Integral of the form $\int \frac{g'(x)}{g(x)} = \log_e g(x) + c$ (4)

!!! Note let $u = g(x)$

Examples

1) $\int \left(\frac{2x+2}{x^2+2x+8}\right) dx = \log_e (x^2+2x+8) + C \Rightarrow$

Let $u = x^2+2x+8$

$\frac{du}{dx} = 2x+2$

$du = (2x+2)dx$

$\int \frac{du}{u} = \ln u + c$

2) $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \log_e(\sin x) + c$.

3) $\int \frac{4x-8}{x^2-4x+5} dx$

Let $u = \cancel{2x} x^2-4x+5$

$\frac{du}{dx} = 2x-4$

$du = (2x-4)dx$

The integral can be written as

$\int \frac{2(2x-4)}{x^2-4x+5} dx$

$\int \frac{2du}{u} = 2\ln u + c$

$= 2\ln(x^2-4x+5) + c$.

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* $\int \frac{(x-4)}{(x^2-8x+4)} dx$

Let $u = x^2-8x+4$

$\frac{du}{dx} = 2x-8$

$\frac{du}{dx} = 2(x-4)$

$du = 2(x-4)dx$

$(x-4)dx = \frac{du}{2}$

$\int \frac{du}{2} \div u = \int \frac{du}{2} \times \frac{1}{u}$

$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u + c$

$= \frac{1}{2}\ln(x^2-8x+4) + c$

Of the form $\int g(x) g'(x) dx$

Example

1) $\int (x^2-x+1)(2x-1)dx$

Let $u = x^2-x+1$

$\frac{du}{dx} = 2x-1$

$du = (2x-1)dx$

$\Rightarrow \int u\,du = \frac{u^2}{2} + c$

$= \frac{(x^2-x+1)^2}{2} + c$

2) $\int (3x^2-4x+2)(6x-4)dx$.

$u = 3x^2-4x+2$

$\frac{du}{dx} = 6x-4$

$du = (6x-4)dx$

$\int u\,du = \frac{u^2}{2} + c$

$=$

(5)

Consider integral of the form

$\int \frac{dx}{x^2+a^2}$

2. $\int \frac{dx}{\sqrt{a^2-x^2}}$

3. $\int \frac{-dx}{\sqrt{a^2-x^2}}$

1. When we have an integral of the form

$\int \frac{dx}{a^2+x^2}$, we will

Let $x = a\tan\theta$

$\frac{dx}{d\theta} = a\sec^2\theta$

$dx = a\sec^2\theta\, d\theta$

$a^2+x^2 = a^2 + a^2\tan^2\theta$

$= a^2(1+\tan^2\theta)$

Recall $1+\tan^2\theta = \sec^2\theta$ (trig identities)

$a^2+x^2 = a^2\sec^2\theta$

$\int \frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta} = \frac{1}{a}\int d\theta$

$= \frac{1}{a}[\theta] + c$

And

$\theta = \tan^{-1} x/a$

$= \frac{1}{a}\tan^{-1} x/a.$

Example

Evaluate $\int \frac{dx}{x^2+25}$

$\int \frac{dx}{x^2+25} = \int \frac{dx}{x^2+5^2}$

$x = 5\tan\theta$

$\frac{dx}{d\theta} = 5\sec^2\theta$

$dx = 5\sec^2\theta\, d\theta$

$x^2+5^2 = 5^2\tan^2\theta + 5^2 = 5^2(\tan^2\theta+1)$

$= 25\sec^2\theta$

$\Rightarrow \int \frac{5\sec^2\theta\, d\theta}{25\sec^2\theta} = \int \frac{d\theta}{5} = \frac{1}{5}\int d\theta$

$= \frac{1}{5}[\theta] + c$

$= \frac{1}{5}\tan^{-1} x/5 + c.$

(2) $\int \frac{dx}{x^2+4x+13}$

$= \int \frac{dx}{x^2+4x+4+9} = \int \frac{dx}{(x+2)^2+3^2}$

$(x+2) = 3\tan\theta$

$x = 3\tan\theta - 2$

$\frac{dx}{d\theta} = 3\sec^2\theta \Rightarrow dx = 3\sec^2\theta\, d\theta$

$(x+2)^2 + 3^2 = 3^2\tan^2\theta + 3^2 = 3^2\sec^2\theta$

$\int \frac{3\sec^2\theta\, d\theta}{3^2\sec^2\theta} = \frac{1}{3}\int d\theta$

$= \frac{1}{3}[\theta] + c$

$= \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + c$

Integral of the form

$$\int \frac{dx}{\sqrt{a^2-x^2}}$$

$x = a\sin\theta$

$x^2 = a^2\sin^2\theta$

$\frac{dx}{d\theta} = a\cos\theta$

$dx = a\cos\theta \, d\theta$

$a^2-x^2 = a^2 - a^2\sin^2\theta$

$= a^2(1-\sin^2\theta) \ldots\ldots (1)$

Recall

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1-\sin^2\theta$

~~from~~ In (1), we have

$a^2\cos^2\theta$

$$\int \frac{a\cos\theta \, d\theta}{\sqrt{a^2\cos^2\theta}} = \int \frac{a\cos\theta \, d\theta}{a\cos\theta}$$

$$= \int d\theta = \theta + c$$

$\theta = \sin^{-1} x/a$

$\Rightarrow \sin^{-1} x/a + c$

Examples 1

$$\int \frac{dx}{\sqrt{25-x^2}} = \int \frac{dx}{\sqrt{5^2-x^2}}$$

$x = 5\sin\theta$

$\frac{dx}{d\theta} = 5\cos\theta$

$dx = 5\cos\theta \, d\theta$

$5^2-x^2 = 5^2 - 5^2\sin^2\theta$

$= 5^2\cos^2\theta$

$\sqrt{5^2\cos^2\theta} = 5\cos\theta$

$$\int \frac{5\cos\theta \, d\theta}{5\cos\theta} = \int d\theta = \theta + c$$

$= \sin^{-1} x/5 + c$

(2) *

$$\int \frac{5\,dt}{\sqrt{4-t^2}} = 5\int \frac{dt}{\sqrt{2^2-t^2}}$$

$t = 2\sin\theta$

$\frac{dt}{d\theta} = 2\cos\theta$

$dt = 2\cos\theta \, d\theta$

$$\Rightarrow 5\int \frac{2\cos\theta \, d\theta}{\sqrt{2^2\cos^2\theta}} = 5\int \frac{2\cos\theta \, d\theta}{2\cos\theta}$$

$= 5\theta + c \qquad \theta = \sin^{-1} t/2$

$= 5\sin^{-1}\frac{t}{2} + c$

Integral of the form **

$$\int \frac{-dx}{\sqrt{81-x^2}} = \int \frac{-dx}{\sqrt{9^2-x^2}}$$

$x = 9\cos\theta$

$\frac{dx}{d\theta} = -9\sin\theta$

$dx = -9\sin\theta \, d\theta$

$-dx = 9\sin\theta \, d\theta$

$\sqrt{9^2-9^2\cos^2\theta} = \sqrt{9^2\sin^2\theta} = 9\sin\theta$

$$= \int \frac{9\sin\theta \, d\theta}{\sqrt{9^2\sin^2\theta}} = \int \frac{9\sin\theta \, d\theta}{9\sin\theta}$$

$= \int d\theta = \theta + c$

$= \cos^{-1} x/9$

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Integration by Partial fraction

Simplify

(1) $\int \frac{7x-4}{2x^2-3x-2}\,dx$

$2x^2-3x-2$
$2x^2-4x+x-2$
$2x(x-2)+1(x-2)$
$(2x+1)(x-2)$

$$\frac{7x-4}{2x^2-3x-2} = \frac{7x-4}{(2x+1)(x-2)} = \frac{A}{2x+1} + \frac{B}{x-2} \Rightarrow \frac{A(x-2)+B(2x+1)}{(2x+1)(x-2)}$$

multiply all by $(2x+1)(x-2)$

$\cancel{A(2x+1)}$ $A(x-2)+B(2x+1) = 7x-4$

At $x=2$, we have

$B(5) = 14-4$

$B=2$

At $x=-\frac{1}{2}$, we have $A(-\frac{1}{2}-2) = 7(-\frac{1}{2})-4$

$-\frac{5}{2}A = -\frac{15}{2}$

$5A=15$

$A=3$

OR

$Ax-2A+2Bx+B = 7x-4$
$(A+2B)x+(B-2A) = 7x-4$
$A+2B = 7 \quad \times 2$
$-2A+B = -4 \quad \times 1$
$2A+4B = 14$
$-2A+B = -4$
$5B = 10 \Rightarrow B=2$
$A+4=7 \Rightarrow A=3$

We can now write

$$\int \frac{3}{2x+1}\,dx + \int \frac{2}{x-2}\,dx = \int \frac{7x-4}{2x^2-3x-2}\,dx$$

$$\Rightarrow \int \frac{3\,dx}{2x+1} + \int \frac{2\,dx}{x-2} = \int \frac{7x-4}{2x^2-3x-2}\,dx$$

Let $u=2x+1$
$du = 2dx$
$dx = du/2$
$\Rightarrow \int \frac{3du/2}{u}$
$= \frac{3}{2}\ln u$

$u = x-2$
$du = dx$
$2\int \frac{du}{u}$
$= 2\ln u$

$$\Longrightarrow \frac{3}{2}\ln(2x+1) + 2\ln(x-2)$$

$$\int \frac{3x+1}{(x^2+1)(x+2)}\,dx$$

$$\frac{Ax+B}{x^2+1} + \frac{C}{x+2} = \frac{3x+1}{(x^2+1)(x+2)}$$

$$\frac{(Ax+B)(x+2)+C(x^2+1)}{(x^2+1)(x+2)} = \frac{3x+1}{(x^2+1)(x+2)}$$

multiply b.s $(x^2+1)(x+2)$

$$(Ax+B)(x+2)+C(x^2+1) = 3x+1$$

$$Ax^2+2Ax+Bx+2B+Cx^2+C = 3x+1$$

$$(A+C)x^2 + x(2A+B)+(2B+C) = 3x+1$$

Comparing/ Equating coefficients

$A+C=0$ ——— (1)

$2A+B=3$ ——— (2)

$2B+C=1$ ——— (3)

From (1) put $A=-C$ in (2)

$$2(-C)+B=3$$

$$B-2C=3 \times 2$$

$$2B+C=1\times 1$$

$$2B-4C=6$$

$$2B+C=1$$

$$-5C=5$$

$$C=-1$$

~~fx~~ from (4) $A=-C \Rightarrow A=1$

And $2A+B=3 \Rightarrow B=1$

$$\int \frac{x+1}{x^2+1}dx - \int \frac{dx}{x+2} \Rightarrow \int \frac{x}{x^2+1}dx + \int \frac{dx}{x^2+1} - \int \frac{dx}{x+2}$$

$$\frac{1}{2}\int \frac{du}{u} + \arctan x - \ln(x+2)+c$$

Let $u=x^2+1$

$\frac{du}{dx}=2x$

$du=2x\,dx$

$x\,dx=du/2$

$$\Rightarrow \frac{1}{2}\ln(x^2+1)+\arctan x-\ln(x+2)+c$$

(6) $\int \frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}dx$

$4\ln(x+1)-3\ln(x-2)+\ln(x+3)$

4

$$\int \frac{11-3x}{x^2+2x-3} = 2\ln(x-1)-5\ln(x+3)$$

(5) $\int \frac{4x-16}{x^2-2x-3}$

$$\int \frac{2x^2-10x}{(x+3)(x-1)^2}\,dx$$

$$\frac{2x^2-10x}{(x+3)(x-1)^2} = \cancel{A(x+1)^2} + \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\frac{2x^2-10x}{(x+3)(x-1)^2} = \frac{A(x-1)^2+B(x-1)(x+3)+C(x+3)}{(x+3)(x-1)^2}$$

$$2x^2-10x = A(x^2-2x+1)+B(x^2+2x-3)+C(x+3)$$

$$2x^2-10x = Ax^2-2Ax+A+Bx^2+2Bx-3B+Cx+3C$$

$$2x^2-10x = Ax^2+Bx^2-2Ax+2Bx+Cx+A-3B+3C$$

$$2x^2-10x = (A+B)x^2+x(-2A+2B+C)+(A-3B+3C)$$

$$A+B=2 \text{ --- } (1)$$

$$-2A+2B+C=-10 \text{ --- } (2)$$

$$A-3B+3C=0 \text{ --- } (3)$$

From (1) $A=2-B$ --- (4) And put (4) in (2) & (3)

$$-4+2B+2B+C=-10 \Rightarrow 4B+C=-6$$

$$2-B-3B+3C=0 \qquad -4B+3C=-2$$

$$4C=-8$$

$$C=-2$$

Also

$$4B+C=-6$$

$$4B-2=-6$$

$$4B=-4$$

$$B=-1$$

Finally

$$A+B=2$$

$$A-1=2$$

$$A=3.$$

$$\int\frac{2x^2-10x}{(x+3)(x-1)^2}dx = \int\frac{3}{x+3}dx+\int\frac{-1\,dx}{x-1}+\int\frac{-2}{(x-1)^2}dx.$$

$$= 3\ln(x+3)-\ln(x-1)+2(x-1)^{-1}+C$$

$$= 3\ln(x+3)-\ln(x-1)+\frac{2}{x-1}+C.$$

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