7.4 THE ANGLE BETWEEN TWO STRAIGHT LINES

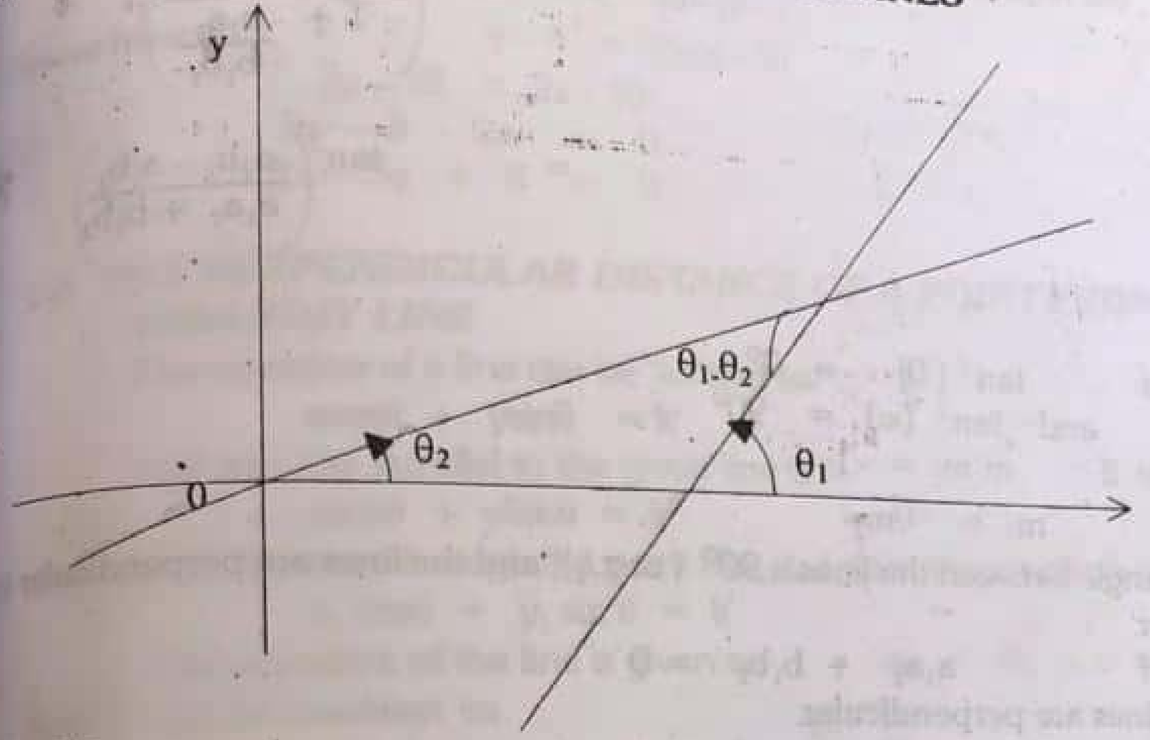

Fig. 7. 7

If the equation of the lines are given in the from

$y = m_1x + c_1$ and $y = m_2x + c_2$

and θ_1, and θ_2 are the angles the lines make with x - axis then

$m_1 = \tan\theta_1$, $m_2 = \tan\theta_2$

Considering

the diagram in figure 7.7 the required angle is $\theta_1 - \theta_2$

Also, from trigonometry

$$\tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$\tan(\theta_1 - \theta_2) = \frac{m_1 - m_2}{1 + m_1m_2}$$

$$\theta_1 = \theta_2 = \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$$

If the lines are given in the form $a_1 x + b_2y + c_1 = 0$, and $a_2x + b_1y + c_2 = 0$ then

$$m_1 = \frac{-a_1}{b_1} \quad \text{and} \quad m_2 = \frac{-a_2}{b_2}$$

Hence, the required angle is $\theta_1 - \theta_2 = \tan^{-1}\left(\dfrac{\dfrac{-a_1}{b_1} + \dfrac{a_2}{b_2}}{1 + \dfrac{a_1 a_2}{b_1 b_2}}\right)$

$$= \tan^{-1}\left(\frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2}\right)$$

Note that $\tan^{-1}(0) = 0^\circ$

and $\tan^{-1}(\infty) = 90^\circ$

Therefore if $m_1 m_2 = -1$

That is $m_1 = {}^{-1}/m_2$

then the angle between the lines is 90° (or $\pi/_2$), and the lines are perpendicular to each other.

similarly if $a_1 a_2 + b_1 b_2 = 0$

then the lines are perpendicular.

Examples:

1. Find the acute angle between the lines $-3x + y - 4 = 0$ and $-2x + y - 1 = 0$

Solution:

$m_1 = 3/1$, $m_2 = 2/1$

The angle between the lines $\theta = \tan^{-1}\left(\dfrac{m_1 - m_2}{1 + m_1 m_2}\right)$

$$\tan^{-1}\left(\frac{3-2}{1+3(2)}\right)$$

$$\tan^{-1} {}^1/_7 = 8.1^\circ$$

2. Find the equation of the line perpendicular to the line $3x + 2y + 4 = 0$ and passing through the point ((5, 6).

Solution:

$3x + 2y + 4 =- 0$

$m_1 = -3/2$

Let m_2 be the scope of the line perpendicular to $3x + 2y + 4 = 0$.

then $m_2 = -1/m_1$

$$m_2 = \frac{-1}{-3/2} = \frac{1}{3/2} = {}^2/_3$$

The equation of a line with scope m passing through a point is

$$y - y_1 = m(x - x_1)$$

so we have

$$y - 6 = 2/3(x - 5)$$

$$3y - 18 = 2x - 10$$

$$3y - 8 - 2x = 0$$

$$2x - 3y + 8 = 0$$

7.5 THE PERPENDICULAR DISTANCE OF A POINT FROM A STRAIGHT LINE

The equation of a line can be written as

$$x\cos\theta + y\sin\theta = k$$

And any line parallel to the given line is

$$x\cos\theta + y\sin\theta = k'$$

and if this line passes through the point $P(x_1, y_1)$ then the equation of the line is

$$x_1 \cos\theta + y_1 \sin\theta = k'$$

If the equation of the line is given as $ax + by + c = 0$

then it may be rewritten as

$$\left(\frac{a}{\sqrt{a^2+b^2}}\right)x + \left(\frac{b}{\sqrt{a^2+b^2}}\right)y + \frac{c}{\sqrt{a^2+b^2}} = 0$$

which is the perpendicular form because

$$\frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \frac{b}{\sqrt{a^2+b^2}}$$

are the sine and cosine of an angle since the sum of their squares is unity. The length of the perpendicular is

$$\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}$$

The length of the perpendicular from any point on the positive side of the line will be positive and from any point on the negative side of the line it will be negative.

Example:

Find the length of the perpendicular from the point $P(2, -4)$, to the line $3x + 2y - 5 = 0$ and state which side of the line P is on.

Solution:

$$3x + 2y - 5 = 0, \quad P(2, -4)$$

The length of the perpendicular is

$$\frac{ax_1 + by_1 + c}{\sqrt{a_2 + b_2}}$$

$$= \frac{3(2) + 2(-4) - 5}{\sqrt{3^2 + 2^2}} = \frac{6 - 8 - 5}{\sqrt{9 + 4}} = -7/\sqrt{13}$$

Thus the length of the perpendicular is $7/\sqrt{13}$ on the negative side of the line, i.e. below the line.

7.6 THE EQUATION OF A CIRCLE

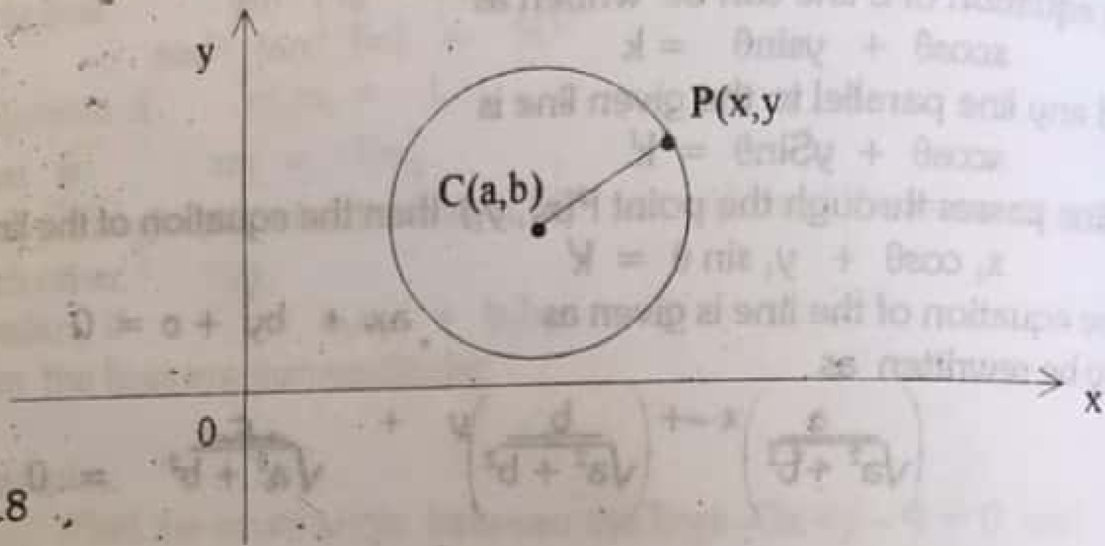

Fig 7.8

Let c(a, b) be the centre of the circle and r the radius of the circle.
Let P(x, y) be any point on the circumference of the circle.

The length $CP = r = (x - a)^2 + (y - b)^2$

$r^2 = (x - a)^2 + (y - b)^2$ .... (*)

equation (*) is called the equation of a circle with radius r centred at the point (a, b)

if $a = b = 0$ ie the circle is centred at the origin then

$r^2 = (x - 0)^2 + (y - 0)^2$

$r^2 = x^2 + y^2$ ....(**)

The equation (*) above can expressed in another form.

i.e.

$(x - a)^2 + (y - b)^2 = r^2$

$x^2 - ax + y^2 - 2by + b^2 = r^2$

$x^2 + y^2 - 2ax - 2by + a^2 + b^2 = r^2$

Which can be written as

$x^2 + y^2 - 2ax - 2by + c = 0$

where $c = a^2 + b^2 - r^2$

In general, the equation of a circle is such that:

(i). The coefficients of x^2 and y^2 are equal

(ii). There is no term in xy .

Examples:

1. Find the equation of a circle centred at (3,7) with radius 5.

Solution:

$$(x - a)^2 + (y - b)^2 = 6^2$$
$$(x - 3)^2 + (y - 7)^2 = 5^2$$
$$x^2 - 6x + 9 + y^2 - 14y + 49 = 25$$
$$x^2 + y^2 - 6x - 14y + 33 = 0$$

2. Find the equation of the circle centred at (4, -7) which touches the line $3x + 4y - 9 = 0$

Solution:

The line which touches a circle is a tangent to the circle, and so the radius of the circle meets the tangent at a perpendicular. The perpendicular distance is given by.

$$\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \qquad \frac{3(4) + 4(-7) - 9}{\sqrt{3^2 + 4^2}}$$

$$\frac{12 - 28 - 9}{\sqrt{9 + 16}} = \frac{-25}{\sqrt{25}} = \frac{-25}{5} = -5$$

The distance is 5 units below the line therefore the radius of the circle is 5 and the equation of the circle is

$$(x - 4)^2 + (y + 7)^2 = 5^2$$
$$(x - 4)^2 + (y + 7)^2 = 25$$