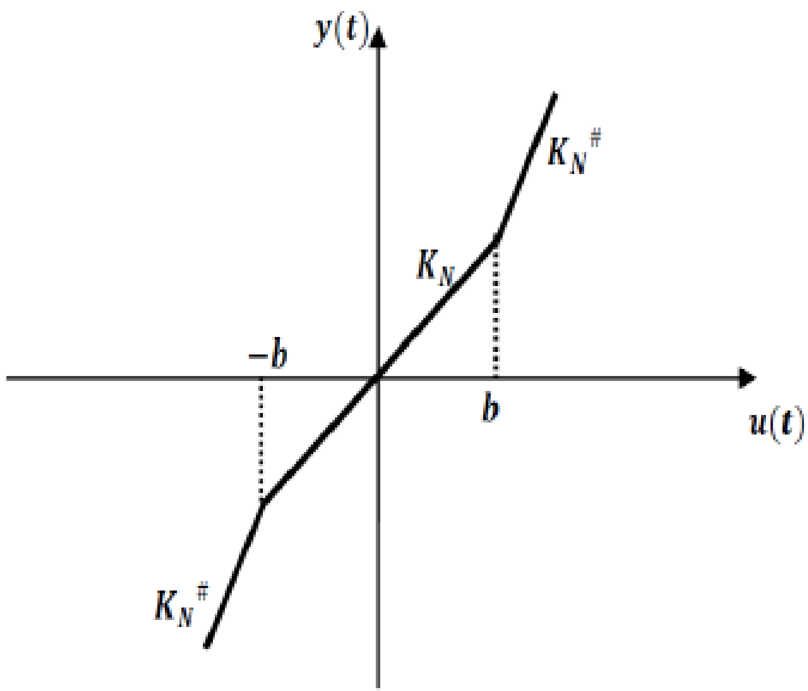


- The describing function for the Negative-Deficiency nonlinearity is given by

$$\boldsymbol{N(U,j\omega) = \left[\frac{4K}{\pi U} + K_N\right]\langle 0^0}$$

for all possible values of $\boldsymbol{U}$

- **Nonlinearity 8:**
  Variable Gain



- The describing function for the Variable-Gain nonlinearity is given by

$$N(U, j\omega) = K_N \langle 0^0 \qquad \text{if } U < b$$

$$N(U, j\omega) = \left(K_N^{\#} - \frac{2}{\pi}\left(K_N^{\#} - K_N\right)\left(\alpha + \frac{\sin 2\alpha}{2}\right)\right) \langle 0^0$$

$$\text{if } \quad U > b$$

where α is given by

$$\alpha = \sin^{-1}\left(\frac{b}{U}\right)$$