Consider an integral of the form

$\int \frac{dx}{x^2+a^2}$

2. $\int \frac{dx}{\sqrt{a^2-x^2}}$

3. $\int \frac{-dx}{\sqrt{a^2-x^2}}$

1. When we have an integral of the form

$\int \frac{dx}{a^2+x^2}$, we will

Let $x = a\tan\theta$

$\frac{dx}{d\theta} = a\sec^2\theta$

$dx = a\sec^2\theta\, d\theta$

$a^2+x^2 = a^2+a^2\tan^2\theta$

$= a^2(1+\tan^2\theta)$

Recall $1+\tan^2\theta = \sec^2\theta$ (trig Identities)

$a^2+x^2 = a^2\sec^2\theta$

$\int \frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta} = \frac{1}{a}\int d\theta$

$= \frac{1}{a}[\theta] + c$

And

$\theta = \tan^{-1} x/a$

$= \frac{1}{a}\tan^{-1} x/a$.

Example

Evaluate $\int \frac{dx}{x^2+25}$

$\int \frac{dx}{x^2+25} = \int \frac{dx}{x^2+5^2}$

$x = 5\tan\theta$

$\frac{dx}{d\theta} = 5\sec^2\theta$

$dx = 5\sec^2\theta\, d\theta$

$x^2+5^2 = 5^2\tan^2\theta + 5^2 = 5^2(\tan^2\theta + $

$= 25\sec^2\theta$

$\Rightarrow \int \frac{5\sec^2\theta\, d\theta}{25\sec^2\theta} = \int \frac{d\theta}{5} = \frac{1}{5}\int$

$= \frac{1}{5}[\theta] + c$

$= \frac{1}{5}\tan^{-1} x/5 + c$

(2) $\int \frac{dx}{x^2+4x+13}$

$= \int \frac{dx}{x^2+4x+4+9} = \int \frac{dx}{(x+2)^2+}$

$(x+2) = 3\tan\theta$

$x = 3\tan\theta - 2 \Rightarrow dx = 3\sec^2\theta$

$\frac{dx}{d\theta} = 3\sec^2\theta$

$(x+2)^2 + 3^2 = 3^2\tan^2\theta + 3^2 = 3^2 se$

$\int \frac{3\sec^2\theta\, d\theta}{3^2\sec^2\theta} = \frac{1}{3}\int d\theta$

$= \frac{1}{3}[\theta] + c$

$= \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + c$

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Integral of the form

$$\int \frac{dx}{\sqrt{a^2-x^2}}$$

$x = a\sin\theta$

$x^2 = a^2\sin^2\theta$

$\frac{dx}{d\theta} = a\cos\theta$

$dx = a\cos\theta\, d\theta$

$a^2-x^2 = a^2 - a^2\sin^2\theta$

$= a^2(1-\sin^2\theta)$ 1

Recall

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1-\sin^2\theta$

from 1, we have

$a^2\cos^2\theta$

$$\int \frac{a\cos\theta\, d\theta}{\sqrt{a^2\cos^2\theta}} = \int \frac{a\cos\theta\, d\theta}{a\cos\theta}$$

$= \int d\theta = \theta + c$

$\theta = \sin^{-1} x/a$

$\Rightarrow \sin^{-1} x/a + c$

Examples 1

$$\int \frac{dx}{\sqrt{25-x^2}} = \int \frac{dx}{\sqrt{5^2-x^2}}$$

$x = 5\sin\theta$

$\frac{dx}{d\theta} = 5\cos\theta$

$dx = 5\cos\theta\, d\theta$

$5^2-x^2 = 5^2 - 5^2\sin^2\theta$

$= 5^2\cos^2\theta$

$\sqrt{5^2\cos^2\theta} = 5\cos\theta$

$$\int \frac{5\cos\theta\, d\theta}{5\cos\theta} = \int d\theta = \theta + c$$

$= \sin^{-1} x/5 + c$

*

$$\int \frac{5\,dt}{\sqrt{4-t^2}} = 5\int \frac{dt}{\sqrt{2^2-t^2}}$$

$t = 2\sin\theta$

$\frac{dt}{d\theta} = 2\cos\theta$

$dt = 2\cos\theta\, d\theta$

$$\Rightarrow 5\int \frac{2\cos\theta\, d\theta}{\sqrt{2^2\cos^2\theta}} = 5\int \frac{2\cos\theta\, d\theta}{2\cos\theta}$$

$= 5\theta + c \qquad \theta = \sin^{-1} t/2$

$= 5\sin^{-1}\frac{t}{2} + c$

Integral of the form **

$$\int \frac{-dx}{\sqrt{81-x^2}} = \int \frac{-dx}{\sqrt{9^2-x^2}}$$

$x = 9\cos\theta$

$\frac{dx}{d\theta} = -9\sin\theta$

$dx = -9\sin\theta\, d\theta$

$-dx = 9\sin\theta\, d\theta$

$\sqrt{9^2-9^2\cos^2\theta} = \sqrt{9^2\sin^2\theta} = 9\sin\theta$

$$= \int \frac{9\sin\theta\, d\theta}{\sqrt{9^2\sin^2\theta}} = \int \frac{9\sin\theta\, d\theta}{9\sin\theta}$$

$= \int d\theta = \theta + c$

$= \cos^{-1} x/9$.

Integration by Partial fraction

Simplify $\int \frac{7x-4}{2x^2-3x-2}\,dx$.

$2x^2-3x-2$
$2x^2-4x+x-2$
$2x(x-2)+1(x-2)$
$(2x+1)(x-2)$

$$\frac{7x-4}{2x^2-3x-2} = \frac{7x-4}{(2x+1)(x-2)} = \frac{A}{2x+1} + \frac{B}{x-2} \Rightarrow \frac{A(x-2)+B(2x+1)}{(2x+1)(x-2)}$$

multiply all by $(2x+1)(x-2)$

$$\cancel{A(2x+1)}\ A(x-2)+B(2x+1) = 7x-4$$

At $x=2$, we have

$B(5) = 14-4$

$B = 2$

At $x=-\frac{1}{2}$, we have $A\left(-\frac{1}{2}-2\right) = 7\left(-\frac{1}{2}\right)-4$

$-\frac{5}{2}A = -\frac{15}{2}$

$5A = 15$

$A = 3$

OR

$Ax-2A+2Bx+B = 7x-4$
$(A+2B)x+(B-2A) = 7x-4$
$A+2B = 7 \quad \times 2$
$-2A+B = -4 \quad \times 1$
$2A+4B = 14$
$-2A+B = -4$
$5B = 10 \Rightarrow B = 2$
$A+4 = 7 \Rightarrow A = 3$

We can now write

$$\int \frac{3}{2x+1}\,dx + \int \frac{2}{x-2}\,dx = \int \frac{7x-4}{2x^2-3x-2}\,dx$$

$$\Rightarrow \int \frac{3\,dx}{2x+1} + \int \frac{2\,dx}{x-2} = \int \frac{7x-4}{2x^2-3x-2}\,dx$$

Let $u = 2x+1$
$du = 2dx$
$dx = du/2$
$\Rightarrow \int \frac{3du/2}{u}$
$= \frac{3}{2}\ln u$

$u = x-2$
$du = dx$
$2\int \frac{du}{u}$
$= 2\ln u$

$$\Longrightarrow \frac{3}{2}\ln(2x+1) + 2\ln(x-2)$$

$$\int \frac{3x+1}{(x^2+1)(x+2)}\,dx$$

$$\frac{Ax+B}{x^2+1} + \frac{C}{x+2} = \frac{3x+1}{(x^2+1)(x+2)}$$

$$\frac{(Ax+B)(x+2)+C(x^2+1)}{(x^2+1)(x+2)} = \frac{3x+1}{(x^2+1)(x+2)}$$

multiply b.s $(x^2+1)(x+2)$

$(Ax+B)(x+2)+C(x^2+1) = 3x+1$

$Ax^2+2Ax+Bx+2B+Cx^2+C = 3x+1$

$(A+C)x^2 + x(2A+B) + (2B+C) = 3x+1$

Comparing/ Equating coefficients

$A+C=0$ ———— ①

$2A+B=3$ ———— ②

$2B+C=1$ ———— ③

From ① put $A=-C$ in ②

$2(-C)+B=3$

$B-2C=3 \times 2$

$2B+C=1\times 1$

$2B-4C=6$

$2B+C=1$

$-5C=5$

$C=-1$

from ④ $A=-C \Rightarrow A=1$

And $2A+B=3 \Rightarrow B=1$

$$\int \frac{x+1}{x^2+1}dx - \int \frac{dx}{x+2} \Rightarrow \int \frac{x}{x^2+1}dx + \int \frac{dx}{x^2+1} - \int \frac{dx}{x+2}$$

$$\frac{1}{2}\int \frac{du}{u} + \arctan x - \ln(x+2)+$$

$$\Rightarrow \frac{1}{2}\ln(x^2+1) + \arctan x - \ln(x+2) +$$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$dx = du/2$

⑥ $$\int \frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}dx$$

$4\ln(x+1) - 3\ln(x-2) + \ln(x+3)$

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$$\int \frac{11-3x}{x^2+2x-3} = 2\ln(x-1) - 5\ln(x+3)$$

⑧ $$\int \frac{4x-16}{x^2-2x-3}$$

1 $\displaystyle\int \frac{2x^2-10x}{(x+3)(x-1)^2}\,dx$

$$\frac{2x^2-10x}{(x+3)(x-1)^2} = \cancel{A(x+1)^2}\;\frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\frac{2x^2-10x}{(x+3)(x-1)^2} = \frac{A(x-1)^2+B(x-1)(x+3)+C(x+3)}{(x+3)(x-1)^2}$$

$$2x^2-10x = A(x^2-2x+1)+B(x^2+2x-3)+C(x+3)$$

$$2x^2-10x = Ax^2-2AX+A+BX^2+2BX-3B+CX+3C$$

$$2x^2-10x = AX^2+BX^2-2AX+2BX+CX+A-3B+3C$$

$$2x^2-10x = (A*B)x^2+x(-2A+2B+C)+(A-3B+3C)$$

$A+B=2$ —— ①

$-2A+2B+C=-10$ —— ②

$A-3B+3C=0$ —— ③

From ① $A=2-B$ —— ④ And put ④ in ② & ③

$-4+2B+2B+C=-10$ $\Rightarrow$ $4B+C=-6$

$2-B-3B+3C=0$ $\quad$ $-4B+3C=-2$

$4C=-8$

$C=-2$

Also $4B+C=-6$

$4B-2=-6$

$4B=-4$

$B=-1$

Finally

$A+B=2$

$A-1=2$

$A=3.$

$$\int\frac{2x^2-10x}{(x+3)(x-1)^2}\,dx = \int\frac{3}{x+3}\,dx+\int\frac{-1\,dx}{x-1}+\int\frac{-2}{(x-1)^2}\,dx.$$

$$= 3\ln(x+3)-\ln(x-1)+2(x-1)^{-1}+c$$

$$= 3\ln(x+3)-\ln(x-1)+\frac{2}{x-1}+C.$$

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Short Note on Integration Of Rational Algebraic fractions (Before example

We now consider the integration of rational algebraic fractions by which we mean fractions whose numerator and denominator each contain only positive integral powers of x with constant coefficients. In all cases, if the numerator is of the same/higher degree than the denominator, we first divide out. Thus, we shall have one or more terms (in x, x^2 etc or a constant) which can be immediately integrated and a fraction whose numerator is of lesser degree than the denominator.

Integration of rational algebraic fractions (polynom...

* Give short note from previous note

Examples

1. $\int \frac{2x^3 - x^2 - x}{2x-3}\,dx$

$$
\begin{array}{r|l}
 & x^2 + x + 1 \\
\hline
2x-3 & 2x^3 - x^2 - x \\
 & 2x^3 - 3x^2 \\
\hline
 & 2x^2 - x \\
 & 2x^2 - 3x \\
\hline
 & 2x \\
 & 2x - 3 \\
\hline
 & 3
\end{array}
$$

Which can now be writtenas

$$\int (x^2 + x + 1)\,dx + \int \frac{3}{2x-3}\,dx$$

$$\Rightarrow \frac{x^3}{3} + \frac{x^2}{2} + x + \frac{3}{2}\ln(2x-3) + C$$

Given then $\frac{d^2y}{dx^2}$ (Exple in ...)

2. $\int \frac{2x^2+11}{x^2+4}\,dx$

first divide out to have

$$
\begin{array}{r|l}
 & 2 \\
\hline
x^2+4 & 2x^2+11
\end{array}
$$

$$\int 2\,dx + \int \frac{3}{x^2+4}\,dx$$

$$= 2x + \frac{3}{2}\arctan x/2 + C$$

3. $\int \frac{x^2}{x-1}\,dx$

$$
\begin{array}{r|l}
 & x + 1 \\
\hline
x-1 & x^2 \\
 & x^2 - x \\
\hline
 & x \\
 & x - 1 \\
\hline
 & 1
\end{array}
$$

$$\int (x+1)\,dx + \int \frac{1}{x-1}\,dx$$

$$\Rightarrow \frac{x^2}{2} + x + \ln(x-1) + C$$

4. $\int \frac{2\theta - 3\theta^2}{1-\theta}\,d\theta$

$... - 3xe + 6xe^x - 6e^x + C$