ENGINEERING MATHEMATICS II (ENG 282) LECTURE NOTE

MODELLING AND SIMULATION OF FIRST ORDER SYSTEMS

RADIOACTIVITY

Experiments have shown that a radioactive substance decomposes at each instant of time in a manner that is proportional to the amount of the substance present.

If the amount of substance present at any time t is denoted by $y(t)$, applying the physical law, the time rate of change of the substance is proportional to $y(t)$. That is,

$$\frac{dy(t)}{dt} \propto y(t) \tag{1}$$

$$\frac{dy}{dt} \propto y \tag{2}$$

$$\frac{dy}{dt} = ky \tag{3}$$

$$\frac{dy}{y} = kdt \tag{4}$$

$$\int \frac{dy}{y} = \int kdt \tag{5}$$

$$\ln(y) = kt + C \tag{6}$$

$$y = e^{kt+C} \tag{7}$$

$$y = e^{kt} \times e^{C} \tag{8}$$

$$y = e^{kt} \times y_0 \tag{9}$$

$$y = y_0 e^{kt} \tag{10}$$

For example, given that the initial amount of the substance is 0.5 g,

$$0.5g = y_0 e^{k(0)} \tag{11}$$

$$e^{k(0)} = e^0 = 1 \tag{12}$$

Therefore,

$$0.5g = y_0 \times 1 \tag{13}$$

$$y_0 = 0.5g \tag{14}$$

So,

$$y = (0.5g)e^{kt} \tag{15}$$

The constant, *k*, is positive for exponential growth while it is negative for decay.

Assuming that $k = -1.5hr^{-1}$ (a decay problem),

$$y = (0.5g)e^{(-1.5hr^{-1})t} \tag{16}$$

MATLAB *mfile* for Simulation

```
commandwindow
clearvars
clc
close all

t = 0:0.01:5;
y = 0.5*exp(-1.5*t);
plot(t,y)
xlabel('Time (hr)')
ylabel('Amount of substance present (g)')
grid on
grid minor
```

Graphical Output

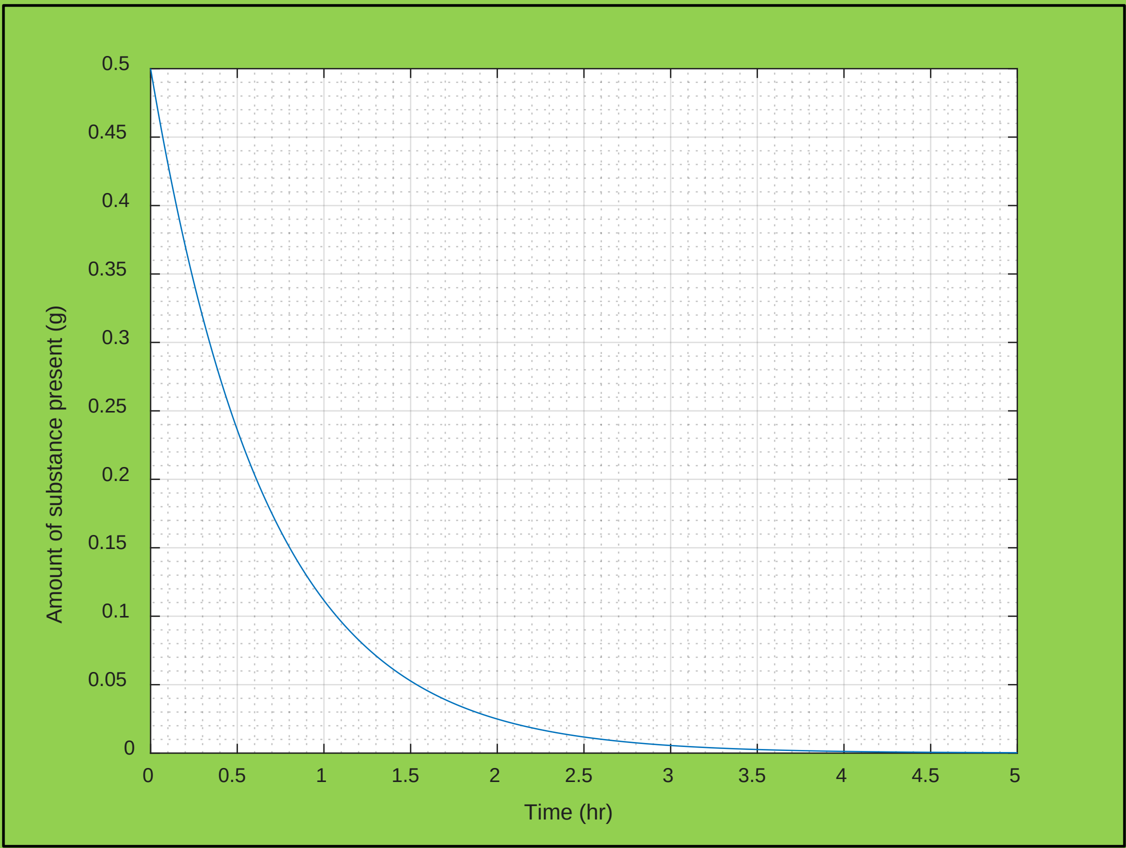

Figure 1: Plot of dynamic response of amount of substance present at t = 5 hr

MIXING

Mixing problems occur quite frequently in industries. Problems of this nature can be solved by taking material balance across the system. To take the material balance, balance law is applied, and it is expressed mathematically as given in Equation (17).

$$\left\{\begin{array}{l}\text{Accumulation rate}\\ \text{within a system}\end{array}\right\} = \left\{\begin{array}{l}\text{Input rate into}\\ \text{the system}\end{array}\right\} - \left\{\begin{array}{l}\text{Output rate from}\\ \text{the system}\end{array}\right\} \tag{17}$$

For instance, considering the tank shown in Figure 2 that contains 1000 gal of water in which 100 lb of salt is dissolved initially. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissolved salt. The mixture in the tank is kept uniform by stirring.

Brine runs out at 10 gal/min. Derive an expression for finding the amount of salt in the tank at any time *t*. Also, plot the dynamic response of the system for $0 \le t \le 700\text{min}$.

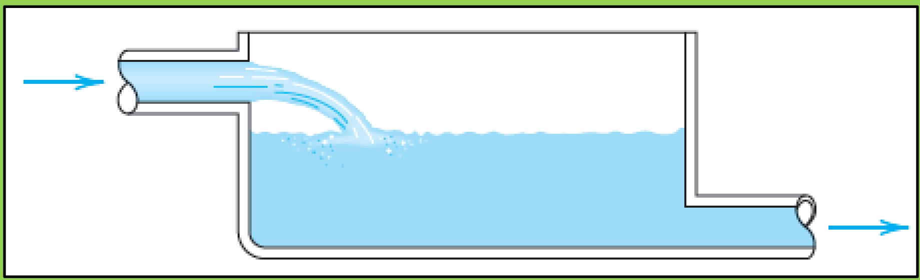

Figure 2: A tank system for mixing (Source: Kreyszig, 2011)

Applying the balance law,

$$\left\{\begin{matrix} \textit{Accumulation rate of} \\ \textit{salt within a system} \end{matrix}\right\} = \left\{\begin{matrix} \textit{Input rate of salt} \\ \textit{into the system} \end{matrix}\right\} - \left\{\begin{matrix} \textit{Output rate of salt} \\ \textit{from the system} \end{matrix}\right\} \qquad (18)$$

Denoting the amount of salt present in the tank at any time *t* as *y*, its time rate of change is given as:

$$\frac{dy}{dt} = \dot{y}_{in} - \dot{y}_{out} \qquad (19)$$

Since 10 gallons enter per minute and one gallon contains 5 lb of salt, it means that the amount of salt entering the tank is:

$$\dot{y}_{in} = 10\frac{gal}{\min} \times 5\frac{lb}{gal} = 50\frac{lb}{\min} \qquad (20)$$

The tank contains 1000 gal of water with the dissolved salt, and 10 gallons of the solution leave the tank per minute. That is, $\frac{10gal}{1000gal} = 0.01 = 1\%$ of the content of the tank. If that is the case, 1% of the salt present in the tank will also leave the tank per minute. In other words,

$\dot{y}_{out} = 1\% of\ y$.

Therefore, from Equation (19),

$$\frac{dy}{dt}\frac{lb}{\min} = 50\frac{lb}{\min} - 1\% y\frac{lb}{\min} \quad (21)$$

$$\frac{dy}{dt} = 50 - 0.01y \quad (22)$$

$$\frac{dy}{dt} = -0.01y + 50 \quad (23)$$

$$\frac{dy}{dt} = -0.01\left(\frac{-0.01y}{-0.01} + \frac{50}{-0.01}\right) \quad (24)$$

$$\frac{dy}{dt} = -0.01(y - 5000) \quad (25)$$

$$\frac{dy}{(y - 5000)} = -0.01dt \quad (26)$$

$$\int \frac{dy}{(y - 5000)} = \int -0.01dt \quad (27)$$

$$\int \frac{dy}{(y - 5000)} = -0.01\int dt \quad (28)$$

$$\ln(y - 5000) = -0.01t + C \quad (29)$$

$$y - 5000 = e^{-0.01t + C} \quad (30)$$

$$y - 5000 = e^{-0.01t}e^{C} \quad (31)$$

$$y - 5000 = e^{-0.01t}y_0 \quad (32)$$

$$y - 5000 = y_0e^{-0.01t} \quad (33)$$

$$y = y_0e^{-0.01t} + 5000 \quad (34)$$

Given that when $t = 0\min(initially)$, $y = 100\,lb$,

$$100 = y_0e^{-0.01(0)} + 5000 \quad (35)$$

$$100 - 5000 = y_0 \times 1 \qquad (36)$$

$$y_0 = -4900 \qquad (37)$$

So,

$$y = -4900e^{-0.01t} + 5000 \qquad (38)$$

$$y = 5000 - 4900e^{-0.01t} \qquad (39)$$

With the aid of Microsoft Excel and MathCAD, the dynamic response has been obtained to be as shown in Figure 3.

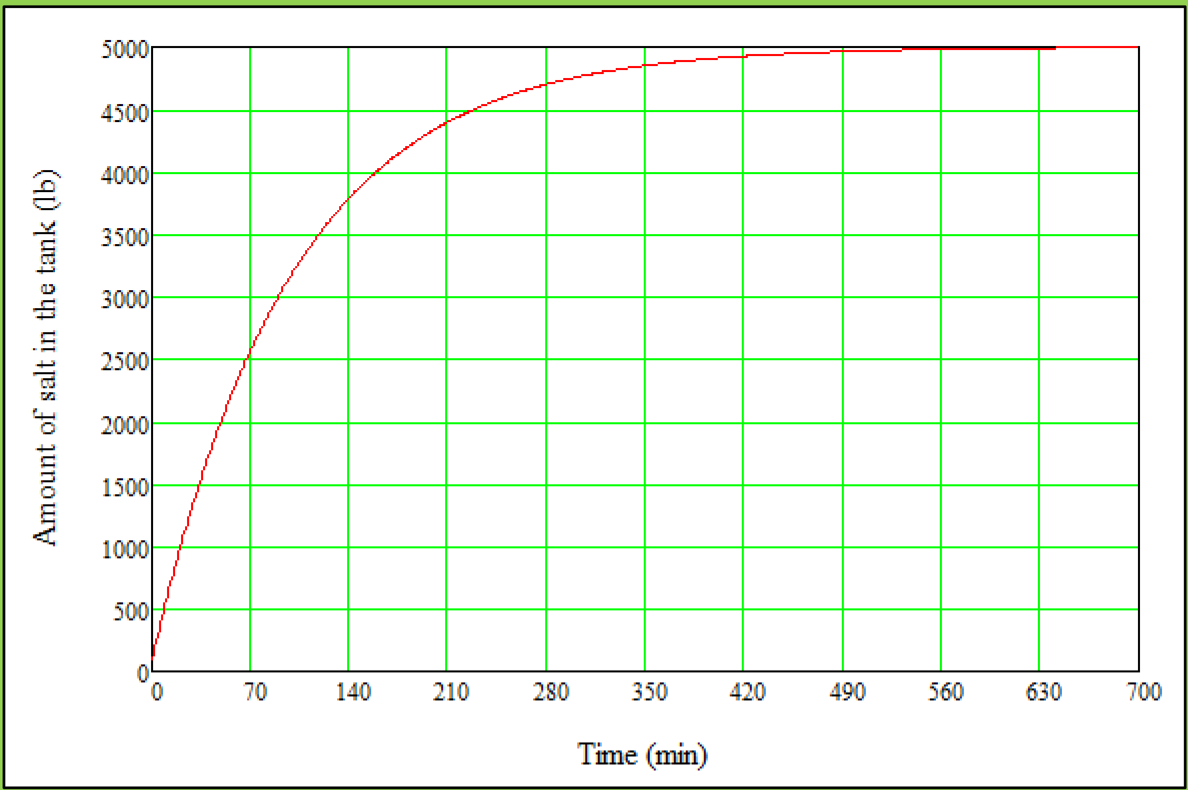

Figure 3: Dynamic response of the mixing system

HEATING

Systems involving temperature changes can be modelled by applying Newton's law of cooling that states that "the time rate of change of the temperature of a body, which conducts heat well, is proportional to the difference between the temperature of that body and that of the surrounding medium". Denoting the temperature of the body by T and that of the surrounding medium by T_a, the law can be expressed mathematically as in Equation (40) to be:

$$\frac{dT}{dt} \propto \left(T - T_a\right) \tag{40}$$

which gives

$$\frac{dT}{dt} = k\left(T - T_a\right) \tag{41}$$

Example

The daytime temperature in a certain office building was maintained at 70 °F during winter. The heating was shut off at 10 p.m. and turned on again at 6 a.m. On a certain day, the temperature inside the building at 2 a.m. was found to be 65 °F. The outside temperature was 50 °F at 10 p.m. and dropped to 40 °F by 6 a.m. Obtain the dynamic response of the temperature of the system for $0 \le t \le 11\,hr$ and, using the graph, write the temperature inside the building when the heat was turned on at 6 a.m.

Solution

Taking the time (10 a.m.) when the heating was shut off while it was being maintained at 70 °F as the initial time, it means that $T\left(0\right) = 70\,^{o}F$.

Since the outside (surrounding) temperature was not constant, an average value can be used. Therefore,

$$T_a = \frac{50\,^{o}F + 40\,^{o}F}{2} = 45\,^{o}F \tag{42}$$

Applying Newton's law of cooling, Equation (41) becomes

$$\frac{dT}{dt} = k(T - 45) \quad (43)$$

$$\frac{dT}{T - 45} = kdt \quad (44)$$

$$\ln(T - 45) = kt + C \quad (45)$$

$$e^{\ln(T-45)} = e^{(kt+C)} \quad (46)$$

$$(T - 45) = e^{(kt+C)} \quad (47)$$

$$T - 45 = e^{kt}e^{C} \quad (48)$$

$$T - 45 = e^{kt}T_o \quad (49)$$

$$T = T_o e^{kt} + 45 \quad (50)$$

At this point, there is a need to find the values of T_o and k. To find T_o, knowing that when t = 0 hr, T = 70 ^oF $\left(T(0) = 70\,^oF\right)$ means that

$$70 = T_o e^{k(0)} + 45 \quad (51)$$

$$70 = T_o(1) + 45 \quad (52)$$

$$70 - 45 = T_o \quad (53)$$

$$T_o = 25 \quad (54)$$

Therefore,

$$T = 25e^{kt} + 45 \quad (55)$$

To find the value of k, considering the statement that: "On a certain day, the temperature inside the building at 2 a.m. was found to be 65 °F" implies that, as given in Table 1, when t = 4 hr, T = 65 ^oF,

Table 1: Clock, time and temperature of the building

Clock	Time (hr)	Temperature (°F)
10:00 PM	0	70
11:00 PM	1	
12:00 AM	2	
1:00 AM	3	
2:00 AM	4	65
3:00 AM	5	
4:00 AM	6	
5:00 AM	7	
6:00 AM	8	?

and,

$$65 = 25e^{k(4)} + 45 \quad (56)$$

$$-25e^{4k} = +45 - 65 \quad (57)$$

$$-25e^{4k} = -20 \quad (58)$$

$$25e^{4k} = 20 \quad (59)$$

$$e^{4k} = \frac{20}{25} \quad (60)$$

$$e^{4k} = 0.8 \quad (61)$$

$$\ln\left(e^{4k}\right) = \ln\left(0.8\right) \quad (62)$$

$$4k = \ln\left(0.8\right) \quad (63)$$

$$k = \frac{\ln\left(0.8\right)}{4} \quad (64)$$

$$k = \frac{-0.2231}{4} \quad (65)$$

$$k = -0.05578 \quad (66)$$

Therefore,

$$T = 25e^{-0.05578t} + 45 \quad (67)$$

With the aid of Microsoft Excel, the dynamic response was obtained as shown in Figure 4.

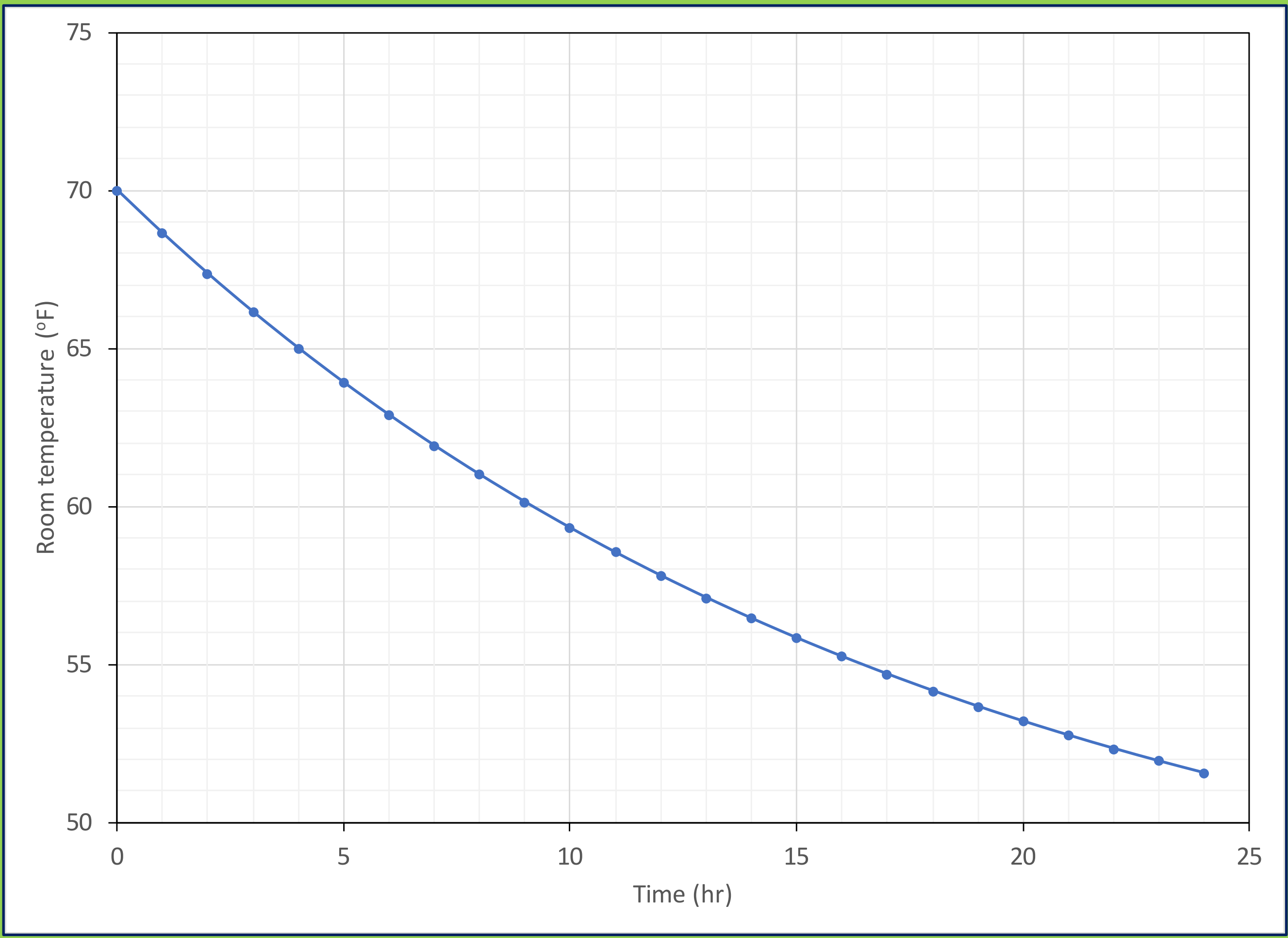

Figure 4: Dynamic response of the building temperature

From the graph, the temperature inside the building when the heat was turned on at 6 a.m. (t = 8 hr) was determined to be 61 °F.

ELECTRIC CIRCUIT

A current in a circuit causes a voltage drop, IR, across a resistor (Ohm's law) and a voltage drop, $L\frac{dI}{dt}$, across a conductor. The sum of these two voltage drops gives electromotive force (EMF). This concept is referred to as Kirchhoff's Voltage Law.

Generally, Kirchhoff's Voltage Law states that "The voltage (that is, the electromotive force) impressed on a closed loop is equal to the sum of the voltage drops across all the other elements of the loop." For this circuit involving a resistor and a conductor, mathematically,

$$E(t) = IR + L\frac{dI}{dt} \tag{68}$$

Equation (68) can be written as given in Equation (69).

$$\frac{E(t)}{L} = \frac{dI}{dt} + \frac{R}{L}I \tag{69}$$

Example

Formulate the model of the resistor-inductor circuit given in Figure 5. Assuming that the circuit contains a battery of $E = 48V$ as a constant electromotive force, a resistor of $R = 11\Omega$ and an inductor of $L = 0.1H$, solve the resulting ordinary differential equation and plot the current flowing as a function of time for $0 \le t \le 0.09\,s$. Take the initial value of the current to be zero.

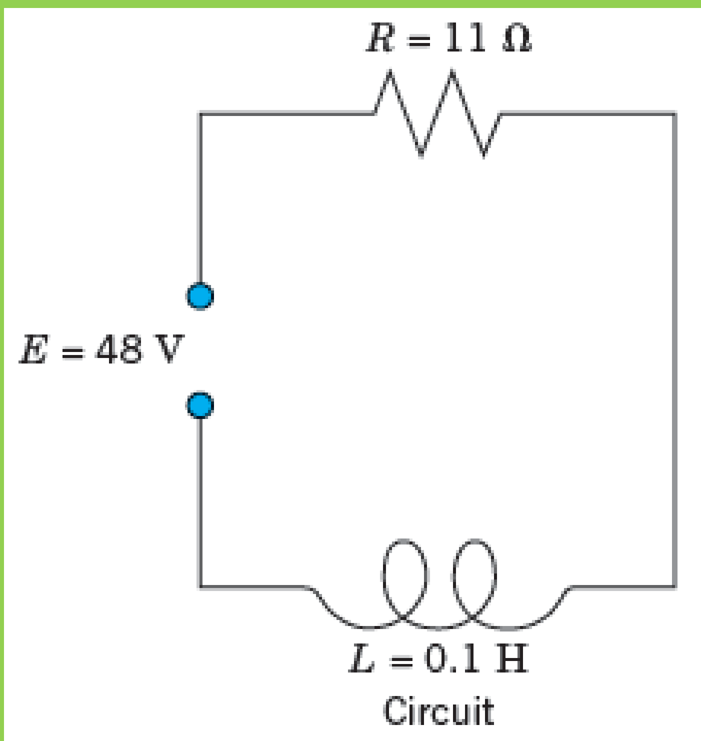

Figure 5: A resistor-inductor circuit

Solution

Using Kirchhoff's Voltage Law given in Equation (69) and written as in Equation (70),

$$\frac{dI}{dt}+\frac{R}{L}I=\frac{E}{L} \qquad (70)$$

the solution to the problem can be obtained by applying the method of using integrating factor for solving linear equations of the form

$$\frac{dy}{dx}+Py=Q \qquad (71)$$

and whose solution is given as

$$y\,IF=\int Q\,IF\,dx \qquad (72)$$

where

$$IF=e^{\int P\,dx} \qquad (73)$$

In this case, comparing Equations (70) and (71),

$$P=\frac{R}{L} \qquad (74)$$

$$Q=\frac{E}{L} \qquad (75)$$

$$x=t \qquad (76)$$

$$y=I\left(Current\right) \qquad (77)$$

Therefore,

$$\int P\,dx=\int\frac{R}{L}dt=\frac{R}{L}t \qquad (78)$$

$$IF=e^{\int P\,dx}=e^{\int P\,dt}=e^{\frac{R}{L}t} \qquad (79)$$

Substituting, Equation (72) becomes

$$I\,e^{\frac{R}{L}t}=\int\frac{E}{L}e^{\frac{R}{L}t}\,dt \qquad (80)$$

$$I e^{\frac{R}{L}t} = \frac{E}{L}\frac{e^{\frac{R}{L}t}}{R/L} + C \qquad (81)$$

$$I e^{\frac{R}{L}t} = \frac{E}{\cancel{L}}\frac{e^{\frac{R}{L}t}}{R/\cancel{L}} + C \qquad (82)$$

$$I e^{\frac{R}{L}t} = \frac{E}{R}e^{\frac{R}{L}t} + C \qquad (83)$$

"Take the initial value of the current to be zero" implies that when t = 0 s, I = 0 A. That is,

$$(0)e^{\frac{R}{L}(0)} = \frac{E}{R}e^{\frac{R}{L}(0)} + C \qquad (84)$$

$$0 = \frac{E}{R}(1) + C \qquad (85)$$

$$-\frac{E}{R} = +C \qquad (86)$$

$$C = -\frac{E}{R} \qquad (87)$$

As such, Equation (83) becomes

$$I e^{\frac{R}{L}t} = \frac{E}{R}e^{\frac{R}{L}t} - \frac{E}{R} \qquad (88)$$

$$I = \frac{\frac{E}{R}e^{\frac{R}{L}t}}{e^{\frac{R}{L}t}} - \frac{\frac{E}{R}}{e^{\frac{R}{L}t}} \qquad (89)$$

$$I = \frac{E}{R} - \frac{E}{R}e^{-\frac{R}{L}t} \qquad (90)$$

$$I = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right) \qquad (91)$$

Substituting $E = 48V$, $R = 11\Omega$ and $L = 0.1H$,

$$I=\frac{48}{11}\left(1-e^{-\frac{11}{0.1}t}\right) \quad (92)$$

$$I=\frac{48}{11}\left(1-e^{-110t}\right) \quad (93)$$

MathCAD Worksheet for the Simulation

$$I(t) := \frac{48}{11}\cdot\left(1-e^{-110\cdot t}\right)$$

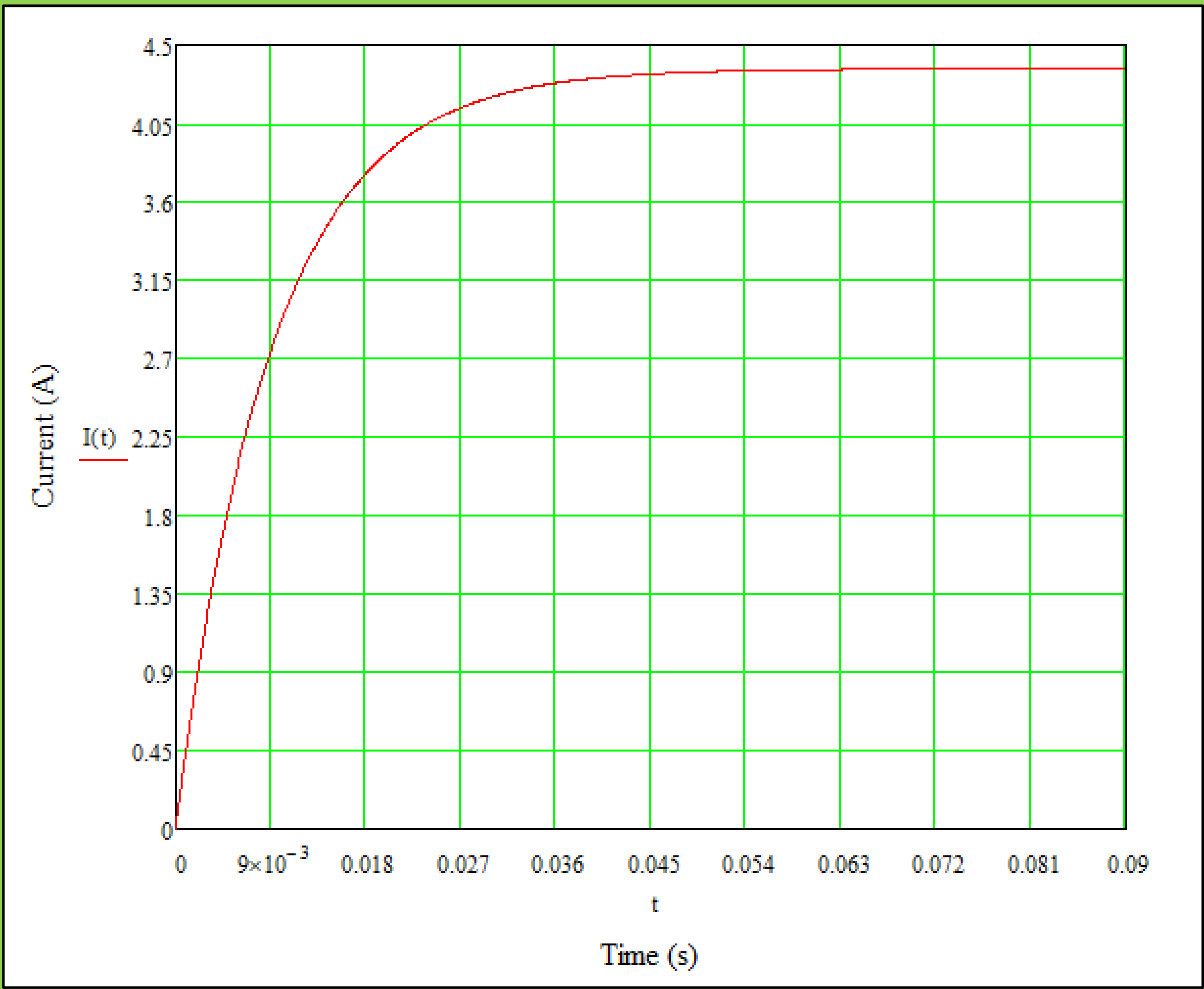

Figure 6: Plot of the current flowing as a function of time

Always wishing you success.

I care about you.

Please, stay safe.

Thank you!