MEE 322

FLOW THROUGH A CIRCULAR PIPE

Hagen Poiselle's theory is based on the following assumptions :

(1) The fluid flow follows Newton's Law of viscosity.
Newton's law of viscosity states that , the shear stress $\tau \; in \; fluid$ element is directly proportional to the rate of shear strain i.e $\tau = \mu \frac{\partial u}{\partial y}$

(2) There is no slip of fluid particles at the boundary (fluid particles adjacent to the pipe wall have zero viscosity).

Let us consider a horizontal pipe as shown in figure 2.

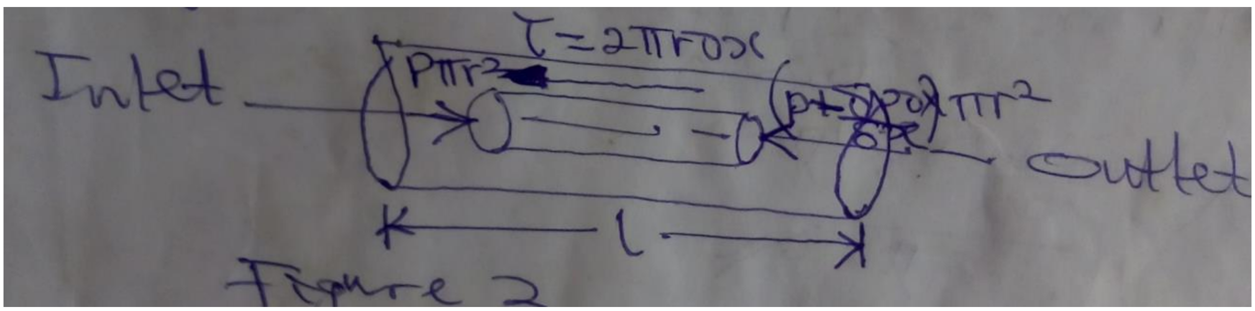

Figure 2

Figure 2 shows a horizontal circular pipe of radius 'R" having laminar flow of fluid through it. Also consider a small concentric cylinder (fluid element) of radius "r" and distance "x" as a free body shear force "F" is given as $F = \tau . 2\pi r . \delta x$

Let P be the intensity of the pressure at the left end and $P + \frac{\partial P}{\partial x}$ be the pressure at the right hand side end.

Forces acting on the fluid element are as follows :

(1) The shear force $F = \tau . 2\pi r . \delta x$ on the surface of fluid element .
(2) Pressure force $P \, \pi r^2$ on the left end .
(3) Pressure force $(P + \frac{\partial P}{\partial x} \delta x)\pi r^2$ on the right end.

For a steady flow, the net forces on the cylinder must be equal to zero.

$$P \, \pi r^2 - \tau . 2\pi r . \delta x - (P + \frac{\partial P}{\partial x} \delta x)\pi r^2 = 0 \quad \text{-----------------------------} \; (1)$$

$$P \, \pi r^2 - \tau . 2\pi r . \delta x - P\pi r^2 - \frac{\partial P}{\partial x} \delta x \, \pi r^2 = 0 \quad \text{----------------------------} \; (2)$$

$$-\tau . 2\pi r . \delta x = \frac{\partial P}{\partial x} \delta x \, \pi r^2 \quad \text{-----------------------------} \; (3a)$$

$$-\tau = \frac{\frac{\partial P}{\partial x}\delta x\,\pi r^2}{2\pi r.\delta x}$$

$$-\tau = \frac{\partial P}{\partial x}\frac{r}{2}$$

$$\tau = -\frac{\partial P}{\partial x}\frac{r}{2} \quad \text{---------------------------------} \quad (3b)$$

Recall from the Newton's law of viscosity

$$\tau = \mu\frac{\partial u}{\partial y} \quad \text{--------------------------------------} \quad (4)$$

$$y = R - r \quad \text{-------------------------------------} (5a)$$

When radius R = 0

$$y = -r \quad \text{----------------------------} \quad (5b)$$

$$\delta y = -\delta r \quad \text{-----------------------------} \quad (5c)$$

Substitute equation (5c) into (4)

$$\tau = -\mu\frac{\partial u}{\partial r} \quad \text{------------------------------------} (6)$$

Comparing (3b) and (6)

$$-\mu\frac{\partial u}{\partial r} = -\frac{\partial P}{\partial x}\frac{r}{2}$$

$$\mu\delta u = \frac{\partial P}{\partial x}\frac{r}{2}\delta r \quad \text{-----------------------------------} \quad (7)$$

$$\delta u = \frac{\partial P}{\partial x}\frac{r}{2\mu}\delta r \quad \text{------------------------------------} \quad (8)$$

Integrate with respect to r

$$u = \frac{1}{2\mu}\frac{\partial P}{\partial x}\frac{r^2}{2} + C$$

$$u = \frac{1}{4\mu}\frac{\partial P}{\partial x}r^2 + C \quad \text{----------------------------} \quad (9)$$

At the boundary condition r = R and u = 0

$$0 = \frac{1}{4\mu}\frac{\partial P}{\partial x}R^2 + C$$

$$C = -\frac{1}{4\mu}\frac{\partial P}{\partial x}R^2 \quad \text{------------------------------} \quad (10)$$

Substitute equation (10) and (9)

$$u = \frac{1}{4\mu}\frac{\partial P}{\partial x}r^2 - \frac{1}{4\mu}\frac{\partial P}{\partial x}R^2 \quad \text{--------------------} \quad (11)$$

$$u = \frac{1}{4\mu}\frac{\partial P}{\partial x}(r^2 - R^2) \quad \text{-------------------------} \quad (12a)$$

$$u = -\frac{1}{4\mu}\frac{\partial P}{\partial x}(R^2 - r^2) \text{ ----------------------- (12b)}$$

For maximum velocity ; r = 0

$$u_{max} = -\frac{1}{4\mu}\frac{\partial P}{\partial x}R^2 \text{ ------------------------- (12c)}$$

Equation (12b) and that of (12c) are fore velocity and maximum velocity for the fluid in a circular pipe respectively .

Divide equation (12b) by (12c)

$$\frac{u}{u_{max}} = \frac{-\frac{1}{4\mu}\frac{\partial P}{\partial x}(R^2 - r^2)}{-\frac{1}{4\mu}\frac{\partial P}{\partial x}R^2}$$

$$\frac{u}{u_{max}} = \frac{(R^2 - r^2)}{R^2}$$

$$\frac{u}{u_{max}} = 1 - \frac{r^2}{R^2}$$

$$u = u_{max}\left(1 - \frac{r^2}{R^2}\right) \text{ ------------------------ (13)}$$

LAMINAR FLOW IN CIRCULAR PIPE

For discharge Q

The discharge through an elementary ring of thickness " δr " at radial discharge " r " is given by :

$$\delta Q = 2\pi u r \delta r \text{ ---------------------(14)}$$

Substitute (13) into (14)

$$\delta Q = u_{max}\left(1 - \frac{r^2}{R^2}\right) 2\pi r \delta r \text{ ------------------------------------ (15a)}$$

$$\int dQ = \int_0^R u_{max}\left(1 - \frac{r^2}{R^2}\right) 2\pi r \delta r \text{ ------------------------------ (15b)}$$

$$Q = u_{max} 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R \text{ -- (16)}$$

$$Q = u_{max} 2\pi \left[\frac{R^2}{2} - \frac{R^4}{4R^2}\right]$$

$$Q = 2\pi u_{max} \left[\frac{2R^4 - R^4}{4R^2}\right]$$

$$Q = 2\pi u_{max} \left[\frac{R^4}{4R^2}\right]$$

$$Q = 2\pi u_{max} \left[\frac{R^2}{4}\right]$$

$$Q = \frac{\pi}{2} u_{max\,R^2} \text{ -------------------------------- (17)}$$

THE AVERAGE VELOCITY OF FLOW

$\overline{u} = \frac{Q}{A}$ **(continuity equation)**

$\overline{u} = \frac{\frac{\pi}{2} u_{max\,R^2}}{\pi R^2}$

$\overline{u} = \frac{u_{max}}{2}$ **------------------------------ (18) For a circular pipe**

By substituting equation (12c) into equation (18)

$\overline{u} = \frac{-\frac{1}{4\mu}\frac{\partial P}{\partial x} R^2}{2}$

$\overline{u} = -\frac{1}{8\mu}\frac{\partial P}{\partial x} R^2$ **----------------------------------- (19)**

$-\delta P = \frac{8\mu\,\overline{u}}{R^2}\frac{\delta x}{}$ **---------------------------------- (20)**

The pressure difference between 2 sections 1 and 2 at a distance x1 and x2 is given as :

$-\int_{P1}^{P2} \delta P = \int_{X1}^{X2} \frac{8\mu\,\overline{u}}{R^2}\frac{\delta x}{}$

P1 – P2 = $\frac{8\mu\,\overline{u}}{R^2}[X2 - X1]$ **------------------------- (21)**

Note : X2 – X1 = L and P1 – P2 = ΔP **Hence :**

$\Delta P = \frac{8\mu\,\overline{u}\,L}{R^2}$ **-- (22)**

Also note that ; $\frac{D}{2} = R$

$\Delta P = \frac{8\mu\,\overline{u}\,L}{[\frac{D}{2}]^2}$

$\Delta P = \frac{32\mu\,\overline{u}\,L}{D^2}$ **--------------------------------------- (23)**

$\overline{u} = \frac{Q}{A}$

$\Delta P = \frac{32\mu\,\overline{\frac{Q}{A}}\,L}{D^2}$

$\Delta P = \frac{32\mu\,\overline{Q}L}{D^2 A}$ **---------------------------------- (24)**

$A = \frac{\pi D^2}{4}$

$\Delta P = \frac{32\mu\,\overline{Q}L}{D^2 . \frac{\pi D^2}{4}}$

$\Delta P = \frac{128\,\mu\,\overline{Q}L}{\pi D^4}$ **------------------------------- (25)**

Equation (23) and (24) are referred to as Hagen Poiseiulle's equation .

Example 1

Oil of viscosity 700 centipoise and specific gravity 0.9 flow through a horizontal pipe of diameter 80 mm. If the pressure drop in 100m length of pipe is 1800 kN/m^2 . Calculate :

- **(a) Rate of flow of oil**
- **(b) Centre line velocity (maximum velocity)**
- **(c) Total frictional drag over 100m length**
- **(d) Power required to maintain the flow**
- **(e) The velocity gradient at the pipe wall**
- **(f) The velocity and shear stress at 80 mm from the wall**

SOLUTION

Coefficient of dynamic viscosity " μ" = 700 cp

1000cp = 1 Ns/m^2

700 cp = μ

$\mu = \frac{700}{1000}$ **= 0.7 Ns/m^2**

$-\frac{\partial P}{\partial x} = \frac{1800 \times 10^3}{100}$

$\frac{\partial P}{\partial x} = -\frac{1800 \times 10^3}{100}$

Pipe diameter D = 80 mm = 0.08 m

Specific gravity of oil = 0.9

Density of the oil $\rho = 0.9 \times 10^3$ kg/m^3

Area of the pipe A = $\frac{\pi D^2}{4}$

A = $\frac{3.142 \times 0.08^2}{4}$ **= 0.0050 m^2**

(a) Rate of flow of oil = $Q = A\,\bar{u}$

$\bar{u} = -\frac{1}{8\mu}\frac{\partial P}{\partial x}R^2$

$\bar{u} = (-\frac{1}{8\times 0.07})(-\frac{1800\times 10^3}{100})\,(0.04^2)$

$\bar{u}$ **= 5.14 m/s**

$Q =$ **0.0050** $\times\ 5.14$

$Q =$ **0.0257 m^3/s**

Alternatively

$\Delta P = \frac{32\mu\,\bar{u}\,L}{D^2}$

$1800 = \frac{32\times 0.7\,\mu\,\bar{u}\times 100}{0.08^2}$

$\bar{u}$ **= 5.14 m/s**

Q = A$\bar{u}$

Q = 0.0050 $\times 5.14$

= 0.0257 m^3/s

Nature of flow

Re = $\frac{\rho u D}{\mu}$

Re = $\frac{0.9\times 10^3 \times 5.14\times 0.08}{0.7}$

Re = 528.69

Since Re < 2000

The flow is laminar

(b) Centre line velocity

u_{max} **= 2** $\bar{u}$

= 2 $\times$ **5.14**

u_{max}**= 10.28 m/s**

(c) Total frictional drag F_D

F_D **=** $\pi_o\,\pi D l$

$\tau_o = -\frac{\partial P}{\partial x}\frac{r}{2}$

$\tau_o = \frac{1800\times 10^3\ \times 0.04}{100\times\ 2}$

$\tau_o = 360$ **N/m^2**

F_D **=** $360\ \times 3.142\ \times 0.08\ \times 100$

F_D **= 9047.7 N**

$F_D \approx =$ **9.048 kN**

(d) Power required to mantainthe flow

$P = F_D . \overline{u}$

P = 9047.7 $\times$ 5.14

P= 46505 Watts

P =46.s kW

(e) Velocity gradient at the pipe wall

$\tau_o = \mu \frac{\partial u}{\partial y}$ **@ y= 0**

$\frac{\partial u}{\partial y} = \frac{\tau_o}{\mu}$

$\frac{\partial u}{\partial y} = \frac{360}{0.7}$

$\frac{\partial u}{\partial y}$ **= 514.2857** s^{-1}

(f) Velocity at 80 mm from the wall

$u = -\frac{1}{4\mu} \frac{\partial P}{\partial x} (R^2 - r^2)$

y = 0.08m

y = R – r

0.08 = 0.04 – r

-0.04 = r

$u = -\frac{1}{4 \times 0.7} (- \frac{1800 \times 10^3}{100}) (0.04^2 - (-0.04)^2)$

u = 0 m/s

(g) Shear stress at 80 mm from the wall

$\frac{\tau_{mm}}{r} = \frac{\tau_o}{R}$

$\frac{\tau_{mm}}{0.04} = \frac{360}{0.04}$

τ_{mm} **= 360 N/m^2**