Example 6

If $\underline{w} = 3t^2\underline{i} + \cos 2t\,\underline{j}$, find

(a) $\frac{d\underline{w}}{dt}$ (b) $\left|\frac{d\underline{w}}{dt}\right|$ (c) $\frac{d^2\underline{w}}{dt^2}$

Solution

(a) If $\underline{w} = 3t^2\underline{i} + \cos 2t\,\underline{j}$, then differentiation with respect to t yields: $\frac{d\underline{w}}{dt} = 6t\underline{i} - 2\sin 2t\,\underline{j}$

(b) $\left|\frac{d\underline{w}}{dt}\right| = \sqrt{(6t)^2 + (-2\sin 2t)^2} = \sqrt{36t^2 + 4\sin^2 2t}$

(c) $\frac{d^2\underline{w}}{dt^2} = 6\underline{i} - 4\cos 2t\,\underline{j}$

t is possible to differentiate more complicated expressions involving vectors provided certain rules re adhered to as summarized in the following Key Point.

Key Point 9

If $\underline{w}$ and $\underline{z}$ are vectors and c is a scalar, all these being functions of time t, then:

$$\begin{aligned}
\frac{d}{dt}(\underline{w} + \underline{z}) &= \frac{d\underline{w}}{dt} + \frac{d\underline{z}}{dt} \\
\frac{d}{dt}(c\underline{w}) &= c\frac{d\underline{w}}{dt} + \frac{dc}{dt}\underline{w} \\
\frac{d}{dt}(\underline{w} \cdot \underline{z}) &= \underline{w} \cdot \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \cdot \underline{z} \\
\frac{d}{dt}(\underline{w} \times \underline{z}) &= \underline{w} \times \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \times \underline{z}
\end{aligned}$$

Example 7

If $\underline{w} = 3t\underline{i} - t^2\underline{j}$ and $\underline{z} = 2t^2\underline{i} + 3\underline{j}$, verify the result

$$\frac{d}{dt}(\underline{w} \cdot \underline{z}) = \underline{w} \cdot \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \cdot \underline{z}$$

Solution

$\underline{w} \cdot \underline{z} = (3t\underline{i} - t^2\underline{j}) \cdot (2t^2\underline{i} + 3\underline{j}) = 6t^3 - 3t^2.$

Therefore $\quad \frac{d}{dt}(\underline{w} \cdot \underline{z}) = 18t^2 - 6t \qquad (1)$

Also $\quad \frac{d\underline{w}}{dt} = 3\underline{i} - 2t\underline{j} \quad$ and $\quad \frac{d\underline{z}}{dt} = 4t\underline{i}$

so

$$\begin{aligned} \underline{w} \cdot \frac{d\underline{z}}{dt} + \underline{z} \cdot \frac{d\underline{w}}{dt} &= (3t\underline{i} - t^2\underline{j}) \cdot (4t\underline{i}) + (2t^2\underline{i} + 3\underline{j}) \cdot (3\underline{i} - 2t\underline{j}) \\ &= 12t^2 + 6t^2 - 6t \\ &= 18t^2 - 6t \qquad (2) \end{aligned}$$

We have verified $\frac{d}{dt}(\underline{w} \cdot \underline{z}) = \underline{w} \cdot \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \cdot \underline{z}$ since (1) is the same as (2).

Example 8

If $\underline{w} = 3t\underline{i} - t^2\underline{j}$ and $\underline{z} = 2t^2\underline{i} + 3\underline{j}$, verify the result

$$\frac{d}{dt}(\underline{w} \times \underline{z}) = \underline{w} \times \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \times \underline{z}$$

Solution

$$\underline{w} \times \underline{z} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3t & -t^2 & 0 \\ 2t^2 & 3 & 0 \end{vmatrix} = (9t + 2t^4)\underline{k} \quad \text{implying} \quad \frac{d}{dt}(\underline{w} \times \underline{z}) = (9 + 8t^3)\underline{k} \qquad (1)$$

$$\underline{w} \times \frac{d\underline{z}}{dt} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3t & -t^2 & 0 \\ 4t & 0 & 0 \end{vmatrix} = 4t^3\underline{k} \qquad (2)$$

$$\frac{d\underline{w}}{dt} \times \underline{z} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2t & 0 \\ 2t^2 & 3 & 0 \end{vmatrix} = (9 + 4t^3)\underline{k} \qquad (3)$$

We can see that (1) is the same as (2) + (3) as required.

Exercises

1. If $r = 3t\underline{i} + 2t^2\underline{j} + t^3\underline{k}$, find (a) $\dfrac{d\underline{r}}{dt}$ (b) $\dfrac{d^2\underline{r}}{dt^2}$

2. Given $\underline{B} = te^{-t}\ \underline{i} + \cos t\ \underline{j}$ find (a) $\dfrac{d\underline{B}}{dt}$ (b) $\dfrac{d^2\underline{B}}{dt^2}$

3. If $\underline{r} = 4t^2\underline{i} + 2t\underline{j} - 7\underline{k}$ evaluate $\underline{r}$ and $\dfrac{d\underline{r}}{dt}$ when $t = 1$.

4. If $\underline{w} = t^3\underline{i} - 7t\underline{k}$ and $\underline{z} = (2+t)\underline{i} + t^2\underline{j} - 2\underline{k}$

   (a) find $\underline{w}\cdot\underline{z}$, (b) find $\dfrac{d\underline{w}}{dt}$, (c) find $\dfrac{d\underline{z}}{dt}$, (d) show that $\dfrac{d}{dt}(\underline{w}\cdot\underline{z}) = \underline{w}\cdot\dfrac{d\underline{z}}{dt} + \dfrac{d\underline{w}}{dt}\cdot\underline{z}$

5. Given $\underline{r} = \sin t\ \underline{i} + \cos t\ \underline{j}$

   (a) find $\dot{\underline{r}}$, (b) find $\ddot{\underline{r}}$, (c) find $|\underline{r}|$

   (d) Show that the position vector $\underline{r}$ and velocity vector $\dot{\underline{r}}$ are perpendicular.

Answers

1. (a) $3\underline{i} + 4t\underline{j} + 3t^2\underline{k}$ (b) $4\underline{j} + 6t\underline{k}$
2. (a) $(-te^{-t} + e^{-t})\underline{i} - \sin t\underline{j}$ (b) $e^{-t}(t-2)\underline{i} - \cos t\underline{j}$
3. $4\underline{i} + 2\underline{j} - 7\underline{k},\ 8\underline{i} + 2\underline{j}$
4. (a) $t(t^3 + 2t^2 + 14)$ (b) $3t^2\underline{i} - 7\underline{k}$ (c) $\underline{i} + 2t\underline{j}$
5. (a) $\cos t\underline{i} - \sin t\underline{j}$ (b) $-\sin t\underline{i} - \cos t\underline{j}$ (c) 1 (d) Follows by showing $\underline{r}\cdot\dot{\underline{r}} = 0$.