ENGINEERING MATHEMATICS II (ENG 282) LECTURE NOTE

MODELLING AND SIMULATION OF FIRST ORDER SYSTEMS

RADIOACTIVITY

Experiments have shown that a radioactive substance decomposes at each instant of time in a manner that is proportional to the amount of the substance present.

If the amount of substance present at any time t is denoted by $y(t)$, applying the physical law, the time rate of change of the substance is proportional to $y(t)$. That is,

$$\frac{dy(t)}{dt} \propto y(t) \qquad (1)$$

$$\frac{dy}{dt} \propto y \qquad (2)$$

$$\frac{dy}{dt} = ky \qquad (3)$$

$$\frac{dy}{y} = kdt \qquad (4)$$

$$\int \frac{dy}{y} = \int kdt \qquad (5)$$

$$\ln(y) = kt + C \qquad (6)$$

$$y = e^{kt+C} \qquad (7)$$

$$y = e^{kt} \times e^{C} \qquad (8)$$

$$y = e^{kt} \times y_0 \qquad (9)$$

$$y = y_0 e^{kt} \qquad (10)$$

For example, given that the initial amount of the substance is 0.5 g,

$$0.5g = y_0 e^{k(0)} \tag{11}$$

$$e^{k(0)} = e^0 = 1 \tag{12}$$

Therefore,

$$0.5g = y_0 \times 1 \tag{13}$$

$$y_0 = 0.5g \tag{14}$$

So,

$$y = (0.5g)e^{kt} \tag{15}$$

The constant, k, is positive for exponential growth while it is negative for decay.

Assuming that $k = -1.5hr^{-1}$ (a decay problem),

$$y = (0.5g)e^{\left(-1.5hr^{-1}\right)t} \tag{16}$$

MATLAB *mfile* for Simulation

```
commandwindow
clearvars
clc
close all

t = 0:0.01:5;
y = 0.5*exp(-1.5*t);
plot(t,y)
xlabel('Time (hr)')
ylabel('Amount of substance present (g)')
grid on
grid minor
```

Graphical Output

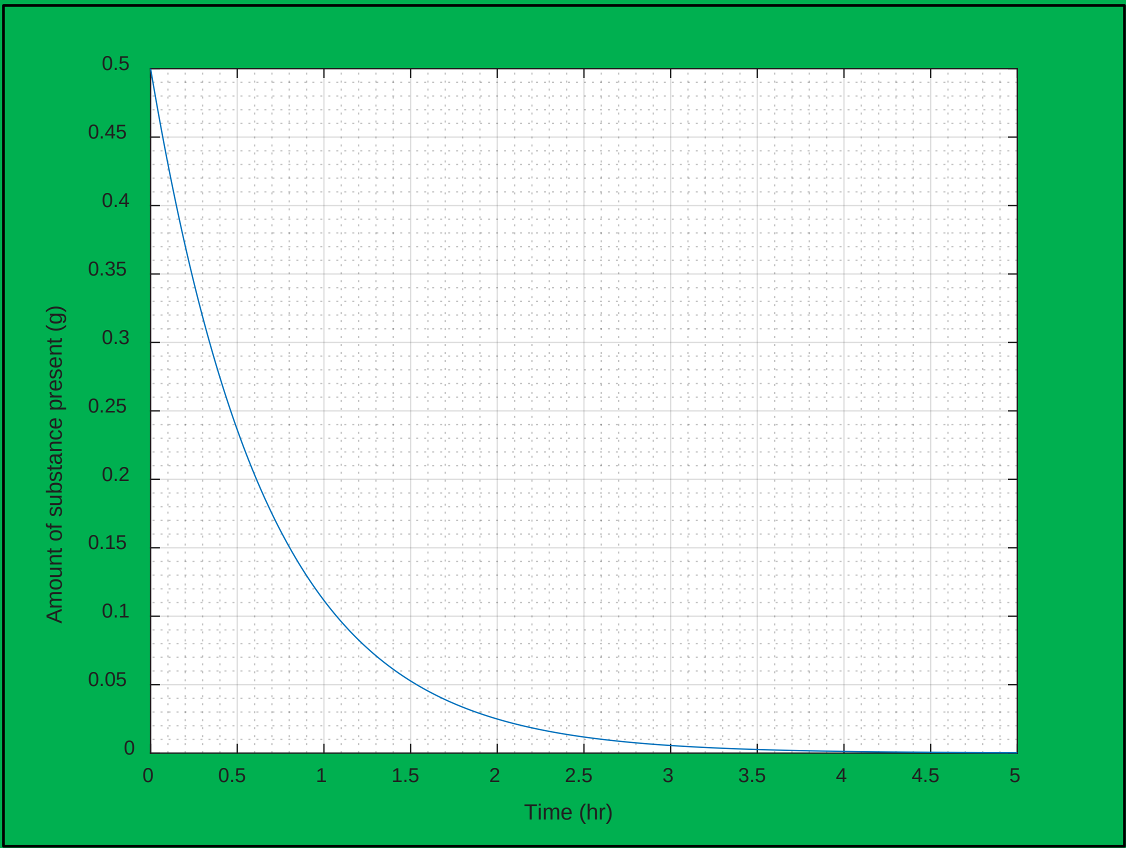

Figure 1: Plot of dynamic response of amount of substance present at t = 5 hr

MIXING

Mixing problems occur quite frequently in industries. Problems of this nature can be solved by taking material balance across the system. To take the material balance, balance law is applied, and it is expressed mathematically as given in Equation (17).

$$\begin{Bmatrix} \text{Accumulation rate} \\ \text{within a system} \end{Bmatrix} = \begin{Bmatrix} \text{Input rate into} \\ \text{the system} \end{Bmatrix} - \begin{Bmatrix} \text{Output rate from} \\ \text{the system} \end{Bmatrix} \tag{17}$$

For instance, considering the tank shown in Figure 2 that contains 1000 gal of water in which 100 lb of salt is dissolved initially. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissolved salt. The mixture in the tank is kept uniform by stirring.

Brine runs out at 10 gal/min. Derive an expression for finding the amount of salt in the tank at any time *t*. Also, plot the dynamic response of the system for $0 \le t \le 700 \text{min}$.

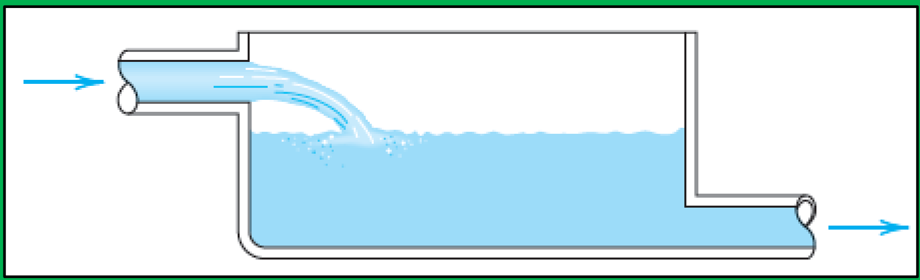

Figure 2: A tank system for mixing (Source: Kreyszig, 2011)

Applying the balance law,

$$\left\{ \begin{matrix} Accumulation\ rate\ of \\ salt\ within\ a\ system \end{matrix} \right\} = \left\{ \begin{matrix} Input\ rate\ of\ salt \\ into\ the\ system \end{matrix} \right\} - \left\{ \begin{matrix} Output\ rate\ of\ salt \\ from\ the\ system \end{matrix} \right\} \quad (18)$$

Denoting the amount of salt present in the tank at any time *t* as *y*, its time rate of change is given as:

$$\frac{dy}{dt} = \dot{y}_{in} - \dot{y}_{out} \quad (19)$$

Since 10 gallons enter per minute and one gallon contains 5 lb of salt, it means that the amount of salt entering the tank is:

$$\dot{y}_{in} = 10\frac{gal}{\min} \times 5\frac{lb}{gal} = 50\frac{lb}{\min} \quad (20)$$

The tank contains 1000 gal of water with the dissolved salt, and 10 gallons of the solution leave the tank per minute. That is, $\frac{10gal}{1000gal} = 0.01 = 1\%$ of the content of the tank. If that is the case, 1% of the salt present in the tank will also leave the tank per minute. In other words,

$\dot{y}_{out} = 1\% of\ y$.

Therefore, from Equation (19),

$$\frac{dy}{dt}\frac{lb}{\min} = 50\frac{lb}{\min} - 1\%y\frac{lb}{\min} \tag{21}$$

$$\frac{dy}{dt} = 50 - 0.01y \tag{22}$$

$$\frac{dy}{dt} = -0.01y + 50 \tag{23}$$

$$\frac{dy}{dt} = -0.01\left(\frac{-0.01y}{-0.01} + \frac{50}{-0.01}\right) \tag{24}$$

$$\frac{dy}{dt} = -0.01(y - 5000) \tag{25}$$

$$\frac{dy}{(y - 5000)} = -0.01dt \tag{26}$$

$$\int \frac{dy}{(y - 5000)} = \int -0.01dt \tag{27}$$

$$\int \frac{dy}{(y - 5000)} = -0.01\int dt \tag{28}$$

$$\ln(y - 5000) = -0.01t + C \tag{29}$$

$$y - 5000 = e^{-0.01t + C} \tag{30}$$

$$y - 5000 = e^{-0.01t}e^{C} \tag{31}$$

$$y - 5000 = e^{-0.01t}y_0 \tag{32}$$

$$y - 5000 = y_0e^{-0.01t} \tag{33}$$

$$y = y_0e^{-0.01t} + 5000 \tag{34}$$

Given that when $t = 0\min(initially)$, $y = 100\,lb$,

$$100 = y_0e^{-0.01(0)} + 5000 \tag{35}$$

$$100 - 5000 = y_0 \times 1 \quad (36)$$

$$y_0 = -4900 \quad (37)$$

So,

$$y = -4900e^{-0.01t} + 5000 \quad (38)$$

$$y = 5000 - 4900e^{-0.01t} \quad (39)$$

With the aid of Microsoft Excel and MathCAD, the dynamic response has been obtained to be as shown in Figure 2.

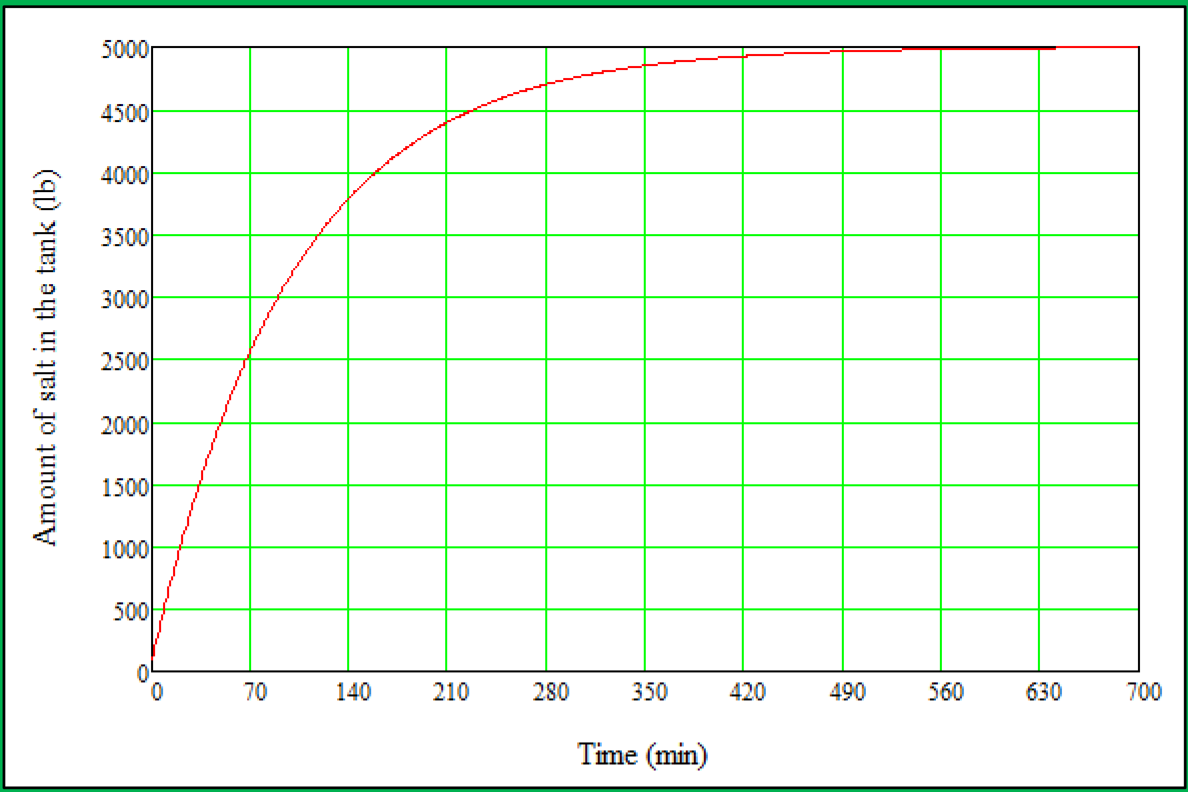

Figure 2: Dynamic response of the mixing system