3.6 The Maximum and Minimum values:

Stationary Points.

Definition. Consider a curve $y = f(x)$, a point on the curve at which $\frac{dy}{dx} = 0$ is called a

Stationary Point: The value of the function at which $\frac{dy}{dx} = 0$ is called its Stationary value.

The simplest way of determining the stationary points is to set $\frac{dy}{dx} = 0$ and then solve the resulting equation.

Example 1.

Find the stationary points of the function

$f(x) = 4x^3 + 15x^2 - 18x + 7$

Solution

Let $y = 4x^3 + 15x^2 - 18x + 7$

$$\frac{dy}{dx} = 12x^2 + 30x - 18$$

put $\frac{dy}{dx} = 0$, that is

$12x^2 + 30x - 18 = 0$

This is a quadratic equation which can be solved by factorization. Therefore,

$12x^2 + 30x - 18 = 6(2x - 1)(x + 3) = 0$

This implies that either $2x - 1 = 0$ or $x + 3 = 0$

i.e $2x = 1$, $x = ½$ or $x = -3$

The two stationary points are ½ and – 3

To determine the stationary value, we return to the original function; $f(x) = 4x^3 + 15x^2 - 18x + 7$ to determine f(½) and f(-3).

$f(½) = 4(½)^3 + 15(½)^2 - 18(½) + 7$

$= \frac{4}{8} + \frac{15}{4} - 9 + 7$

$= ½ + \frac{15}{4} - 2 = \frac{2 + 15 - 8}{4}$

$= \frac{17 - 8}{4} = \frac{9}{4}$

Also $f(-3) = 4(-3)^3 + 15(-3)^2 - 18(-3) + 7$

$= 88$

Hence the two stationary values are $\frac{9}{4}$ and 88

Maximum and Minimum Points.

Let us consider the following diagram (stationary Point)

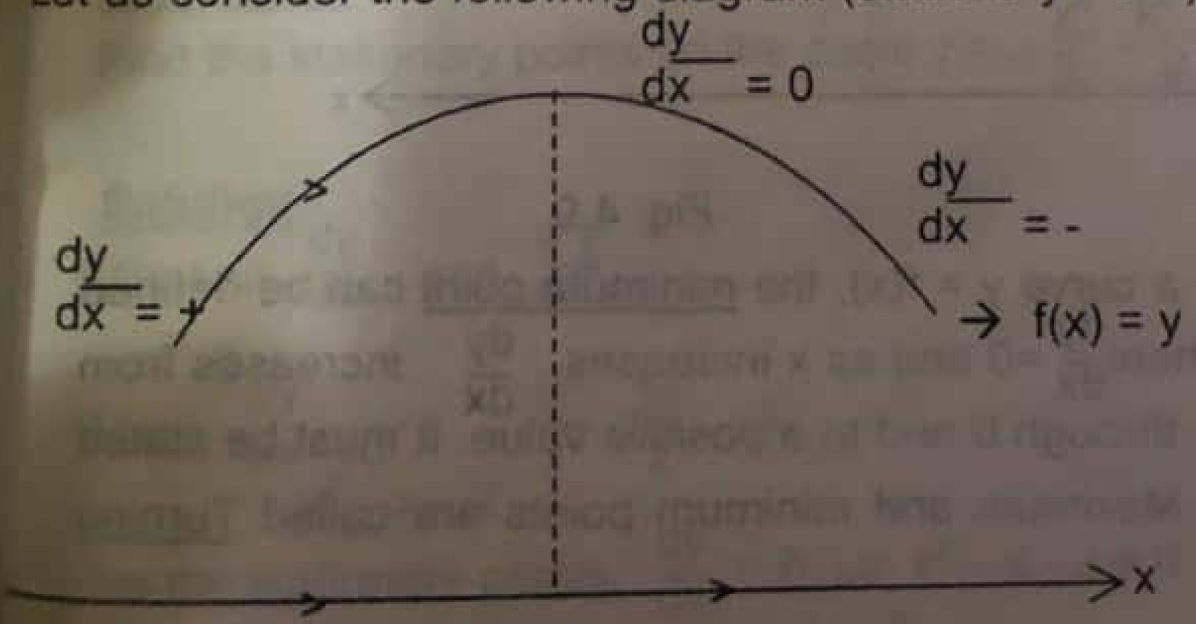
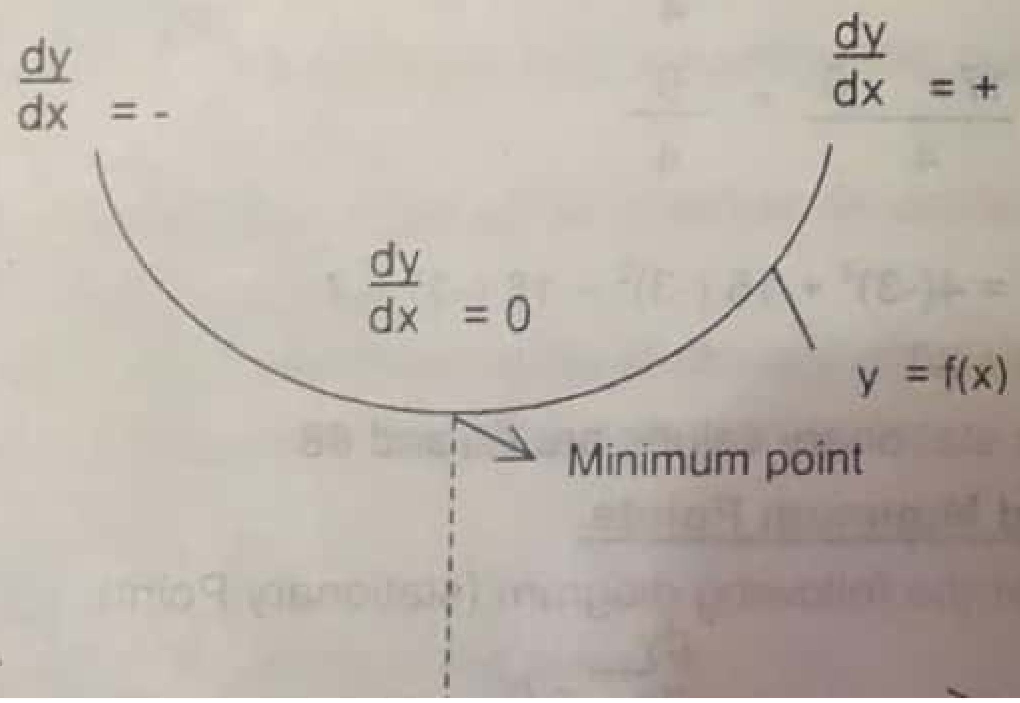


fig. 4.8

figure 4.8 represent a curve y= f(x). The curve passes through a stationary point and reach a maximum value at the point where

$\frac{dy}{dx} = 0$ As we can see in the diagram, as x increases, $\frac{dy}{dx}$ decreases from tve through 0 to –ve (This is indicated in the above diagram.)

<u>Definition;</u> consider a given curve y = f(x) at a <u>Maximum point</u> $\frac{dy}{dx} = 0$ and as x increases, the gradient of y = f(x) decreases from a + ve value through 0 to a negative value for the minimum value, we consider the following diagram



The questions is; how do we test for

Stationary points

From figure 4.8, we see that $\frac{dy}{dx}$ changes from positive to negative. this implies that $\frac{dy}{dx}$ is decreasing as it goes from +ve to -ve. So we define $\frac{dy}{dx}$ as a deceasing function. So the derivative $\frac{d^2y}{dx^2} < 0$.

Also in figure (4.9), $\frac{dy}{dx}$ increases from –ve to +ve and so $\frac{dy}{dx}$ is an increasing function at this point and so $\frac{d^2y}{dx^2} > 0$

We can summarize all the above facts as follows;

(1) At Maximum point $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$

(2) At Minimum point $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

We apply the above principles to test for the stationary points of a given curve

Example 2:

Find the stationary points on the curve $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 5$

Solution

We find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Here $\frac{dy}{dx} = \frac{3x^3}{3} \frac{2x}{2} -2$

$= x^2 - x - 2$

for the stationary points, $\frac{dy}{dx} = 0$; i.e. $x^2 - x - 2 = 0$

This implies that (x-2) (x+1) = 0

X = 2 or x =-1

The stationary point are x = 2, x = - 1

We now proceed to find the nature of the stationary points.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(x^2 - x - 2) = 2x - 1$$

$$\therefore \quad \frac{d^2y}{dx^2} = 2x - 1$$

We now investigate how $\frac{d^2y}{dx^2}$ changes at the neighborhood of x=2 and x = -1 at x = 2, $\frac{d^2y}{dx^2} = 2(2) - 1 = 4-1 = 3= +Ve$

At x = - 1,

$$\frac{d^2y}{dx^2} = 2(-1) - 1 = -2-1 = -3 \text{ (-ve)}$$

At x = 2 $\frac{d^2y}{dx^2} > 0$, hence, x = 2 gives Minimum point

At x = -1 $\frac{d^2y}{dx^2} < 0$, this lead to maximum point.

To calculate the value of y_{min} and y_{max}, we proceed as follows.

$$y_{min} = \frac{2^3}{3} - \frac{2^2}{2}\ 2(2) + 5$$

$$= \frac{8}{3} - \frac{4}{2} - 4 + 5 = \frac{8}{3} - \frac{4}{2} + 1$$

$$= \frac{16 - 12 + 6}{6} = \frac{10}{6} = 1\frac{4}{6} = 1\frac{2}{3}$$

Hence

$$y_{min} = 1\tfrac{2}{3}$$

$$y_{mcx} = \frac{(-1)^2}{3} - \frac{(-1)^2}{2} - 2(-1) + 5$$

$= \frac{1}{27} - ¼ + 3 + 5$

$= \frac{1}{27} - ¼ + 8 = 6$ '/ₙ

Example 3

Find the maximum and minimum value of the function y = f(x) = 3 sin x + 4 cos x and the value of x (0° ≤ x ≤ 360°) which they occur

Solution

Let f(x) = 3 sin x + 4 cosx = y

Here $f^1(x) = \frac{dy}{dx}$ = 3 cos x – 4- sin x and

$f^{11}(x) = \frac{d^2y}{dx^2}$ = - 3 sin x – 4 cos x

For the stationary points $\frac{dy}{dx} = 0$

Hence 3 cos x – 4 sin x = 0

i.e. 3 cos x = 4 sin x and ¾ cos x = sin x

This implies that ¾ = $\frac{\text{Sin x}}{\text{Cos x}}$ = tan x

Hence **tan x = ¾** . In the range 0° to 360°,

Tan x = ¾

X = 36° 52¹ or 216° 52¹

With x = 36°, 52¹, $\frac{d^2y}{dx^2} < 0$. This a maximum point.

point

Example 4 The cost ₦c of running a ship on a certain voyage is given by $C = av + \frac{b}{v}$, where v is the average speed in km/h, a, b are constants. Find the value of v (in terms of a and b) to give the minimum cost and find the cost.

Solution

Given that $C = av + \frac{b}{v^2}$, at the minimum, $\frac{dc}{dv} = 0$, then

implies that $\frac{dc}{dv} = a - \frac{b}{v^2} = 0$

$\therefore \quad V = \sqrt{\frac{b/}{a}}$

We check for $\frac{d^2c}{dv^2} = \frac{2b}{v^3}$, put $v = \sqrt{\frac{b}{a}}$

We observe that $\frac{d^2c}{dv^2} > 0$

the minimum cost is

$₦\left(av + \frac{b}{v}\right) = ₦\left(a\left(\sqrt{\frac{b}{a}}\right) + \frac{b}{\sqrt{\frac{b}{a}}}\right)$

$= ₦2\sqrt{ab}$